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Numerical Solutions of Asymmetric, First-Price, Independent Private Values Auctions

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Abstract We propose a powerful, fully automated, and numerically robust algorithm to compute (inverse) equilibrium bid functions for asymmetric, Independent Private Values, First-Price auctions. The algorithm relies upon a built-in algebra of local Taylor-series expansions in order to compute highly accurate solutions to the set of differential equations characterizing first order conditions. It offers an extensive user friendly menu whereby one can assign commonly-used distributions to bidders and can also create arbitrary (non-inclusive) coalitions. In addition to (inverse) bid functions, the algorithm also computes a full range of auxiliary statistics of interest (expected revenues, probabilities of winning, probability of retention under reserve pricing and, on request, optimal reserve price). The algorithm also includes a built-in numerical procedure designed to automatically produce local Taylor-series expansions for any user-supplied distribution, whether analytical or tabulated (empirical, parametric, semi- or non-parametric). It provides a tool of unparalleled flexibility for the numerical investigation of theoretical conjectures of interest and/or for easy implementation within any numerical empirical inference procedure relying upon inverse bid functions.

Keywords Auctions · Numerical solution · Asymmetric first price · Independent private values · Ex-ante heterogeneity · Expected revenues · Optimal reserve · Collusion

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1 Introduction

In this article, we propose a powerful numerical algorithm to solve first-price, single-object auctions, where bidders draw Independent and Private Values (hereafter IPV) from heterogeneous distributions, allowing for subsets of bidders to collude and for a set reserve price. We also provide operational univariate quadratures to evaluate probabilities of winning, as well as expected revenues for the bidders and the auctioneer. The latter is used to compute optimal reserves under asymmetric environments. This also enables us to provide insights as to whether collusion among subsets of bidders is sustainable.

We first review some of the relevant literature. Much of the earlier auction literature assumed that bidders draw their signals from a common underlying distribution. Pioneering contributions include Riley and Samuelson (1981), Milgrom and Weber (1982), Mathews (1983), and Maskin and Riley (1984). Important theoretical results such as revenue equivalence theorems obtain under symmetry. However, the assumption of symmetry is often far too restrictive for many empirical applications.

Relaxing the symmetry assumption prevents analytical derivation of (first-price) bid functions and, therefore, considerably complicates revenue comparisons. Nevertheless, important results have been derived under asymmetry. For example, existence and uniqueness results under asymmetry can be found in Lebrun (1996, 1999, 2006) or Maskin and Riley (2000a,b). Furthermore, under stochastic dominance, Maskin and Riley (2000a) show that the high-bid auction dominates the open-bid auction in terms of seller revenue and that the strong bidder (with the stochastically dominant distribution) shades his bid more than the weak bidder. They also provide examples of situations where the seller revenue is higher in open auctions than in high auctions. From a numerical perspective, a pioneering contribution which lead to the present article is found in Marshall et al. (1994) (hereafter MMRS). They proposed a numerical algorithm to compute first-price, equilibrium-bid functions in a two (subgroups of) players asymmetric environment under uniform distributions. Actually, the MMRS framework also implicitly assumes stochastic dominance. Marshall and Schulenberg (1998) modified MMRS to accommodate reserve prices set by the auctioneer.

Other numerical algorithms have been proposed in the literature. Of particular interest is the program BIDCOMP[2] by Li et al.[1] The program is user-friendly but restricted to (truncated) power and/or Normal density functions. It has allowed Riley and Li (1997) to make several strong claims relative to a variety of asymmetric scenarios. Two quotes from the authors are relevant for the present article: "Possibly there are other important classes of distributions for which quite different conclusions may emerge. We strongly encourage any suggestions along these lines;" and, "Because the differential equations which define the equilibrium bid functions are very poorly behaved at the lower end-point, there are some quite complex technical issues which had to be dealt with before we could develop such a program." The algorithm we present here addresses these two key issues at an unparalleled level of generality and provides, therefore, a powerful numerical tool to reassess Riley and Li's main findings

[1] See Riley and Li (1997) for comments and applications. The program can be downloaded from the site http://www.econ.ucla.edu/riley/bidcomp/.

in a much broader context. The illustrations presented below contribute a first step toward a more systematic analysis, which goes beyond the objectives of the present article, but does belong to our research agenda.

Another relevant article is that of Bajari (2001) who proposes several numerical methods to solve asymmetric auctions (and procurements) in order to compare competition and collusion. The first algorithm is essentially that of Riley and Li (1997) which is found to be often slow to converge, an issue of significant concern in applied work where it may be necessary to evaluate the inverse bid functions a large number of times. A second algorithm makes the initial guess that participants bid their values and then iterates on the best responses. This algorithm is faster than the first one, but convergence is not guaranteed as pointed out by Bajari himself and confirmed by our own earlier experimentation with iterated best responses. Athey (1997) confirms numerical instability within a similar framework using discretized strategies and types spaces (though, as pointed out by Bajari, the use of closed form derivatives—as available e.g., under normality—can significantly alleviate, but not eliminate numerical instability). Bajari's third algorithm assigns parametric flexible functional forms to the bid functions (typically low order polynomials), and solves for equilibrium within such constrained strategy spaces. It is closely related to the concept of constrained strategic equilibrium (CSE) proposed by Armantier and Richard (1997) and will often produce accurate numerical approximations to the (unknown) equilibrium bid functions in view of their typical smoothness. It is found to be fast, at least with good starting values. It remains, however, an approximation whose accuracy needs to be carefully assessed. Armantier and Richard (1997) suggest comparison of the individual best responses to CSE strategies, which can be numerically evaluated point by point. Note that the algorithm we propose offer such verifications as an option.

To some extent, Bajari's third algorithm is the one which is most closely related to ours since both rely upon polynomial approximations. There is, however, a fundamental difference between the two. Bajari's algorithm produces approximate solutions (as emphasized by Bajari himself) based on a single *global* polynomial approximation for each (inverse) bid function. Ours relies upon a fine grid sequence of *local* polynomial approximations in order to produce highly accurate and exact solutions to the first-order conditions. Moreover, as explained below, these local (Taylor-series) approximations can also be used for highly accurate numerical evaluation of all related statistics of interest, such as expected revenues and probabilities of winning, and the probability of retention under reserve pricing. Bajari does not discuss the computation of such statistics, but does provide first-order approximations for some in the context of a truncated normal distribution example.

The article by Athey (2001) presents powerful existence theorems for a broad class of games of incomplete information under a single crossing condition. She relies upon infinite sequences of increasingly finely discretized games and suggests that a similar approach could be used to compute equilibrium solutions. An earlier working article, Athey (1997), provides examples. However, as discussed above, convergence remains a critical issue since the resulting sequence cannot be constructed into a global contraction mapping. As Riley and Li (1997), as well as Athey (2001) emphasized, the pathological behavior of the system of differential equations at the origin is important.

All in all, the literature clearly offers several innovative proposals for the numerical evaluation of (inverse) bid functions for single object IPV first-price auctions. Nevertheless, there does not currently exist user-friendly, fully-automated algorithms of sufficient generality either to allow broad investigations of theoretical issues of interest or, foremost, to meet the needs of practitioners (uniform, power functions and/or normal distributions are typically far too restrictive to provide adequate empirical characterizations of bidders' value distributions). Riley and Li's BIDCOMP[2] software contributes a pioneering step toward user-friendliness, but remains too restrictive in its selection of distributions, nor can it accommodate cartel formation (since the distributions it currently incorporates are not closed form order statistics). The other algorithms we discussed require significant custom programming by their potential users. The algorithm we propose in the present article fills that gap.

In the present article, we generalize the MMRS algorithm to a much broader class of first-price, asymmetric IPV auction and procurement problems, allowing for arbitrary numbers of (subgroups of) players independently drawing their valuations from arbitrary distributions. Common distributions (exponential, Weibull, beta, normal, log-normal, . . .) are offered as options in the program. Additional distributions can easily be added by users in the form of a subroutine. Our program takes care of constructing Taylor-series expansions for these distributions. The only (standard) restriction is that these distributions have common support. Stochastic dominance is not required. This will enable us to investigate whether existing results generalize when stochastic dominance no longer holds. As in MMRS, we are actually computing numerical solutions to a system of ordinary differential equations (ODEs) characterizing the first-order conditions for a Nash equilibrium. The solution belongs to a class of two-point boundary value problems and is evaluated by recursive application of (low order) Taylor-series expansions. As highlighted by previous authors (Marshall et al. 1994; Riley and Li 1997; as well as Athey 2001) the problem is ill-defined at the origin and requires backward extrapolation starting from the end-point. Since the latter is unknown it needs to be determined by iteration. Uniqueness of this end-point follows from Lebrun (2006) and is also proven by backward extrapolation. Based on Lebrun's derivation, the search for the end-point is very well-behaved, as we shall discuss further below. We note that Bajari's (2001) third algorithm, which relies upon global polynomial approximation, does not require an explicit end-point search, but iterates instead on individual best responses.

For ease of implementation, our algorithm currently relies upon equal spacing subdivisions of the support of the component distributions. While this has proved to be numerically stable for most distributions, occasional pathologies (specifically, densities which are not bounded away from zero on their supports) would require smarter adaptive selection of step size. Such robustification goes beyond the objective of the present article and are currently addressed by increasing the number of points in the grid as much as needed.

The key advantage offered by our algorithm relative to MMRS lies in its capability to accommodate arbitrary distributions, providing us with a powerful tool to investigate whether classic results (revenue equivalence, a.s.o.) extend to situations where symmetry and/or stochastic dominance are no longer assumed. This feature also provides broad flexibility for the analysis of (sub)coalitions.

The article is organized as follows. The baseline model and the solution algorithm are described in Sect. 2; expected revenue calculations are provided in Sect. 3 for first-price auctions and in Sect. 4 for second-price auctions; numerical examples are presented in Sect. 5; numerical accuracy is discussed in Sect. 6 ; Sect. 7 presents the procurement version of the algorithm and Sect. 8 concludes.

2 The Algorithm

2.1 Baseline Model

We consider here a single object IPV first-price auction. Bidders submit sealed bids, the highest bidder wins and pays his bid. There are N potential bidders. Only those with private valuations above the reserve price R set by the auctioneer submit competitive bids. Bidders are ex-ante heterogeneous. Each bidder belongs to one of n types. Each type is characterized by a distribution function F_i on a common support $[\underline{v}, \overline{v}]$. There are k_i bidders in group i for a total of $N = \sum_{i=1}^{n} k_i$ (potential) bidders.

Bid functions are denoted by the Greek $\varphi_i, i = 1, \ldots n$. Bidders are assumed to be risk neutral with utility from winning the auction with a bid b given a type v defined as $U_i(v-b) = v-b$. The generalization to constant risk aversion is fairly trivial and will not be discussed here. Clearly, utility from winning the auction is increasing in the individual's signal. Under these assumptions, Proposition 5 of Maskin and Riley (2000b) establishes the existence of a monotonic pure-strategy equilibrium in the standard first-price auction. Indeed, Lebrun (1996) has shown that these bid functions are strictly monotone and increasing, therefore, invertible. Inverse bid functions are denoted by the Greek letter $\lambda_i, i = 1, \ldots, n$. Uniqueness of such equilibrium for the N player game has been derived by Lebrun (1999), Maskin and Riley (2000b) and Bajari (2001).

Here we assume further that F_i is twice continuously differentiable with a density f_i bounded away from zero on $[\underline{v}, \overline{v}]$. Whence, under these assumptions, Lebrun (1999) proves that the equilibrium is unique, and that the inverse bid functions have a common support $[R, t_*]$, where t_* is the bid associated with the valuation $\bar{v}$, and R is the reserve price set by the auctioneer. We show in this article that this equilibrium is amenable to numerical analysis, and presents itself as a natural extension to the methods proposed in MMRS. As such, the (numerical) determination of t_* is a critical component of the problem to be solved.

2.2 The Differential Equations

Let $t = \varphi_i(v)$ denote the equilibrium bid submitted by bidder i with private signal $v \in [\underline{v}, \overline{v}]$. For the ease of notation, bidders with signals lower than the reserve R are assumed to bid their signals, whence $\varphi_i(v) = v$ for $v \leq R$. Let $v = \lambda_i(t)$ denote inverse bid functions. Following Lebrun (1999), the λ_is share a common support $[\underline{v}, t_*]$. For the ease of presentation, we momentarily assume that t_* is known. Bidder i with signal $v \in [R, \overline{v}]$ submits a bid t, which is solution of the optimization problem

$$t = \arg \max_{u \in (R, \overline{v})} (v - u) \cdot [F_i(\lambda_i(u))]^{k_i - 1} \prod_{j \neq i} [F_j(\lambda_j(u))]^{k_j}. \tag{1}$$

The ODEs defined by the first order conditions (FOCs) are given by

$$\prod_{s=1}^{n} F_s(\lambda_s(t)) = (\lambda_i(t) - t) \cdot \left[\sum_{j=1}^{n} k_{i,j}^* f_j(\lambda_j(t))\lambda_j'(t) \prod_{s \neq j} F_s(\lambda_s(t)) \right], \tag{2}$$

where $k_{i,i}^* = k_i - 1$ and $k_{i,j}^* = k_j$ for $j \neq i, i : 1 \to n$. Let $\ell_i(t) = F_i(\lambda_i(t))$. Equation 2 is rewritten as

$$1 = [F_i^{-1}(\ell_i(t)) - t] \cdot \left[\sum_{j=1}^{n} k_{i,j}^* \frac{\ell_j'(t)}{\ell_j(t)} \right], \quad i = 1 \to n. \tag{3}$$

The boundary conditions for λ_i and ℓ_i are given by

$$\lambda_i(R) = R, \quad \lambda_i(t_*) = \overline{v}, \quad i\colon 1 \to n, \text{ and} \tag{4}$$

$$\ell_i(R) = F_i(R), \quad \ell_i(t_*) = 1, \quad i\colon 1 \to n, \tag{5}$$

respectively. As noted by several authors and discussed in our introduction, the system (3) of ODEs is ill-behaved at the lower boundary. If, for example, $R = \underline{v}$ then a recursive application of l'Hospital's rule for $t \to \underline{v}^+$ produces the result that the right derivative of ℓ_i at $\underline{v}$ is given by

$$\lim_{t \to \underline{v}^+} \ell_i'(t) = f_i(\underline{v}) \cdot \frac{N}{N-1}, \quad i\colon 1 \to n, \tag{6}$$

and, most importantly, that all higher-order right derivatives at $\underline{v}$ are zero. If instead $R > \underline{v}$, then

$$\lim_{t \to R^+} \ell_i'(t) = +\infty, \quad i\colon 1 \to n, \tag{7}$$

as in Marshall and Schulenberg (1998). Whence, following MMRS, we shall solve the ODEs (3) backward starting from the right boundary $\ell_i(t_*) = 1$, assuming momentarily that t_* is known.

2.3 The Baseline Algorithm

Our algorithm amounts to constructing piecewise polynomial approximations to the ℓ_is from which (as discussed in Sect. 2.4 below) approximations for the λ_is and φ_is immediately follow. Assuming we just computed $x_{i,0} = \ell_i(t_0)$ where $t_0 \in (R, t_*)$, we describe next how to construct Taylor-series expansions for the ℓ_is at t_0 which are then used to compute $x_{i,1} = \ell_i(t_1)$ at the next point $t_1 = t_0 - \Delta t$, where Δt denotes the selected step size.

Since Eq. 3 involves compositions, product, and sum of functions, brute force Taylor-series expansions would be unnecessarily complicated (nor would it provide the auxiliary expansions required below for expected revenues and probability calculations). Instead, we develop an algebra of Taylor-series expansions by treating separately the individual operations required for Eq. 3. This amounts to introducing the following auxiliary expansions:

$$\ell_i(t) = \sum_{j=0}^{\infty} a_{i,j} \cdot (t - t_0)^j, \tag{8}$$

$$F_i^{-1}(\ell_i(t)) - t = \sum_{j=0}^{\infty} p_{i,j}(t - t_0)^j, \tag{9}$$

$$\ell_i'(t)/\ell_i(t) = \sum_{j=0}^{\infty} b_{i,j} \cdot (t - t_0)^j, \tag{10}$$

$$F_i^{-1}(x) = \sum_{j=0}^{\infty} d_{i,j}(x - x_0)^j. \tag{11}$$

Our algorithm takes as input the Taylor-series expansions of F_i^{-1} (actually, the expansion of all commonly used distributions are pre-programmed and, as discussed in Sect. 2.4.4, our program also includes automated construction of Taylor-series expansion for arbitrary distributions). The algorithm then iterates between Eqs. 8, 9, and 10 in that order, increasing the order of approximation by one unit at a time. The fact that Eq. 10 includes a derivative provides the key to moving from order $J - 1$ to order J. Specifically, given $\{(a_{ij}, b_{ij}, p_{ij}), j : 1 \to J - 1\}$, step J of our algorithm follows the sequence:

(1) Links Eq. 10 from step $J - 1$ to Eq. 8 for step J through the identity $\ell_i'(t) = \ell_i(t) \cdot \left[\ell_i'(t)/\ell_i(t)\right]$ in order to compute a_{iJ};
(2) Applies lemma 1 (composition) in Appendix A to compute p_{iJ} from Eq. 9;
(3) Uses the ODE's linking Eqs. 9 and 10 to compute b_{iJ}.

The relationships between these coefficients and that of other functions of interest, such as the F_is (input) and the φ_is (output), are discussed in Sect. 2.4. The details of the computations we just outlined are given by Eqs. 12–22 below.

- **The computation of $a_{i,J}$ given $\{(a_{i,j}, b_{i,j}); j < J\}$.** The corresponding recurrence relationship obtains from the identities $\ell_i'(t) = \ell_i(t)$. $\ell_i'(t)/\ell_i(t)$ which together with formulae (8) and (9) imply the identities

$$\sum_{j=1}^{\infty} j a_{i,j}(t - t_0)^{j-1} = \left[\sum_{r=0}^{\infty} a_{i,r}(t - t_0)^r\right] \cdot \left[\sum_{s=0}^{\infty} b_{i,s}(t - t_0)^s\right]. \tag{12}$$

Equating the coefficients of order $J-1$ produces the following relationship

$$a_{i,J} = \frac{1}{J}\sum_{r=0}^{J-1} a_{i,r} b_{i,J-r-1}, \quad (i\colon 1 \to n;\ J\colon 1 \to J_M), \tag{13}$$

with initial conditions

$$a_{i,0} = \ell_i(t_0), \quad b_{i,0} = \ell_i'(t_0)/\ell_i(t_0), \quad i\colon 1 \to n. \tag{14}$$

- **The computation of $p_{i,J}$ given $\{(a_{i,j},\ d_{i,j});\ j \le J\}$.** The corresponding relationship obtains by application of Lemma 1 in Appendix A to the composition of F_i^{-1} (input) and ℓ_i (output from (13)), accounting for the additional factor $t = [t_0 + (t - t_0)]$. Whence we have

$$p_{i,J} = \sum_{r=1}^{J} d_{i,r}\theta_{i,r,J} - z_J, \quad (i\colon 1 \to n;\ J\colon 1 \to J_M), \tag{15}$$

$$\theta_{i,r,J} = \sum_{s=1}^{J-r+1} a_{i,s}\theta_{i,r-1,J-s}, \quad (r\colon 1 \to J), \tag{16}$$

with $z_0 = z_1 = 1$, $z_J = 0$ for $J > 1$, and initial conditions

$$p_{i,0} = F_i^{-1}(x_{i,0}), \quad \theta_{i,0,0} = 1 \quad (i\colon 1 \to n). \tag{17}$$

- **The computation of $b_{i,j}$ given $\{p_{i,j};\ j \le J);\ (b_{i,j};\ j < J)\}$.** The corresponding relationships originate from the ODEs themselves. Substituting expansions (9) and (10) into Eq. 3 produces the identities

$$1 = \left[\sum_{r=0}^{\infty} p_{i,r}(t-t_0)^r\right]\left[\sum_{\ell=1}^{n} k_{i,\ell}^{*}\sum_{s=0}^{\infty} b_{\ell,s}(t-t_0)^s\right], \tag{18}$$

for $i\colon 1 \to n$ and $\ell\colon 1 \to n$ Eq. 18 can be rewritten as

$$1 = \sum_{j=0}^{\infty}\left[\sum_{\ell=1}^{n} k_{i,\ell}^{*}\left(\sum_{r=0}^{j} p_{i,r} b_{\ell,j-r}\right)\right](t-t_0)^j. \tag{19}$$

Equating the coefficients of $(t-t_0)^J$ to zero for $J > 1$ and rearranging the corresponding identities into matrix form produces the following vectorial recurrence relationship

$$P_0(I_n - i_n k')b_J = c_J, \tag{20}$$

where $P_0 = \text{diag}(p_{1,0}, \ldots, p_{n,0})$, I_n is the identity matrix of order n, $i'_n = (1 \ldots 1)$, $k' = (k_1, \ldots k_n)$, $b'_J = (b_{1,J}, \ldots, b_{n,J})$, $c_0 = -i_n$ and

$$c_J = \begin{pmatrix} \vdots \\ \sum_{\ell=1}^{n} k^*_{i,\ell} \left(\sum_{r=1}^{J} b_{\ell,J-r} \right) \\ \vdots \end{pmatrix}, \quad J > 0. \tag{21}$$

Standard formulae for partitioned matrices produce the following expressions for the determinant and inverse of $(I_n - i_n k')$:

$$|I_n - i_n k'| = 1 - N, \quad (I_n - i_n k')^{-1} = I_n - \frac{i_n k'}{N-1}. \tag{22}$$

Formulae (12) to (22) for $J : 0 \to J_M$ define our baseline recurrences algorithm for the evaluation of Taylor-series expansions at an arbitrary base point $t_0 \in (R, t_*)$, from which function values at a new point $t_1 = t_0 - \Delta t$ are approximated.

2.4 Additional Details

A number of additional details need to be addressed in order to complete an operational implementation of our baseline algorithm.

2.4.1 Numerical Search for t_*

With very few exceptions, one of which is found in Appendix A of MMRS, t_* cannot be found analytically. Instead, we shall rely on the uniqueness result in Lebrun (1999), together with the initial conditions (5), to define t_* as

$$t_* = \arg \min_{t_f \in (R, \overline{v})} \sum_{i=0}^{n} \left[\ell_i(R|t_f) - F_i(R) \right]^2, \tag{23}$$

where $\ell_i(\cdot|t_f)$ denotes the solutions to the ODEs in (3) under a tentative terminal condition $\ell_i(t_f) = 1$. Note that since

$$\lim_{t \to R^+} [F_i^{-1}(\ell_i(t)) - t] = 0, \tag{24}$$

the coefficients $(p_{i,0}; i : 1 \to n)$ should be zero for $t_0 = R$. This prevents us from solving the system (20) at $t_0 = R$ but we do not need to do so. Instead we compute $\ell_i(R|\cdot)$ from the Taylor-series expansions at $t_0 = R + \Delta t$. Substituting these approximate values in the objective function (23) suffices to produce very accurate estimates of t_* for Δt small enough. Alternatively, once we have an estimate of $a_{i,0} = \ell_i(t_0)$, we

can also compute $p_{i,0} = F_i^{-1}(x_{i,0}) - t_0$ according to Eq. 9, and use the objective function $\sum_{i=1}^{n} p_{i,0}^2$. As for the actual minimization, we rely upon the simplex subroutine AMOEBA which is numerically very efficient for our problem.

2.4.2 Additional Taylor-series Expansions

As described in Sect. 2.4.4 below, our algorithm constructs Taylor-series expansions of F_i^{-1} to compute those of $\ell_i, i\colon 1 \to n$. Very little work is required to reformulate it in terms of the primitives of the problem, the distribution F_i and the bid functions φ_i. First, note that the inverse bid functions λ_i are given by

$$\lambda_i(t) = t + \sum_{j=0}^{\infty} p_{i,j}(t - t_0)^j. \tag{25}$$

Next, we can rely upon Lemma 2 in Appendix A to transform Taylor-series expansions of F_i and λ_i into those of F_i^{-1} and φ_i, respectively.

2.4.3 Support Conditions

As expected from the formulation of Eq. 3, our algorithm can become numerically unstable if the ℓ_is get too close to zero. Our current program implementation requires that tail areas of (very) low probability be truncated away. Note that such truncations are commonly imposed in empirical applications since most estimation techniques for auction models critically rely upon the invertibility of bid functions and lack robustness relative to tail area behavior of the latter. See e.g., Donald and Paarsch (1996), Laffont et al. (1995) or Florens et al. (2004). See also **?** for an empirical application where truncation of an assumed Weibull distribution had to be imposed for estimation purposes. Note, however, that many distributions of interest with tractable higher order statistics (e.g., exponential, Weibull or extreme value distribution) have unbounded support. In practice any such distribution F_i with unbounded support will be replaced by a truncated vesion thereof

$$F_i^*(v) = \frac{F_i(v) - F_i(\underline{v})}{F_i(\overline{v}) - F_i(\underline{v})}, \quad \underline{v} < v < \overline{v}. \tag{26}$$

Transforming the Taylor-series expansion of F_i^{-1} into that of F_i^{*-1} follows by application of Lemma 1 in Appendix to the following composite function

$$F_i^{*-1}(u) = F_i^{-1}\left(F_i(\underline{v}) + u[F_i(\overline{v}) - F_i(\underline{v})]\right), \quad 0 \le u \le 1. \tag{27}$$

Such transformations are automated in our computer programs.

2.4.4 Automated Taylor-series Expansions

Analytical Taylor-series expansions for inverse CDFs are available for a number of standard distributions such as the extreme value distributions which are commonly

assumed in empirical applications. However, there are situations where this is not the case. One such important situation is discussed in Sect. 2.4.5 below where we analyze non-inclusive coalitions. Other important examples would be applications where empirical and/or non-parametric CDFs have been numerically evaluated.

In order to accommodate such situations, our program includes a fully-automated numerical procedure for the computation of (piecewise) Taylor-series expansions for the inverses of arbitrary CDFs. This procedure incorporates the following steps:

1. We construct an equally spaced grid $\{u_j;\ j\colon 1 \to J\}$ for the interval [0, 1];
2. Using a standard root finder we compute the corresponding (unequally spaced) grid for the inverse CDF F^{-1}, $\{v_j;\ v_j = F^{-1}(u_j);\ j\colon 1 \to J\}$;
3. Next, we construct a B-spline interpolator for F^{-1}. Specifically, we invoke the IMSL subroutines DBSNAK (to construct a knot sequence) and DBSINT (to compute B-spline coefficients). See de Boor (1978) for numerical details.
4. Finally, we invoke the IMSL subroutine BSCPP to convert the B-spline interpolator into a piecewise polynomial approximation, which provides the Taylor series expansion needed for our algorithm.

2.4.5 IMSL Subroutines

Our program relies upon the following (double precision) subroutines from the IMSL library, which we have found to be numerically very robust components of our algorithm:

DBETDF : Evaluates the beta probability distribution function;
DNORDF : Evaluates the standard normal distribution function;
DZBREN : Finds a zero of a real function that changes sign in a given interval;
DBSNAK : Computes a "not-a-knot" spline knot sequence;
DBSINT : Computes a cubic spline interpolant with the "not-a-knot" condition;
DBSCPP : Converts a spline in B-spline representation to piecewise polynomial representation;
DPPDER : Evaluates the derivatives of a piecewise polynomial.

A FORTRAN-90 program (source and/or executable) covering all the options described in our article can be obtained from the first author. See Section D for a detailed user's guide. The executable version contains all the necessary compiled modules and does not require access to the IMSL library. A user who plans to modify the source code will need access to the IMSL library to compile the new source. This is particularly relevant for the potential addition of new baseline distributions in addition to those currently offered.[2] Our program is organized in such a way that adding a new baseline distribution is a straightforward operation. All it requires is the addition of a new subroutine to compute its distribution function (in the same calling format as DNORDF for the normal) plus a minor modification of the input sequence to accommodate the new option (steps 9–12 in the sequence as described in

[2] The current version of the program offers the following distributions: two parameter Weibull, beta, normal and lognormal. See Appendix D for details on how to combine these into hybrid distributions in the sense of formula (28) below.

Appendix D). Empirical CDFs are allowed as long as they are strictly monotone and, therefore, invertible. The catch is that such additions will require a new compilation for which access to IMSL is required. In general, we would be interested in assisting in the development of significant new extensions to our program.

2.4.6 Non-inclusive Coalitions

It is not the objective of our article to provide a theoretical investigation of the stability of non-inclusive coalitions within a first-price asymmetric framework (which, in many cases, would likely required repeated games concepts). Nevertheless, we can use our algorithm to investigate numerically whether such non-inclusive coalitions could potentially be incentive compatible, and also, whether a strategic auctioneer could reduce the profitability of collusions. Pioneering examples of such computations under (ex-ante symmetric) uniform distributions can be found in MMRS and Marshall and Schulenberg (1998). See also Marshall and Marx (2005) for an in-depth discussion of incentive compatible mechanisms for non-inclusive cartels, as well as an extensive list of related references.

Short of such theoretical analysis our algorithm can be used to numerically evaluate bid functions and expected revenues in the presence of non-inclusive cartels, as long as one treats such a cartel as a single representative bidder. At a minimum, such computations can provide useful insight on potential incentives to defect and on the auctioneer's capability to reduce cartels' profitability. For example, MMRS had already illustrated the fact that within an ex-ante (uniform) symmetric framework outsiders benefit more than insiders (on a per capita basis) from the presence of a non-inclusive cartel. One would not expect such findings to generalize to asymmetric scenarios. In particular, there exist numerous real-life illustrations of the viability of non-inclusive cartels consisting, for example, of better informed players. One such situation was recently highlighted by the conviction of seven leading stamp dealers and auctioneers who, for several years, had agreed not to compete against one another at estate auctions of stamp collections.

Specifically, in the context of our program, an arbitrary cartel consisting of $u = \sum_{i=1}^{n} u_i$ players, where u_i denotes the number of players of type i, is treated as a single player drawing her signal from the corresponding highest order statistics CDF:

$$F^*(v) = \prod_{j=1}^{n} \left[\frac{F_j(v) - F_j(\underline{v})}{F_j(\overline{v}) - F_j(\underline{v})} \right]^{u_j}. \tag{28}$$

Taylor-series expansions for the inverse of F^* are automatically produced by application of the numerical procedure described in Sect. 2.4.4 above. It is also trivial to verify that all probability and expected revenue calculations described below remain valid under such scenarios, with the only modification that the revenue computed represents the cartel's total expected revenue. As discussed above, we do not discuss allocation rules among cartel's members and only provide per capita comparisons between insiders and outsiders.

3 Probabilities of Winning, Expected Revenues and Optimal Reserve Price

Auxiliary statistics such as expected revenues, probabilities of winning, and the probability of retention under reserve pricing, are of critical importance when discussing issues such as the selection of the auction format by the auctioneer, optimal reserve pricing and, last but not least, coalition formation. With respect to the latter, comparisons between bidders' expected revenues under alternative (non-inclusive) collusive agreements provide key insights on bidders' willingness to collude. In particular, as confirmed further by example 3 below, one often finds that non-inclusive cartels are not incentive compatible in single-object, IPV first-price auctions. This is not the case with second-price auctions. Marshall and Marx (2005) provide in-depth analysis of bidder collusion. Such findings are critical components of the auctioneer's selection of auction format when collusion is suspected, a rather common real life occurrence. They are also directly relevant for the selection of an optimal reserve price as an effective tool to minimize the impact of collusion.

In this section we provide expressions for expected revenues and probabilities of winning when the auctioneer sets a reserve price R, in the form of simple integrals (quadratures). Moreover, the the integrands are products of auxiliary functions evaluated by our algorithm over the interval (R, t_*), where t_* itself is an implicit function of R. All the proofs are regrouped in Appendix B.

The probability that group i wins the object under reserve R is given by

$$P_i(R) = k_i \int_R^{t_*} \frac{\ell_i'(t)}{\ell_i(t)} \cdot \prod_{j=1}^{n} [\ell_j(t)]^{k_j} dt. \tag{29}$$

Note that

$$\sum_{i=1}^{n} P_i(R) = 1 - \prod_{j=1}^{n} \left[F_j(R)\right]^{k_j},$$

confirming the obvious result that the probability that the auctioneer retains the item is given by

$$P_0(R) = \prod_{j=1}^{n} \left[F_j(R)\right]^{k_j}. \tag{30}$$

Group i's expected revenue is given by

$$V_i(R) = k_i \int_R^{t_*} \left[F_i^{-1}(\ell_i(t)) - t\right] \cdot \frac{\ell_i'(t)}{\ell_i(t)} \cdot \prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j}. \tag{31}$$

Per capita expected revenue within group i, accounting for subcoalitions ($u_i \geq 1$), is then given by $V_i(R)/(k_i \cdot u_i)$. Finally, assuming that the auctioneer receives a fixed percentage of all winning bids, her revenue is proportional to

$$V_a(R) = t_* - R \prod_{j=1}^{n} \left[F_j(R)\right]^{k_j} - \int_R^{t_*} \prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j} dt. \tag{32}$$

Note that formulae (29), (31) and (32) all depend upon univariate integrals of products of the functions which are being evaluated by our algorithm over a fine grid of values of t in (R, t_*). Therefore, these integrals can be evaluated by univariate quadrature as immediate byproducts of our algorithm. As we typically use grids with anywhere from $N = 500$ to $N = 10{,}000$ equally spaced points, we can rely upon the extended Simpson's rule with remainder proportional to N^{-4} to numerically compute highly accurate estimates of all relevant probabilities and expected revenues.[3]

Moreover, the use of a fixed number of equally spaced grid points implies that these numerical integrals will be continuous functions of R. Whence numerical simplex maximization of $V_a(R)$ with respect to R will itself be numerically very accurate. Note that t_* in formulae (29) to (32) is an implicit function of R so that our algorithm has to be rerun for each value of R selected by AMOEBA.

4 Asymmetric Second-Price Auctions

One of the immediate intended uses of our new algorithm is that of running comparisons between first- and second-price auctions under a variety of asymmetric environments. In order to do so we need to derive operational expressions for expected revenues under second-price auctions. While Vickrey's logic still applies, whereby bidders bid their private values, expected revenue calculations are more complex than under first-price due to a wider range of scenarios for the price paid by the winner.

The expected revenues and the probabilities of winning in the second-price auction environment are given as follows[4]:

$$P_i(R) = k_i \int_R^{\overline{v}} f_i(v) \prod_{j=1}^{n} \left[F_j(v)\right]^{k^*_{i,j}} dv, \tag{33}$$

$$P_0(R) = \prod_{j=1}^{n} \left[F_j(R)\right]^{k_j}, \tag{34}$$

[3] See Press et al. (1986, Chap. 4) or Abramowitz and Segun (1968, Chap. 25) for discussions on the numerical accuracy of the extended Simpson's rule.

[4] The details of these calculations are found in Appendix C.

$$V_a(R) = \overline{v} - RP_0(R) - \int_R^{\overline{v}} \prod_{j=1}^{n} \left[F_j(v)\right]^{k_j} dv$$
$$- \sum_{i=1}^{n} k_i \int_R^{\overline{v}} [1 - F_i(v)] \prod_{j=1}^{n} \left[F_j(v)\right]^{k^*_{i,j}} dv, \tag{35}$$

$$V_i(R) = k_i \int_R^{\overline{v}} [1 - F_i(v)] \prod_{j=1}^{n} [F_j(v)]^{k^*_{i,j}} dv. \tag{36}$$

As above, per capita expected revenue in group i is given by $V_i(R)/(k_i \cdot u_i)$.

As was the case for the first-price auction, formulae (33), (35) and (36) are numerically evaluated by quadrature. All probabilities and expected revenue calculations for first-price and second-price auctions have been incorporated in our algorithm allowing for automated comparisons between first and second-price auctions under a wide variety of asymmetric scenarios. Examples are provided below.

5 Examples

In this section, we present three numerical illustrations of the capabilities of our program. The parameters and, in particular, the truncation range $[\underline{v}, \overline{v}]$ were selected to produce graphically well-separated bid functions. For the first two examples, all type distributions are truncated Weibull of the form given in formula (26) together with

$$F_i(v) = 1 - \exp\left[-(v/a_i)^{b_i}\right]. \tag{37}$$

As mentioned above, our program then automatically produces the Taylor-series expansion for the inverse truncated distribution function F_i^{*-1} as given by Eq. 27.

5.1 Example 1 (Three Individual Bidders)

We first consider 3 individual bidders (low, high, median types) and compute their first-price asymmetric bid functions without reserve, as well as with optimal reserve. We also compute bidders' (per capita) expected revenues, bidders' probabilities of winning, the auctioneer's expected surplus, and probability of retaining the item under reserve pricing. The same statistics are also computed for second-price auctions. The graphs of the first-price asymmetric bid functions with and without reserve are provided in Fig. 1. Relevant statistics are presented in Table 1. We note that the reserve price impacts the bidders differently, the larger impact being obviously felt by the high-type bidder. We also note that in the absence of reserve, first-price is more profitable for the auctioneer (by about 5%) but that the ordering is reversed under optimal reserve. The high-type bidder always prefers second-price, especially in the absence of a reserve.

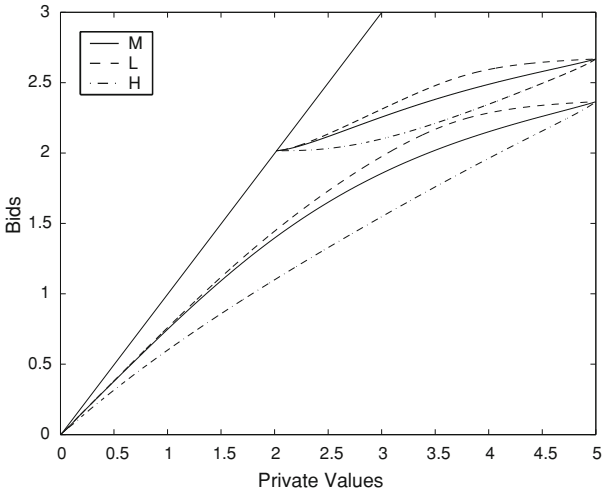

Fig. 1 Three bidders

Table 1 Three bidders (median, low, high)

	Type			Auctioneer
	1	2	3	
k_i	1	1	1	
u_i	1	1	1	
a_i	2.0	1.0	3.39	
b_i	1.0	1.0	2.20	
Mean	1.55	0.966	2.71	
Std. dev.	1.25	0.911	1.15	
First-price, no reserve				
E(revenue)	0.344	0.111	0.912	1.65
Prob 'win'	0.29	0.13	0.58	–
First-price, optimal reserve = 2.016				
E(revenue)	0.225	0.061	0.622	1.851
Prob 'win'	0.22	0.08	0.51	0.18
Second-price, no reserve				
E(revenue)	0.246	0.069	1.16	1.57
Prob 'win'	0.22	0.08	0.70	–
Second-price, optimal reserve = 2.016				
E(revenue)	0.181	0.045	0.692	1.858
Prob 'win'	0.18	0.06	0.58	0.18

$F_i(v) = 1 - e^{-(\frac{v}{a_i})^{b_i}}$, truncated on [0, 5]

5.2 Example 2 (Two Individiual Bidders)

This example illustrates the fact that asymmetric bid functions can cross one another once stochastic dominance no longer applies. Hazard functions are monotone for Weibull distributions. Our choice of shape parameters for this example implies that the

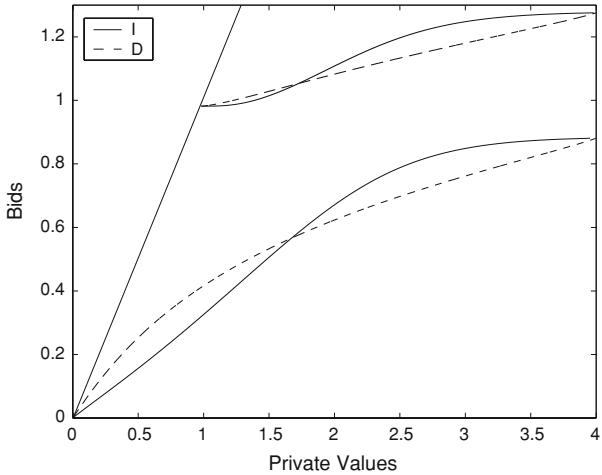

Fig. 2 Two bid functions crossing

hazard function of bidder 1 is increasing ($b_1 = 1.5$) and that of bidder 2 is decreasing ($b_2 = 0.5$). With means close to one another it implies that the distribution functions cross one another at $v = 1.45$. It also implies that, as illustrated by Fig. 2, the two bid functions cross one another at $v = 1.7$. Expected revenues and surpluses, together with probabilities of winning, with and without reserve prices, are reported in Table 2. We note that first-price and second-price auctions are virtually revenue equivalent even though, in the absence of reserve, second-price favors bidder 1. A systematic numerical investigation of whether revenue equivalence holds under a particular asymmetric scenario goes beyond the objectives of the present article but belongs to our research agenda.

5.3 Example 3 (Non-inclusive Cartels)

MMRS offer a numerical investigation of incentive compatibility within subcoalitions when individual bidders all draw their valuations from a common uniform distribution. Within this (single-object) framework, they find that bidders outside the coalitions benefit more than those inside. Here we consider instead an asymmetric scenario where high type bidders collude together in order to protect their informational advantage over low type bidders.

This example is inspired by a recent court case where a group of prominent stamp auctioneers and dealers were found guilty of collusion at estate auctions. While their cartel operated for several years, our example illustrates the fact that such non-inclusive cartels could be incentive compatible even within a single object framework (ignoring, however, proxy defections as analyzed by Marshall and Marx (2005)). We consider two bidders of high type (H) against four (non collusive) bidders of low type (L). Signals are lognormally distributed with a common standard deviation 0.35 and means 1.35 and 0.75, respectively. The common support for signals is the interval [1.5, 6.0].

Table 2 Two bidders

	Type		Auctioneer
	1	2	
Hazard	Increasing	Decreasing	
k_i	1	1	
u_i	1	1	
a_i	1.11	1.50	
b_i	1.50	0.50	
Mean	1.00	0.84	
Std. dev.	0.67	1.01	
First-price, no reserve			
E(revenue)	0.481	0.463	0.440
Prob 'win'	0.58	0.42	–
First-price, optimal reserve = 0.98			
E(revenue)	0.211	0.297	0.656
Prob 'win'	0.33	0.28	0.39
Second-price, no reserve			
E(revenue)	0.55	0.40	0.44
Prob 'win'	0.64	0.36	–
Second-price, optimal reserve = 0.93			
E(revenue)	0.230	0.303	0.660
Prob 'win'	0.37	0.27	0.36

$F_i(v) = 1 - e^{-(\frac{v}{a_i})^{b_i}}$, truncated on [0, 4]

Table 3 No collusion, two high types (H) and four low types (L)

	Mean	Std. dev.	First-price			Second-price		
			Prob.	Rev.	Res.	Prob.	Rev.	Res.
H	3.756	1.030	0.393	0.385		0.415	0.413	
L	2.435	0.724	0.053	0.031		0.042	0.025	
Auc.			0.000	3.557		0.000	3.536	
H	3.756	1.030	0.394	0.386		0.415	0.411	
L	2.435	0.724	0.053	0.031		0.042	0.024	
Auc.			0.000	3.558	2.170	0.001	3.537	2.395

$v_i^H \sim LN(1.35, 0.35)$, $v_i^L \sim LN(0.75, 0.35)$ truncated on [1.5, 6]

Results for the non collusive benchmark scenario are reported in Table 3. Graphs of the corresponding bid functions with and without optimal reserves are provided in Fig. 3. Results for subcoalitions {H,H} and {H,H,L} are reported in Table 4, and Figs. 4 and 5.

The impact of the collusion among high types is greatest under second-price auction. Under first-price, low types also benefit from the presence of the cartel (even more than high types percentage wise). Reserve is most effective under second-price (and would be even more effective if items kept by the auctioneer had a resale value). In the future, we plan to investigate whether the effectiveness of optimal reserve requires precise knowledge of the cartel composition by the auctioneer.

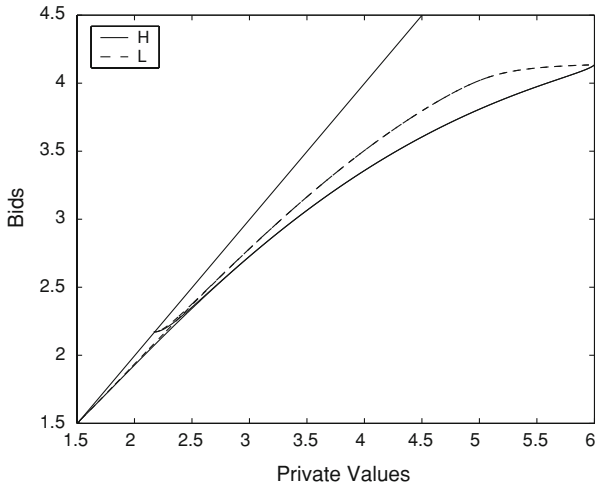

Fig. 3 No collusion

Table 4 Collusion exercise between two high types (H) and four low types (L)

	Mean	Std. dev.	First-Price			Second-price		
			Prob.	Rev.	Res.	Prob.	Rev.	Res.
HH	4.346	0.880	0.668	0.906		0.832	1.227	
L	2.435	0.724	0.083	0.050		0.042	0.025	
Auc.			0.000	3.287		0.000	3.135	
HH	4.346	0.880	0.675	0.857		0.801	0.998	
L	2.435	0.724	0.073	0.044		0.038	0.021	
Auc.			0.026	3.297	2.972	0.048	3.237	3.134
HHL	4.379	0.856	0.706	1.019		0.874	1.398	
L	2.435	0.724	0.098	0.060		0.042	0.025	
Auc.			0.000	3.181		0.000	2.989	
HHL	4.379	0.856	0.709	0.902		0.815	0.977	
L	2.435	0.724	0.077	0.045		0.035	0.020	
Auc.			0.048	3.225	3.134	0.079	3.185	3.300

$v_i^H \sim LN(1.35, 0.35)$, $v_i^L \sim LN(0.75, 0.35)$ truncated on [1.5, 6]

6 Numerical Accuracy and Computational Time

Accuracy of the numerical approximations of the equilibrium bid functions depends primarily on two variables. The first is how fine a grid is chosen on which to evaluate the component distributions. A finer grid leads to higher accuracy of the numerical approximations. The second variable is the order of the Taylor-series approximations chosen to approximate these distributions. A higher order Taylor-series expansion does not necessarily lead to higher accuracy. Indeed, an order of approximation that is too high can lead to significant numerical pathologies.

A reliable method for evaluating accuracy consists of computing pointwise best responses for each individual bidder and comparing them to the NE strategies. Given

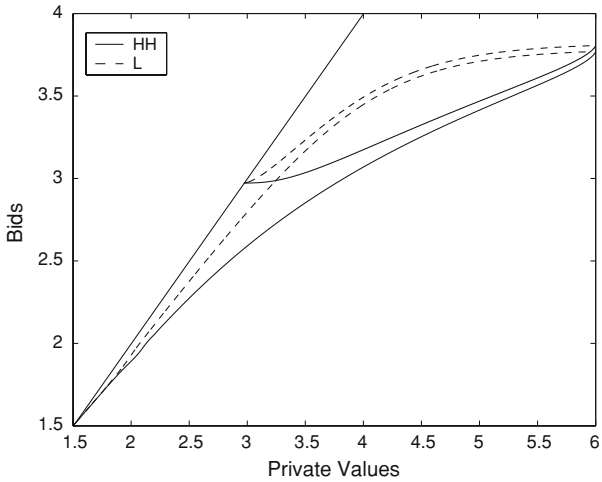

Fig. 4 Two high types colluding

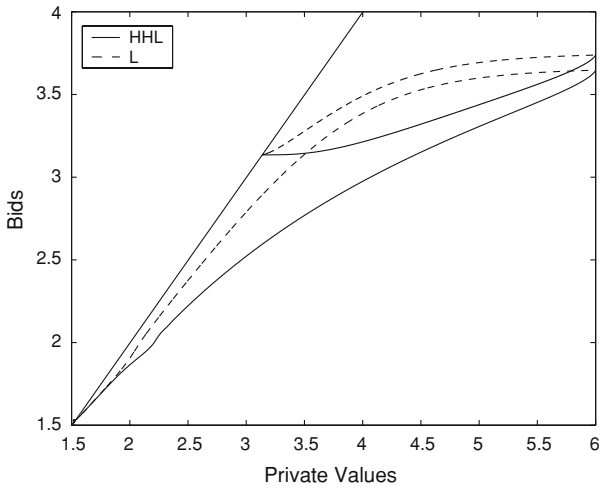

Fig. 5 Two high types and one low type colluding

bidder i's signal, his best response depends on the distribution functions and (inverse) bid functions of his competitors. His best response function does not depend on his own distribution function. This is seen clearly in Eq. 3. Thus, given his competitors' equilibrium strategies and distribution functions, we can use Eq. 3 to compute pointwise the best response of bidder i. His best response function can then be matched against the equilibrium function computed by the algorithm and the difference between these two functions provides a measure of the accuracy of the algorithm. A reasonable metric, and the one we use in this article, is the root of the mean squared deviation (RMSE) between the equilibrium bid function and the best response function.

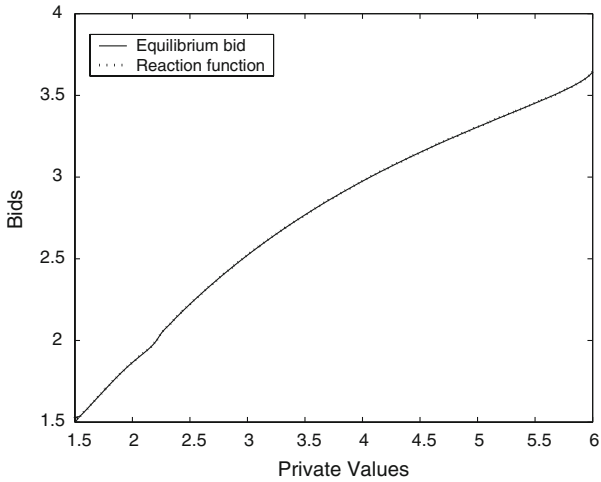

Fig. 6 Comparison of equilibrium bid function and reaction function

An important illustration of the usefulness of such comparisons is provided by example 3. Figure 5 reveals a curious "blip" in the bid function of the coalition. The bid function dips down between private values of 2.0 and 2.5. This gives rise to the question of whether this blip is the result of a numerical error, or if it is a rational response by the collusion to the strategies of the outsiders. This question can be answered by the method of verification we just described. Figure 6 reproduces the equilibrium bid function of the coalition and also plots the best response of the coalition computed as described in Sect. 5. The reaction function (bold dotted line) coincides exactly with the computed bid function (solid line). This confirms that the blip is indeed an equilibrium reaction by the coalition to the strategies of its competitors.

Higher accuracy of the numerical approximations to the equilibrium bid functions comes at the cost of increased computational time. For a small number of types of bidders, one can be liberal with the size of the grid and the order of the Taylor-series expansions. However, for models with a large number of types of bidders, the potential computational time increases significantly. Though this is not a significant obstacle in our current applications, computational time can quickly become a problem in others. An important example is that of empirical applications where the algorithm would be instrumental in the estimation of the underlying private values distributions. In this case, the model would have to be solved for each trial value of the vector of parameters of the private values distributions, and one might have to be conservative with the size of the chosen grid. In this section, we present a small study of the trade off between accuracy and speed as controlled by these two variables.

Table 5 reports the computational time and the RMSE between the equilibrium bids and reaction functions of two bidders. The first panel fixes the order of Taylor-series expansions to 5, and evaluates these statistics for the number of grid points being 500, 1,000, 1,500, and 2,000. The table reveals that the computational time increases linearly with the number of grid points. The computational time increases by 0.11 seconds with a one point increase in the number of grid points. The RMSE for each bidder

Table 5 Study of the trade off between numerical accuracy and computational

Grid	Time (s)	RMSE 1	RMSE 2
Order of expansion, J = 5			
500	37.1880	0.4000	0.0882
1,000	95.3590	0.3988	0.0868
1,500	146.1250	0.3984	0.0864
2,000	202.2500	0.3982	0.0862
J	**Time (s)**	**RMSE 1**	**RMSE 2**
Number of grid points = 500			
2	18.1880	0.4000	0.0882
3	32.2810	0.4000	0.0882
4	38.7030	0.4000	0.0882
5	46.4690	0.4000	0.0882
Number of grid points = 2,000			
2	93.4530	0.3982	0.0862
3	139.4530	0.3982	0.0862
4	173.8120	0.3982	0.0862
5	202.2500	0.3982	0.0862

$F_1(v) = 1 - e^{-v}$, $F_2(v) = 1 - e^{-(\frac{v}{3.39})^{2.2}}$, truncated on [0, 5]

is decreasing and concave in the number of grid points. Indeed, the decrease in the RMSE is very small for large increases in the number of grid points. This suggests that there is not a lot to gain in terms of accuracy by increasing the number of grid points. A relative small number of grid points like 500 provides almost the same numerical accuracy as a larger number of grid points like 2,000.

The second panel of Table 5 fixes the number of grid points to 500 and increases order of Taylor-series expansions incrementally from 2 to 5. The computational time increases linearly by approximately 7 s with each increase in the order of the Taylor-series expansions. Interestingly, the numerical accuracy of the bid functions are invariant to the order of Taylor-series expansion. The third panel of Table 5 bolsters this conclusion. The modest local curvature of the Weibull distribution function could possibly explain the invariance of the accuracy of the program to the order of the Taylor-series expansion.

The conclusion of this exercise is that the investigator looses very little in terms of numerical accuracy by using a relatively small number of grid points and a low order Taylor-series expansion. The time savings is, however, significant.

7 Procurements

Our algorithm can also be applied to procurements. The necessary modifications are minor and are described briefly in this section. For the procurements problem, bidder i with signal $v \in [\underline{v}, R]$ submits a bid t which is solution of the optimization problem

$$t = \arg \max_{u \in (\underline{v}, R)} (u - v) \cdot [H_i(\lambda_i(u))]^{k_i - 1} \prod_{j \neq i} [H_j(\lambda_j(u))]^{k_j}, \tag{38}$$

where $H_i(x) = 1 - F_i(x)$. The ODEs resulting from the FOCs are given by

$$-1 = [H_i^{-1}(\ell_i(t)) - t] \cdot \left[\sum_{j=1}^{n} k_{i,j}^{*} \frac{\ell_j'(t)}{\ell_j(t)} \right], \quad i = 1 \to n, \tag{39}$$

where $\ell_i(t) = H_i(\lambda_i(t))$. The boundary conditions for λ_i and ℓ_i are given by

$$\lambda_i(R) = R, \quad \lambda_i(t_*) = \underline{v}, \quad i\colon 1 \to n, \text{ and} \tag{40}$$

$$\ell_i(R) = H_i(R), \quad \ell_i(t_*) = 1, \quad i\colon 1 \to n, \tag{41}$$

respectively. Under this setup, the algorithm to compute equilibrium bids here mimics exactly the one derived in Sect. 2 to compute equilibrium bids in the auction environment, except for two changes. The first is that the RHS of Eq. 21 is now i_n instead of $-i_n$. The second is that the recursion on the grid of t is a forward iteration instead of a backward iteration. The probabilities of winning and expected revenues in the low-price procurement environment, are given as follows:

$$P_i(R) = -k_i \int_{t_*}^{R} \frac{\ell_i'(t)}{\ell_i(t)} \cdot \prod_{j=1}^{n} [\ell_j(t)]^{k_j} dt, \tag{42}$$

$$P_0(R) = \prod_{j=1}^{n} \left[H_j(R)\right]^{k_j}, \tag{43}$$

$$V_i(R) = -k_i \int_{t_*}^{R} \left[H_i^{-1}(\ell_i(t)) - t\right] \cdot \frac{\ell_i'(t)}{\ell_i(t)} \cdot \prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j}, \tag{44}$$

$$V_a(R) = t_* - R \prod_{j=1}^{n} \left[H_j(R)\right]^{k_j} + \int_{t_*}^{R} \prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j} dt. \tag{45}$$

The corresponding probabilities of winning and expected revenues in the second-price, procurement environment, are given as follows:

$$P_i(R) = -k_i \int_{t_*}^{R} \frac{\ell_i'(t)}{\ell_i(t)} \cdot \prod_{j=1}^{n} [\ell_j(t)]^{k_j} dt, \tag{46}$$

$$P_0(R) = \prod_{j=1}^{n} \left[H_j(R)\right]^{k_j}, \tag{47}$$

$$V_a(R) = \underline{v} - RP_0(R) + \int_{\underline{v}}^{R} \prod_{j=1}^{n} \left[H_j(v)\right]^{k_j} dv$$

$$+ \sum_{i=1}^{n} k_i \int_{\underline{v}}^{R} [1 - H_i(v)] \prod_{j=1}^{n} \left[H_j(v)\right]^{k^*_{i,j}} dv, \tag{48}$$

$$V_i(R) = k_i \int_{\underline{v}}^{R} [1 - H_i(v)] \prod_{j=1}^{n} [H_j(v)]^{k^*_{i,j}} dv. \tag{49}$$

8 Conclusion

In this article, we developed a new, powerful and fully-automated numerical algorithm to solve single-object, asymmetric IPV, first-price auctions. The algorithm allows for a wide variety of (non-inclusive) coalitions, as well as, for the auctioneer to set a reserve price. We provide operational univariate quadratures to evaluate probabilities of winning, the probability that the auctioneer retains the object, as well as the expected revenues to the bidders and the auctioneer. The expected revenue to the auctioneer is maximized with respect to the reserve price to obtain the auctioneer's optimal reserve price.

The article also provides operational expressions for expected revenues and probabilities of winning for corresponding second-price auctions. As in the asymmetric first-price auctions environment, these expressions only require univariate integration of functions evaluated by the program. This results in high numerical accuracy, which is essential when comparing first-price to second-price auctions, as the differences in expected revenues across these environments may be small in many cases. Furthermore, all of the options and quantities that the algorithm provide for the first- and second-price auction environments are replicated for first- and second-price procurements.

Pointwise best responses for each bidder type, given the equilibrium bid of all other types are also calculated by the program. This provides a convenient check of the numerical accuracy of the algorithm, since the best responses function should coincide with their corresponding equilibrium bid functions. This check of accuracy is particularly important in cases where the densities of the private value distribution functions are close to zero at the support points. Our experience with the program is that, in many of these cases, the algorithm still converges. However, due to the possible instability of the algorithm at the support points, the solution may not be accurate. The best response functions therefore provide a convenient and important check of the accuracy of the solution in these cases.

The algorithm proposed in this article relies on accurate piecewise Taylor-series expansions of the inverse of the private values CDFs. While analytical Taylor-series expansions for the inverse CDFs are available for some distributions, there are many

situations in which this is not the case. Relying on analytical Taylor-series expansions would therefore limit the variety of CDFs that could be employed. We avoid this restriction by including in the program a fully-automated numerical procedure for the computation of piecewise Taylor-series expansions for the inverse of arbitrary CDFs. This numerical procedure currently relies B-spline interpolators and piecewise polynomial approximations taken from the IMSL libraries. Readers using the executable version of our program do not need access to the IMSL library. Modifications to the source program will require recompilation with access to IMSL. We can provide assistance for significant new extensions of our program.

Appendix

A Lemmas on Taylor-series Expansions

We prove here two lemmas which are used in the article to evaluate Taylor series expansions for composite and inverse functions. Lemma 1 is taken from MMRS but is included here for the ease of reference.

Lemma A.1 *Let*

$$f(u) = \sum_{j=0}^{\infty} f_j (u - u_0)^j, \quad g(t) = \sum_{j=0}^{\infty} g_j (t - t_0)^j, \tag{A1}$$

together with $u_0 = g(t_0)$. *Then*

$$(f \circ g)(t) = \sum_{j=0}^{\infty} a_j (t - t_0)^j, \tag{A2}$$

where $a_0 = f_0$ *and for* $j \geq 1$

$$a_j = \sum_{k=1}^{j} f_k \theta_{k,j}, \tag{A3}$$

and where the θ*s are evaluated recursively as follows*

$$\theta_{k,j} = \sum_{s=1}^{j-k+1} g_s \theta_{k-1,j-s}, \quad 1 \leq k \leq j, \tag{A4}$$

with $\theta_{0,0} = 1$.

Proof We have

$$(f \circ g)(t) = \sum_{k=0}^{\infty} f_k \left[\sum_{s=1}^{\infty} g_s (t - t_0)^s \right]^k. \tag{A5}$$

Whence a_j is given by formula (A3) where $\theta_{k,j}$ denotes the coefficient of $(t-t_0)^j$ in the k-th power of the factor in brackets. Formula (A4) follows from the identity

$$\sum_{j=k}^{\infty} \theta_{k,j}(t-t_0)^j = \left[\sum_{r=k-1}^{\infty} \theta_{k-1,r}(t-t_0)^r\right] \cdot \left[\sum_{s=1}^{\infty} g_s(t-t_0)^s\right]. \tag{A6}$$

□

Lemma A.2 *Let f^{-1} denote the inverse of f*

$$f^{-1}(x) = \sum_{j=0}^{\infty} h_j(x-x_0)^j, \tag{A7}$$

with $x_0 = f(t_0)$. Then

$$h_0 = x_0, \quad h_1 = f_1^{-1}, \tag{A8}$$

$$h_j = -f_1^{-j} \cdot \left[\sum_{k=1}^{j-1} h_k \theta_{k,j}\right]. \tag{A9}$$

Proof We apply Lemma 1 together with $g = f^{-1}$, whence

$$(g^{-1} \circ g)(t) = t = t_0 + (t-t_0).$$

This implies that $a_0 = a_1 = 1$ and $a_j = 0$ for $j > 1$ in Formula (A3). The proof follows from formulae (A3) and (A4), with the latter implying that $\theta_{j,j} = f_1^g$. □

B Derivation of Expected Revenues and the Probability of Winning in the First-price IPV Auctions

The following conditions have to be met for a bidder from group i to win

$$R < v_i < \overline{v} \quad \text{and} \quad v_j < \lambda_j(\lambda_i^{-1}(v_i)) \quad \text{for} \quad j \neq i. \tag{B1}$$

Whence the probability that group i wins is given by

$$P_i(R) = k_i \int_R^{\overline{v}} f_i(v) \prod_{j=1}^{n} \left[F_j(\lambda_j(\lambda_i^{-1}(v)))\right]^{k_{i,j}^*} dv, \tag{B2}$$

where $k_{i,i}^* = k_i - 1$ and $k_{i,j}^* = k_j$ for $j \neq i$, as in Sect. 2 above. Introducing the change of variable $t = \lambda_i^{-1}(v)$ and rearranging terms yields the following operational expression

$$P_i(R) = k_i \int_R^{t_*} \frac{\ell_i'(t)}{\ell_i(t)} \cdot \prod_{j=1}^{n} [\ell_j(t)]^{k_j} dt. \tag{B3}$$

Summing over groups gives

$$\sum_{i=1}^{n} P_i(R) = \int_{i=1}^{t_*} k_i \ell_i'(t) \prod_{j=1}^{n} [\ell_j(t)]^{k^*_{i,j}}$$

$$= \int_R^{t_*} \left[\prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j} \right]' dt = 1 - \prod_{j=1}^{n} \left[F_j(R)\right]^{k_j} \tag{B4}$$

Group i's expected revenue is given by

$$V_i(R) = k_i \int_R^{\overline{v}} [v - \varphi_i(v)] \cdot f_i(v) \cdot \prod_{j=1}^{n} \left[F_j(\lambda_j(\lambda_i^{-1}(v)))\right]^{k^*_{i,j}} dv, \tag{B5}$$

which can be rewritten as

$$V_i(R) = k_i \int_R^{t_*} \left[F_i^{-1}(\ell_i(t)) - t\right] \cdot \frac{\ell_i'(t)}{\ell_i(t)} \cdot \prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j}. \tag{B6}$$

The auctioneer's revenue is proportional to

$$V_a(R) = \sum_{i=1}^{n} k_i \int_R^{\overline{v}} \varphi_i(v) f_i(v) \prod_{j=1}^{n} \left[F_j(\lambda_j(\lambda_i^{-1}(v)))\right]^{k^*_{i,j}} dv \tag{B7}$$

$$= \int_R^{t_*} t \cdot \left[\prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j} \right]' dt. \tag{B8}$$

Integration by parts produces the following expression

$$V_a(R) = t_* - R \prod_{j=1}^{n} \left[F_j(R)\right]^{k_j} - \int_R^{t_*} \prod_{j=1}^{n} \left[\ell_j(t)\right]^{k_j} dt. \tag{B9}$$

C Derivation of Expected Revenues and the Probability of Winning in the Second-price IPV Auctions

Several pricing scenarios need to be considered. Focusing our attention on group i, let $v_1 > v_2$ denote the two highest order statistics in group i (implicitly assuming that $k_i > 1$, but one verifies that the formulae derived below also apply for $k_i = 1$) and let w_j denote the highest order statistic in group j ($j \neq i$). The following pricing scenarios are relevant:

$$\begin{aligned}
&E_{i,R}\text{: price is } R\text{; i.e., } v_1 > R,\ v_2 < R,\ w_j < R, \quad \text{for } j \neq i;\\
&E_{i,i}\text{: price is } v_2\text{; i.e., } v_2 > R,\ v_2 > w_j, \quad \text{for } j \neq i\text{; and}\\
&E_{i,j}\text{: price is } w_j\text{; i.e., } w_j > R,\ w_j > v_2,\ w_j > w_\ell, \quad \text{for } \ell \neq j, i.
\end{aligned}$$

Probabilities and expected revenues are indexed conformably. The relevant densities are

$$k_i(v_1, v_2) = k_i(k_i - 1) f_i(v_1) f_i(v_2) \left[F_i(v_2)\right]^{k_i - 2}, v_1 > v_2, \tag{C1}$$

$$k_j(w) = k_j f_j(w) \left[F_j(w)\right]^{k_j - 1}. \tag{C2}$$

Note that by relying upon the $k^*_{i,j}$ notation introduced in formula (2), a common treatment applies to scenario $E_{i,i}$ and $E_{i,j}(j \neq i)$. The probability that group i wins and pays either v_2 or w_j is given by

$$\sum_{j=1}^{n} P_{i,j}(R) = k_i \left\{ \sum_{j=1}^{n} k^*_{i,j} \int_R^{\overline{v}} f_i(v_1) \cdot \left\{ \int_R^{v_1} f_j(v) \cdot \left[F_j(v)\right]^{k^*_{i,j} - 1} \right.\right.$$
$$\left.\left. \times \prod_{\ell \neq j} [F_\ell(v)]^{k^*_{i,\ell}}\, dv \right\} dv_1 \right\}, \tag{C3}$$

where v denotes v_2 for $j = i$ and w_j for $j \neq i$. As in Appendix B above, we first apply integration by parts to the outer integral and regroup terms to obtain the following expression

$$\sum_{j=1}^{n} P_{i,j}(R) = k_i \cdot \int_R^{\overline{v}} [1 - F_i(v)] \cdot \left[\prod_{j=1}^{n} [F_j(v)]^{k^*_{i,j}} \right]' dv. \tag{C4}$$

A second integration by parts produces

$$\sum_{j=1}^{n} P_{i,j}(R) = k_i \int_R^{\overline{v}} f_i(v) \prod_{j=1}^{n} \left[F_j(v)\right]^{k^*_{i,j}} dv$$
$$-k_i \left[1 - F_i(R)\right] \prod_{j=1}^{n} \left[F_j(R)\right]^{k^*_{i,j}}. \tag{C5}$$

Note that the second term in the right hand side of formula (C5) represents $P_{i,R}(R)$. Whence the probability that group i wins is given by

$$P_i(R) = P_{i,R}(R) + \sum_{j=1}^{n} P_{i,j}(R) = k_i \int_R^{\overline{v}} f_i(v) \prod_{j=1}^{n} \left[F_j(v)\right]^{k^*_{i,j}} dv. \tag{C6}$$

Note that

$$\sum_{i=1}^{n} P_i(R) = \int_R^{\overline{v}} \left(\prod_{j=1}^{n} \left[F_j(v)\right]^{k_j} \right)' dv = 1 - \prod_{j=1}^{n} \left[F_j(R)\right]^{k_j} = 1 - P_0(R). \tag{C7}$$

Next, we derive the auctioneer expected revenue which is given by

$$V_a(R) = \sum_{i=1}^{n} \left\{ P_{i,R}(R) + k_i \left\{ \sum_{j=1}^{n} k^*_{i,j} \int_R^{\overline{v}} f_i(v)_1 \left[\int_R^{v_1} v f_j(v) \right.\right.\right.$$
$$\left.\left.\left. \times \left[F_j(v)\right]^{k^*_{i,j}-1} \prod_{\ell \neq j} \left[F_\ell(v)\right]^{k^*_{i,\ell}} dv \right] dv_1 \right\} \right\}. \tag{C8}$$

The same integration by parts sequence as for the probability produces the following expression paralleling formula (C6)

$$V_a(R) = -\sum_{i=1}^{n} k_i \int_R^{\overline{v}} \left(v[1 - F_i(v)]\right)' \prod_{j=1}^{n} \left[F_j(v)\right]^{k^*_{i,j}} dv \tag{C9}$$

$$= \int_R^{\overline{v}} v \left(\prod_{j=1}^{n} \left[F_j(v)\right]^{k_j} \right)' dv - \sum_{i=1}^{n} k_i \int_R^{\overline{v}} \left[1 - F_i(v)\right] \prod_{j=1}^{n} \left[F_j(v)\right]^{k^*_{i,j}} dv, \tag{C10}$$

or equivalently, after a third integration by parts,

$$V_a(R) = \overline{v} - R P_0(R) - \int_R^{\overline{v}} \prod_{j=1}^{n} \left[F_j(v)\right]^{k_j} dv$$
$$- \sum_{i=1}^{n} k_i \int_R^{\overline{v}} \left[1 - F_i(v)\right] \prod_{j=1}^{n} \left[F_j(v)\right]^{k^*_{i,j}} dv. \tag{C11}$$

The expected revenue for group i is derived in the same way. We first have

$$V_i(R) = k_i \left\{ \left[\int_R^{\overline{v}} (v - R) f_i(v) dv \right] \prod_{j=1}^{n} \left[F_j(R) \right]^{k^*_{i,j}} + \sum_{j=1}^{n} k^*_{i,j} \right.$$
$$\left. \times \int_R^{\overline{v}} f_i(v_1) \left[\int_R^{v_1} (v_1 - v) f_j(v) \left[F_j(v) \right]^{k^*_{i,j}-1} \prod_{\ell \neq j} [F_\ell(v)]^{k^*_{i,\ell}} dv \right] dv_1 \right\}. \tag{C12}$$

Integration by parts of the first integral in v and of the outer integral in v_1 produces the simpler expression

$$V_i(R) = k_i \left\{ \left[(\overline{v} - R) - \int_R^{\overline{v}} F_i(v) dv \right] \prod_{j=1}^{n} \left[F_j(R) \right]^{k^*_{i,j}} \right\}$$
$$+ \int_R^{\overline{v}} (\overline{v} - v) \left(\prod_{j=1}^{n} \left[F_j(v) \right]^{k^*_{i,j}} \right)' dv + \int_R^{\overline{v}} F_i(v) \left[\prod_j^n [F_j(v)]^{k^*_{i,j}} \right.$$
$$\left. - \prod_j^n [F_j(R)]^{k^*_{i,j}} \right] dv \Bigg\}. \tag{C13}$$

Integration by parts of the second factor in the right hand side of formula (C13) and cancelations produce the following operational expression for V_i

$$V_i(R) = k_i \int_R^{\overline{v}} [1 - F_i(v)] \prod_{j=1}^{n} [F_j(v)]^{k^*_{i,j}} dv. \tag{C14}$$

D User's Guide

Our FORTRAN-90 Program includes all the options described in the article. It currently includes the following four options for the (baseline) private-values CDFs: two parameter Weibull, beta, normal and lognormal. The most powerful feature of our program is that it allows for combining these baseline CDFs into hybrid ones, in the sense of formula (28). Such hybrid CDFs can be used to characterize coalitions or individual players. As explained in Sect. 2.5, coalitions are treated as single players for the computation of equilibrium bids. The only difference is that in the case of coalitions, expected revenues are those of the coalition, since we have not hypothesized distribution mechanisms.

In order to eliminate any confusion, we shall use the prefixes b- and h- to distinguish between baseline and hybrid. In order to illustrate the full flexibility of the program

we provide below the actual input sequence (in italics), together with clarifying comments, for example 3. Multiple inputs are separated by a space. In this example there are two b-types (H, L) each characterized by a lognormal b-CDF and two h-types ((2H, 1L), (1L)) each characterized by the corresponding h-CDF obtained by application of formula (28).

(1) Enter 1 for auctions, or 2 for procurements: 1
(2) Enter number of h-types: 2

These are the hybrid types (2H, 1L) and (1L) which will be defined in steps (5) below.

(3) Enter the order of Taylor-series expansion: 5

We recommend orders between 2 and 5. Theoretically, higher orders produce better approximations, but numerically the computation of higher orders derivatives can have the opposite effect.

(4) Enter the number of grid-points: 2,000

Generally, increasing the number of grid-points increases numerical accuracy. As suggested by Table 5, gains in accuracy are typically small beyond 500 grid-points but can, nevertheless, be important when comparing expected revenues across auctions, as these can be very close to one another.

(5) Enter the number of players of each h-type: 1 3

As mentioned above, we have one single player (coalition) of h-type (2H, 1L) and three players of h-type (1L).

(6) Enter the (common) lower bound of the support of the private-values CDFs: 1.5
(7) Enter the (common) upper bound of the support of the private-values CDFs: 6.0

These are the bounds $\underline{v}$ and $\overline{v}$ in Eq. 28.

(8) Enter a reserve price: 1.5

R should be between $\underline{v}$ and $\overline{v}$. Setting R equal to $\underline{v}$ amounts to no reserve. If one subsequently requests the computation of an optimal reserve (step (14) below), then the search algorithm uses the value entered here as initial value.
The b-CDFs, along with their parameterizations are (see, e.g, Johnson et al. (1994)):

1. Two parameter Weibull $: F(x) = 1 - \exp\left(-\left(\frac{u}{\alpha}\right)^{\beta}\right);$

2. Beta $: F(x) = \int_0^1 \frac{u^{\alpha-1}(1-u)^{\beta-1}}{\int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\mathrm{d}u}\mathrm{d}u;$

3. Normal $: F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{u-\mu}{\sigma}\right)^2\right)\mathrm{d}u;$

4. Lognormal $: F(x) = \int_{-\infty}^{x} \frac{1}{u\,\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln u-\mu}{\sigma}\right)^2\right)\mathrm{d}u.$

Here is a menu of b-CDFs to choose from (see user's guide for parameterizations):
1—two parameter Weibull;
2—beta;
3—normal;
4—lognormal.

(9) *Enter the number of b-CDFs:* 2

These will be the two lognormals characterizing the b-types H and L, respectively.

(10) *Enter the index of each b-CDF :* 4 4

In this example we use two lognormals.

(11) *Enter mu and sigma:* 1.35 0.35
Enter mu and sigma: 0.75 0.35

There is one step (11) for each b-CDFs.

(12) *Enter the exponents of the b-CDF for h-type 1:* 2 1
Enter the exponents of the b-CDF for h-type 2: 0 1

These are the u_i exponents in formula (28) for each h-type. h-types can coincide with b-types, which is the case here for h-type 2 which is also b-type L. After step (12), the program produces the following statistics for each h-type:

MEAN : mean of the truncated distribution;
ST. DEV: standard deviation of the truncated distribution;
LPDF : density function at the lower bound $\underline{v}$;
UPDF : density function at the upper bound $\overline{v}$;
CRIT : lower bound of CDF.

The truncated mean and the standard deviation are evaluated over the support ($\underline{v}$ and $\overline{v}$). LPDF and UPDF allow the user to verify that the densities are (numerically) bounded away from zero as required for uniqueness (Lebrun 1999). Values lower than 0.10D-13 (double precision) are likely to create problems. Since inverse CDFs are critical components of the algorithm, inversion could fail on very low values of the CDF. Based upon repeated experiences, values of CRIT lower than 010.D-07 could destabilize the algorithm.
The output for the current example is as follows:

TYPE	MEAN	ST. DEV	LPDF	UPDF	CRIT
1	4.37929	0.85630	0.45D-12	0.19D+00	0.32D-11
2	2.43531	0.72407	0.56D+00	0.27D-02	0.13D-02

Uniqueness is not guaranteed if any LPDF or UPDF is less than 0.10D-13. The program is numerically unstable and might even crash for values of CRIT less than 0.10D-07. If any of these conditions is violated you might consider shortening the support and/or adjusting the parameters of the b-distributions using the following options:

(13) *Enter 1 to continue, 2 to adjust the parameters and/or support (return to step 6) or 3 to exit:* 1

(14) *Enter 1 if you wish to compute the optimal reserve.*
Enter 2 if you wish to keep the reserve entered in step 8: 1

(15) *Enter an output file name:* illustration.txt

The output file will contain the grid of private values together with the corresponding equilibrium bids for each h-type.

Time taken: 190.7404s
Writing grid points and bids to file: illustration.txt

(16) *Enter 1 to compute expected revenues, and probabilities of winning.*
Enter 2 otherwise: 1

If 1 is entered, the program computes expected revenues, probabilities of winning and probabilities of retention by the auctioneer.

(17) *Enter a revenue file name:* rillustration.txt

This step is skipped if 2 is entered at step 16.

(18) *Enter 1 to compute individual best responses. Enter 2 otherwise:* 1

If 1 is entered, the program computes a best response for each h-type by solving equations (1) point-wise under the equilibrium bid functions for all rival players. Note, in particular that if there are $k_i > 1$ players with h-type i, then $k_i - 1$ rivals are assigned their equilibrium function $l_i(t)$. Comparisons between equilibrium bid functions and individual best responses provide immediate verification of the numerical accuracy of the former. In the present example, the equilibrium bid functions are very accurate in spite of a critical CRIT value of 0.32D-11 for h-type 1.

(19) *Enter a best response file name:* brillustration.txt

This step is skipped if 2 is entered at step 18.

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