Lecture 4. Algebra.

Section 1: . Signature, algebra in a signature. Isomorphisms, homomorphisms, congruences and quotient algebras.

CONTENTS

Reading: Chapter 9: 9.1 – 9.4 of PMW, pp. 247- 253.

0. Why algebra?

Really, why? Algebra is not used widely (at any rate now) in linguistics. But we think that it is very useful to know at any rate a little algebra, to be familiar with some very important basic notions like *homomorphism*, *congruence*, *free algebra*, etc, and with some concrete structures like lattices or Boolean algebras. Algebra gives us another point of view even on formal structures we have known before, for example, on logic. And now algebraic notions are beginning to penetrate in formal linguistic descriptions. You may have heard, for example, about *unification grammars*. Unification is an algebraic notion. To understand it you need to know about congruence, quotient algebras, etc. And you may have heard about the use of *semi-lattice structures* in Godehard Link's and others' work on the semantics of plurals and mass nouns. Lattice structures also seem to be relevant to OT.

The first thing to realize is that "algebra" can be a count noun, not only a proper noun. We will look at a few kinds of algebras and at some fundamental notions used in algebra in general.

1. Algebra in a signature.

1.0. Operations and their names.

Roughly speaking, an algebra is a set and a collection of operations on this set. For example, the set of natural numbers with the operations of addition and multiplication form an algebra. But we would like to consider different sets with "the same" or similar operations on them, for example, addition and multiplication on the set of rational numbers, or on other sets. So we need names for operations. A set of names of operations is called a *signature*.

1.1. Signature and algebra in a signature.

Let **N** = {0,1,2,…} be as usual the set of natural numbers. [Review Partee 1979, Chapter 0]

Definition. A set Ω (of symbols) together with a function $a: \Omega \rightarrow \mathbf{N}$ is called a *signature*. Its elements are called names (symbols) of operations or *operators*. If $a(\omega) = n$, we say that operator ω is n-ary. We write $\Omega(n) = \{\omega \in \Omega \mid a(\omega) = n\}$. [Note that in this case the notation $\Omega(n)$ does *not* represent function-argument application. This use, where the n is 'indexing' a particular subset of Ω, is also fairly common.]

Note that for $n = 0$, an n-ary operator becomes 0-ary. n-ary operators will name n-ary operations of the kind $f: A^n \rightarrow A$ on some set A. In the case $n = 0$ we have 0-ary operations on the set A. Every 0-ary operation just marks some element in A.[1] I.e., a 0-ary operation is what is more familiarly known simply as "a constant".

Definition. A Ω-algebra **A** consists of a set A, called the *carrier* of **A**, together with a family of functions

$$F_n: \Omega(n) \rightarrow \{f \mid f: A^n \rightarrow A\}.$$

Thus, the function F_n maps each operator $\omega \in \Omega(n)$ to an n-ary operation $F_n(\omega)$ on A. We denote the operation $F_n(\omega)$ by $\omega_{\mathbf{A}}$.

If $\langle a_1, ..., a_n\rangle \in A^n$, we will say that $\omega_{\mathbf{A}}(a_1, ..., a_n)$ is (i.e. the expression denotes) the result of the application of the operation $\omega_{\mathbf{A}}$ to the n-tuple $\langle a_1, ..., a_n\rangle$.

Note. The functions F_n play the same role in algebras as the *interpretation function* **I** in models of Predicate Logic which we will discuss later. They assign a particular interpretation (a particular operation[2]) to each operator symbol.

Where there is no ambiguity, we will write $\omega(a_1, ..., a_n)$ for $\omega_{\mathbf{A}}(a_1, ..., a_n)$ and (allowing a commonplace neutralization of the distinction between operator and operation) we will say that $\omega(a_1, ..., a_n)$ is the result of the application of the operator ω to the n-tuple $\langle a_1, ..., a_n\rangle$.

Examples. Numbers

Let us begin with familiar things – numbers and arithmetical operations. Consider the signature Ω_{Numb} = { **zero**, **one**, +, ×} and several algebras in this signature. In this signature **zero** and **one** are 0-ary operators, + and × are binary.

1) Integers can be considered as a Ω_{Numb} -algebra **Int** with the carrier Int = {...-3,-2,-1,0,1,2,3,…}. In this algebra binary operators + and × define respectively operations of addition and multiplication in the usual way and 0-ary operators **zero** and **one** "mark" elements 0 and 1 of our carrier.

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[1] $A^0 = \{\varnothing\}$ [why? Can you find any way of thinking about this to make it a non-arbitrary decision? This can be a good homework problem, and there are several competing solutions to it discussed and debated on the 726 2001 site], so any function $f: A^0 \rightarrow A$ maps the empty set to some element of A (in this way "marking it".)

[2] It is common in mathematics to use *operator* as a syntactic term and *operation* as a semantic term. Addition is an operation; the addition sign "+" is an operator symbol.

2) Natural numbers can be considered as a Ω_{Numb} -algebra **Nat** with the carrier **N** = {0,1,2,3,…}. In this algebra as in **Int** binary operators + and ×define respectively operations of addition and multiplication in the usual way and 0-ary operators **zero** and **one** "mark" 0 and 1.

To discuss: operators and operations and the use of symbols and metasymbols.

3) The third simple example is the algebra of numbers modulo 4 in the same signature Ω_{Numb}, let us call it **Mod4,** on the carrier Mod4 = {0,1,2,3}. Here 0-ary operators are defined in the similar way but binary operators are slightly different. The algebra **Mod4** can be written as follows:

Mod4 = {0,1,2,3}

zero 0

one 1

+

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×

	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

1.2. Subalgebras.

First we introduce some useful notions and notation.

Restriction. Let f: $A \to B$ be a function from A to B, let $C \subseteq A$, and let i be the embedding of C into A, i.e. the one-to-one mapping such that $i(x) = x$ for every $x \in C$. Then the function $f \circ i$: $C \to B$ is called the *restriction* of the function f to C and denoted by $f| C$. So for every $x \in C$ we have $f| C(x) = f(x)$. And for every $x \notin C$, $f| C(x)$ is undefined. [Note about parsing: the constituents of the expression $f| C(x)$ are $f| C$ and (x). This *is* function-argument application. Don't try to parse it into f, | , and $C(x)$; that's incoherent.] [Note about the restriction symbol:

you will find a different symbol, not plain |, in PtMW; that symbol isn't in Word's symbol font. The plain | is also used, so we will use it in these notes.]

Closed sets.

If g: $A^n \to A$ a n-ary operation on A and $B \subseteq A$, we say that B is *closed* with respect to g if for all $x_1, ..., x_n$ in B, $g \,|\, B(x_1, ..., x_n) \in B$.

In our first example, the Ω_{Numb} -algebra **Int,** consider the subset **N** of natural numbers[3], **N** ⊂ Int: **N** is closed under operations + and × (and also under the 0-ary operations corresponding to 0-ary operators **zero** and **one)**. The subset Even ⊂ **N** of even natural numbers is also closed under operations + and × but the subset Odd ⊂ **N** of odd natural numbers is not closed under operation +. (The set Int itself is of course closed under all operation of our algebra; the carrier of an algebra must always be closed under the operations of that algebra.)

Definition. Given two Ω-algebras, **A** and **B**, we say that **B** is a *subalgebra* of **A** if

1) the carrier B of algebra **B** is a subset of the carrier A of algebra **A**;

2) for every operator $\omega \in \Omega(n)$, B is closed with respect to the operation $\omega_{\mathbf{A}}$ (i.e. for every n-tuple $<b_1, ..., b_n> \in B^n$ we have $\omega_{\mathbf{A}}\,(b_1, ..., b_n) \in B$);

3) for every operator $\omega \in \Omega$ we have $\omega_{\mathbf{A}} \,|\, B = \omega_{\mathbf{B}}$.

Thus every subset B of the carrier of a Ω-algebra **A** such that B is closed with respect to every operation $\omega \in \Omega$ can be considered a subalgebra of **A** (using the same operations).

Example. The algebra **Nat** is a subalgebra of algebra **Int.**

Since the empty set ∅ is trivially closed under all operations except for 0-ary operations (why?), ∅ can be considered a subalgebra of **A** if $\omega \in \Omega$ does not contain 0-ary operators. Every algebra is a subalgebra of itself.

It is easy to show that the intesection of two subalgebras of algebra **A** is also a subalgebra of **A.** [This could be an exercise.]

2. Homomorphisms and isomorphisms; congruences and quotient algebras

2.1. Morphisms

Definition. Given Ω-algebras **A** and **B** with carriers A and B respectively and a mapping

f: $A \to B$, we say that f is a *homomorphism* from A to B, if for every $\omega \in \Omega(n)$ we have

$$f(\omega(a_1, ..., a_n)) = \omega(f(a_1), ...,f(a_n)).$$

[3] In the logicians' careful set-theoretic reconstruction of mathematics, the natural numbers are not actually a subset of the integers (see PtMW Appendix A, pp 75-80) but are isomorphic to a subset of the integers (the positive integers plus zero). Here we ignore that distinction and equate the natural numbers to the positive integers plus zero.

We will use the notation f: **A** $\rightarrow$ **B** for the homomorphism f.

A homomorphism may be *into* or *onto*. (In PtMW, only "onto" homomorphisms are considered).

A homomorphism f: **A** $\rightarrow$ **B** is called an *isomorphism* between **A** and **B** if the inverse relation f^{-1}: $B \rightarrow A$ is a function and it is a homomorphism.

[A question. Is it not enough to require only that $f^{-1}: A \rightarrow B$ be a function?].

If there exists an *isomorphism* between **A** and **B**, we say that **A** and **B** are *isomorphic*.

There are special cases of homomorphisms. A one-to-one homomorphism is called a *monomorphism*. A monomorphism of an algebra onto itself is called an *automorphism*.

Examples.

1) Consider Ω_{Numb} –algebras **Nat** and **Mod4**. Let the mapping g: **N** $\rightarrow$ Mod4 be a mapping such that $g(0) = 0$, $g(1) = 1$, $g(2) = 2$, $g(3) = 3$, $g(4) = 0$, $g(5) = 1$, $g(6) = 2$, $g(7) = 3$, $g(8) = 0$, etc.

We see that g is a homomorphism. In fact,

for **zero** in **N**, g(**zero**) = **zero** in **Mod4**

for **one** in **N**, g(**one**) = **one** in **Mod4**

for m and n in **N**, $g(m + n) = g(m) + g(n)$ in **Mod4**

for m and n in **N**, $g(m \times n)$ in $= g(m) \times g(n)$ in **Mod4**[4]

Note: For more on modulo arithmetic, look through Chapter 10, but without worrying about what groups and integral domains are – just look at all the examples as examples of algebras.

2) Let us define another algebra in the same signature Ω_{Numb}, similar to **Mod4**, this time **Mod2**, that of numbers modulo 2, with {0,1} as the carrier and the obvious operations. Let

f: Mod4 $\rightarrow$ Mod2 be a mapping such that $f(0) = 0$, $f(1) = 1$, $f(2) = 0$, $f(3) = 1$.

We see that f is a homomorphism. In fact,

for **zero** in Mod4, f(**zero**) = **zero** in **Mod2**

for **one** in Mod4, f(**one**) = **one** in **Mod2**

for m and n in Mod4, $f(m + n) = f(m) + f(n)$ in **Mod2**

for m and n in Mod4, $f(m \times n)$ in $= f(m) \times f(n)$ in **Mod2**

Question: Why isn't the following mapping a homomorphism? f': Mod4 $\rightarrow$ Mod2 : $f'(0) = 0$, $f'(1) = 1$, $f'(2) = 1$, $f'(3) = 0$.

[4] For addition and multiplication we use traditional *infix* notation instead standard *prefix* notation.

More examples. Now consider a **Parity** algebra in the same signature Ω_{Numb} with the set {*even*, *odd*} as a carrier, 0-ary operators **zero** and **one** "marking" *even* and *odd* respectively (**zero** = *even* and **one** = *odd*) and binary operations + and × with obvious tables, that is *even* + *even* = *even* and so on. Then there is a homomorphism from the algebra of natural numbers, {0,1,2,3,...} to **Parity** mapping even numbers to *even* and odd numbers to *odd*.

The function $f(0)$ = *even* and $f(1)$ = *odd* is clearly a homomorphism from **Mod2** to **Parity**; and its inverse g(*even*) = 0 and g(*odd*) = 1 is a homomorphism from **Parity** to **Mod2**. Thus, f and g (each) constitute an isomorphism between **Mod2** and **Parity**.

The notion of isomorphism is important as an algebraic way of thinking. When we think abstractly, for example, about numbers we do not care what algebra we are talking about "up to within isomorphism". The natural numbers in normal decimal system with addition and multiplication forms the "same algebra" as the natural numbers in binary notation with addition and multiplication. We consider two algebras to be "abstractly the same" if they are isomorphic.

2.2. Congruences.

[We will return to this later; not to be fully covered today]
We will say that any equivalence relation on the carrier of an Ω-algebra holds on the algebra. Some equivalence relations on Ω-algebras are in close relations with homomorphisms.

Given an Ω-algebra **A** with the carrier A, we say that an equivalence relation Q on A *agrees* with the operation $\omega \in \Omega(n)$ when for any n-tuples

$<a_1, ..., a_n>$, $<b_1, ..., b_n> \in A^n$ we have

$<\omega(a_1, ..., a_n), \omega(b_1, ..., b_n)> \in Q$ if $< a_i, b_i> \in Q$ for $i = 1,\ldots, n$.

Definition. We say that equivalence relation Q on Ω-algebra **A** is a *congruence* on **A** if it agrees with every operation $\omega \in \Omega(n)$.

In Lecture 3 we considered notions of a quotient set of the equivalence relations, natural mapping, kernel of a mapping, etc. We will use these notions now.

Theorem. The kernel of a homomorphism is a congruence.

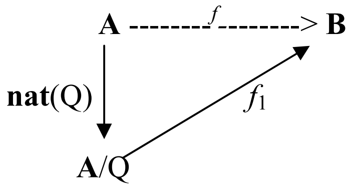
2.3. Quotient algebras.

[We began this in lecture 3, and we will return to it again later; not to be fully covered today]
Theorem. Let **A** be an Ω-algebra and Q a congruence on **A**. Then there exists a unique Ω-algebra on the quotient set A/Q of the carrier A of **A** such that the natural mapping $A \to A/Q$ is a homomorphism.

We will denote such an algebra as **A**/Q and will call it a *quotient algebra* of an algebra **A** by the congruence Q. In this case we have a natural homomorphism **A** → **A**/Q.

Given Ω-algebras **A** and **B** and a homomorphism f: **A** → **B**, consider the equivalence Q = **ker** f. It is easy to show that Q is a congruence on **A** and the natural mapping $A \to A/Q$ is the natural homomorphism **A** → **A**/Q. Consider the one-to-one mapping

f_1: A/Q → B such that f_1([[x]]) = f(x). We can show that mapping f_1: **A**/Q → **B** is a monomorphism and the diagram below is commutative.



Example. Consider the homomorphism f: **Mod4** → **Mod2** from the earlier example. The quotient set Q = **ker** f of its kernel equivalence consists of two classes: {0,2} and {1,3}. The operations of the quotient algebra **Mod4**/Q are defined in a natural way:

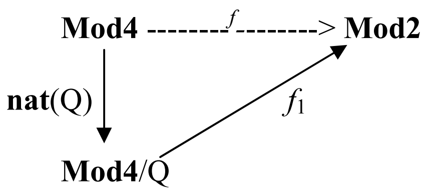
zero = {0,2}

one = {1,3}

plus({0,2}, {0,2}) = {0,2}, **plus**({0,2}, {1,3}) = {1,3}, etc.

And it is easy to see that there exists a monomorphism f_1: **Mod4**/Q → **Mod2**, such that

$f = f_1 \circ$ **nat**(Q), so the following diagram is commutative:



Homework 4

Do at least one of the following exercises.

1. [We may not get through the whole handout today; in either case, doing this problem will help you gain command of the new material. For help and hints, look at some of the solutions to this problem posted on the 2001 726 website.] Consider the homomorphism from the algebra on the set **N** of natural numbers with addition and multiplication to **Parity** (see example above). Write down (a) the kernel equivalence of this homomorphism, (b) the corresponding quotient algebra, and (c) the commutative diagram.
2. Study the algebra of the "symmetries of the square", described in PtMW pp 256-258. You can ignore the fact that it's a group, and you don't have to know what groups are – just think of it as an algebra, that's enough. Then do problems 3a-f on page 272, substituting the word *algebra* for *group* and *subalgebra* for *subgroup.* [A *subgroup* of a group is just a subalgebra which is also a group.] This is a good exercise for the notions of *subalgebra,*

homomorphism, isomorphism, automorphism. There are solutions to this one in the back of the book.

3. There are some questions interspersed in the handout; answer some of them (the more, the better). And look at the back and forth on disputes about what A^0 should be and why, on the 2001 website! It's good not to rest until you've really considered arguments, played devil's advocate, and convinced yourself. And even then you may later discover a flaw in your reasoning. Moral: never hesitate to acknowledge uncertainty about the validity of a particular step in your argument, and never hesitate to question other people's arguments. It's only if we try to be tough on ourselves and each other that we can have some confidence in the final results. When you hit 'thin ice' or a 'dangerous spot', go slower, not faster, and raise every possible question you can think of. – And of course that means be tough on us, too! We could be wrong, or we could be going too quickly over something that needs slowing down. (Sometimes we cite results without doing proofs, just to try to have time left to cover more things, including some interesting linguistic applications. But we should always be able to tell you at least where you could look for fuller discussion or proofs of things.)