Lecture 8. Axioms and theories, more examples. Axiomatic description of properties of relations.

CONTENTS

Reading: Chapter 8: 8.5, Chapter 9: 9.1 – 9.4 of PMW, pp. 206 –211, 247- 253.

Axiomatic description of properties and classes of relations

We can use PL to define properties and classes of relations which we studied in Lectures 1-3. Below we write axioms for different relations using the same binary predicate symbol ***R***. Models in which these axioms (or sets of axioms) will be true will represent corresponding relations. In all cases below, the domain of quantification is the set A on which the relation is defined.

1. Properties of relations

1.1. Reflexivity, etc.

Reflexivity: $\forall \boldsymbol{x} R(\boldsymbol{x}, \boldsymbol{x})$

It is easy to see that in every model in which this axiom holds the relation ***R*** will be reflexive.

Nonreflexivity: $\exists \boldsymbol{x} \neg R(\boldsymbol{x}, \boldsymbol{x})$

Irreflexivity: $\forall \boldsymbol{x} \neg R(\boldsymbol{x}, \boldsymbol{x})$

1.2. Symmetry, etc.

Symmetry: $\forall \boldsymbol{x} \forall \boldsymbol{y}(\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}) \rightarrow \boldsymbol{R}(\boldsymbol{y}, \boldsymbol{x}))$

Nonsymmetry: $\exists \boldsymbol{x} \exists \boldsymbol{y}(\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}) \ \& \ \neg \boldsymbol{R}(\boldsymbol{y}, \boldsymbol{x}))$

Asymmetry: $\forall \boldsymbol{x} \forall \boldsymbol{y}(\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}) \rightarrow \neg \boldsymbol{R}(\boldsymbol{y}, \boldsymbol{x}))$

Antisymmetry: $\forall \boldsymbol{x} \forall \boldsymbol{y}((\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}) \ \& \ \boldsymbol{R}(\boldsymbol{y}, \boldsymbol{x})) \rightarrow (\boldsymbol{x} = \mathbf{y}))$

Note. In Lecture 6 we considered Predicate Logic without equality (we did not introduce the predicate "="). To deal with it we need additional syntactic and semantic rules. The syntactic rule simply introduces formulas of the type $(\boldsymbol{x} = \mathbf{y})$ for any pair of variables $\boldsymbol{x}, \boldsymbol{y} \in$ Var. The natural infix notation is used. The corresponding semantic rule is also natural:

$\|(\boldsymbol{x} = \boldsymbol{y})\|^{M,g} = 1$ iff $g(\boldsymbol{x}) = g(\boldsymbol{y})$

(or in the terms of operations on D_t, $\|(\boldsymbol{x} = \boldsymbol{y})\|^{M,g} = \|\boldsymbol{x}\|^{M,g} \leftrightarrow \|\boldsymbol{y}\|^{M,g}$).

1.3. Transitivity, etc.

Transitivity: $\forall \boldsymbol{x} \forall \boldsymbol{y} \forall z((\boldsymbol{R}(\boldsymbol{x}, \boldsymbol{y}) \ \& \ \boldsymbol{R}(\boldsymbol{y}, z)) \rightarrow \boldsymbol{R}(\boldsymbol{x}, z))$

Nontransitivity: $\exists x \exists y \exists z(R(x, y) \,\&\, R(y, z) \,\&\, \neg R(x, z))$

Intransitivity: $\forall x \forall y \forall z((R(x, y) \,\&\, R(y, z)) \rightarrow \neg R(x, z))$

1.4. Connectedness:

$$\forall x \forall y(R(x, y) \vee R(y, x))$$

2. Classes of relations

2.1. Equivalence relations

We know that this property of relations needs three axioms together: reflexivity, symmetry and transitivity:

1) $\forall x R(x, x)$
2) $\forall x \forall y(R(x, y) \rightarrow R(y, x))$
3) $\forall x \forall y \forall z((R(x, y) \,\&\, R(y, z)) \rightarrow R(x, z))$

2.2. Tolerance relations

Here we need only two first ones from equivalence: reflexivity and symmetry:

1) $\forall x R(x, x)$
2) $\forall x \forall y(R(x, y) \rightarrow R(y, x))$

2.3. Orderings

A relation ***R*** is a **weak order** iff it is transitive, reflexive and antisymmetric**:**

1) $\forall x \forall y \forall z((R(x, y) \,\&\, R(y, z)) \rightarrow R(x, z))$
2) $\forall x R(x, x)$
3) $\forall x \forall y((R(x, y) \,\&\, R(y, x)) \rightarrow (x = y))$

A **strict order** is transitive, irreflexive and asymmetric**:**

1) $\forall x \forall y \forall z((R(x, y) \,\&\, R(y, z)) \rightarrow R(x, z))$
2) $\forall x \neg R(x, x)$
3) $\forall x \forall y(R(x, y) \rightarrow \neg R(y, x))$

For a **linear ordering** (strict or weak) we need the connectedness axiom to be added, so, for example, **linear weak order** is defined by the following four axioms:

1) $\forall x \forall y \forall z((R(x, y) \,\&\, R(y, z)) \rightarrow R(x, z))$
2) $\forall x R(x, x)$
3) $\forall x \forall y((R(x, y) \,\&\, R(y, x)) \rightarrow (x = y))$
4) $\forall x \forall y(R(x, y) \vee R(y, x))$

A **preorder** or **quasi-order** is a relation which is not an order because it violates antisymmetry. For instance, "is at least as old as" is a weak order on the domain of *ages*, but it is not a weak order on the domain of *people*, because "John as at least as old as Peter" and "Peter is at least as old as John" do not together entail that John = Peter, only that John and Peter are the same age. (We can transform this preorder into an order by grouping people of the same age into equivalence classes; the relation "is at least as old as" on these equivalence classes is a weak linear order.)

We can derive the axiomatic definition of weak **preorder** from the definition of weak order by throwing out the condition of antisymmetry. So we have:

1) $\forall x \forall y \forall z((R(x, y) \ \& \ R(y, z)) \rightarrow R(x, z))$

2) $\forall x R(x, x)$

Thought question: Why isn't "is older than" on the set of people a "strong preorder"? What kind of relation is it?

Homework 9 -- mainly algebra review

Do at least one of the following exercises.

1. Answer the thought question at the end of the handout.
2. Continue with the "tree" questions from Homework 7. We'll post "tree discussions" on the website. Send us electronic copies of your tree-work from homeworks 8 and 9 so we can add our own comments and put them on the web.
3. Algebra review. **Algebras in the signature Ω_{Numb} = { zero, one, +, ×}. Homomorphisms, congruences, and quotient algebras.**

Let us return to algebras the signature Ω_{Numb} = { **zero**, **one**, +, ×}: **Nat**, **Parity**, **Mod4, Mod2** and their homomorphisms that we considered in the Lecture 4. (1) and (2) below are just definitions; the questions are in (a) and (b).

(1) The algebra **Nat** with the carrier **N** = {0,1,2,3,…}. In this algebra binary operators + and × define respectively operations of addition and multiplication in the usual way and 0-ary operators **zero** and **one** "mark" 0 and 1.

(2) The algebra **Parity** with the set {*even*, *odd*} as a carrier, 0-ary operators **zero** and **one** "marking" *even* and *odd* respectively (**zero** = *even* and **one** = *odd*) and binary operations + and × with obvious tables, that is *even* + *even* = *even* and so on.

(a) Consider the homomorphism *h*: **Nat** → **Parity** (the function *h*: **N** → {*even*, *odd*} mapping even numbers to *even* and odd numbers to *odd*).

(The first part of this is a restatement of Problem 1 from Lecture 4. Do it now if you didn't do it before. Write down (a) the kernel equivalence of this homomorphism, (b) the corresponding quotient algebra, and (c) the commutative diagram.)

The quotient set **ker** *h* of its kernel equivalence consists of two classes, which we will call "Even" and "Odd": Even = [[0]] = [[36]] (etc) ={0,2,4,…} and Odd = [[17]] = [[1]] ={1,3,5,…}.

Verify that **ker** *h* is a congruence on the algebra **Nat**, i.e. it agrees with operations of the signature Ω_{Numb}. For example, if $x, y \in$ Even and $z, w \in$ Odd then both $(x + z)$ and $(y + w)$ are odd.

(b) Consider the algebras **Mod4** and **Mod2** and the homomorphism *f*: **Mod4** → **Mod2** from Lecture 4. The quotient set Q = **ker** *f* of its kernel equivalence consists of two classes: {0,2} and {1,3}. Verify that **ker** *h* is a congruence on the algebra **Mod4.**