Lecture 7. Model theory. Consistency, independence, completeness, categoricity of axiom systems. Expanded with algebraic view.

CONTENTS

Reading: Chapter 8: 8.1, 8.4, 8.5, of PtMW, pp. 179-189, 192-217. Good supplementary reading: Fred Landman (1991) *Structures for Semantics*. Chapter 1, Sections 1.1 and 1.3., which we draw on heavily here. Another nice resource (thanks to Luis for the suggestion) is Gary Hardegree's online in-progress textbook *Introduction to Metalogic* : http://www-unix.oit.umass.edu/~gmhwww/513/text.htm . (Ch 5: semantics of first-order logic, with notions of consistency, etc.; Ch 14: models and interpretations.) This last reference is still new to us and we haven't studied it. Anyone who has a chance to take Hardegree's Mathematical Logic course would certainly get a thorough grounding in all the notions we are discussing here and more.

1. Syntactic provability and semantic entailment

Proof theory: When we presented first order logic in earlier lectures, we specified only the syntax of well-formed formulas and their semantics. We did not give any *proof theory.* Let us add that now.

Add to our first-order logic a set of *axioms*[1] and *Rules of Inference*, such as those described in Chapters 7 and 8.6 of PtMW or Chapter 1 of Landman. (We are not going to specify these here.) Then we can specify the notion of a *proof* of a formula φ from premises $\varphi_{1, \ldots,} \varphi_n$.

A *proof* of a formula φ from premises $\varphi_{1, \ldots,} \varphi_n$ is a finite sequence of formulas $<\psi_1, \ldots, \psi_m>$ such that $\psi_m = \varphi$, and each ψ_i is either (a) an axiom, (b) a premise, or (c) inferred by means of one of the Rules of Inference from earlier formulas in the sequence.

[1] Here we consider axioms for first order predicate logic itself (PtMW 8.6), which we should distinguish from axioms for theories describing axiomatic classes of models. The former are tautologies which are true in every model. The latter are contingencies which are true in some models and false in other.

Call the resulting system of logic L0. An important aspect of the rules of inference is that they are strictly *formal*, i.e. "syntactic": they apply when expressions are of the right form, with no need to know anything about their semantics.

Corresponding notion of **syntactic derivability or provability**: (Landman p 8)

Let Δ be a set of formulas and ϕ a formula. We write $\Delta \vdash \phi$, meaning ϕ is *provable* from Δ, ϕ is *derivable* from Δ, iff there is an L0-proof of ϕ from premises $\delta_1, ..., \delta_n \in \Delta$. The symbol $\vdash$ is pronounced "turnstile". (In Russian it is called "shtopr", i.e. "corkscrew".)

We write $\vdash \phi$, ϕ is *provable*, ϕ is a *tautology*, for $\varnothing \vdash \phi$, i.e. ϕ is provable from the axioms and inference rules of the logic without further assumptions.

ϕ is a *contradiction* if $\vdash \neg \phi$. It is common to use a special symbol $\perp$ ("bottom") to stand for an arbitrary contradiction.

Note that here we use *syntactic* notion of tautologies and contradictions. Earlier in Lecture 6, and also below, we consider corresponding semantic notions.

1.1. Semantic entailment and validity.

Instead of writing $[[\phi]]^{M,g} = 1$, it is customary when discussing model theory to write $M \models \phi[g]$, meaning ϕ is true in M relative to g. We write $M \models \phi$ (and say that ϕ is *true* in M) iff $M \models \phi[g]$ for all g.

Note that if ϕ is a sentence (i.e., ϕ has no free variables), then for every model M we have ϕ is true in M or ϕ is false in M. In general we are mostly interested in sentences, so we will often restrict ourselves to them.

Let Δ be a set of sentences and ϕ a sentence. Then we define $\Delta \models \phi$, Δ *entails* ϕ, iff for every model M it holds that:

If $M \models \delta$ for every $\delta \in \Delta$, then $M \models \phi$.

In other words, Δ *entails* ϕ if ϕ is true in every model in which all the premises in Δ are true.

We write $\models \phi$ for $\varnothing \models \phi$. We say ϕ is *valid*, or *logically valid*, or a *semantic tautology* in that case. $\models \phi$ holds iff for every M, $M \models \phi$. Validity means truth in all models.

1.2. Soundness and completeness of a logic.

Of course we expect that there should be a correspondence between syntactic provability and semantic entailment. (We can make some such requirements a condition for any logic; see Hardegree's discussion of possible logics in his Chapter 5.) The notions of *soundness* and *completeness* relate to these correspondences.

Soundness (of a logic): If $\Delta \vdash_{L0} \phi$, then $\Delta \models \phi$. I.e. anything you can prove is semantically valid. Our system of proofs doesn't give us anything bad.

Completeness (of a logic): If $\Delta \models \phi$, then $\Delta \vdash_{L0} \phi$. Every argument that is semantically valid can be derived with the L0 rules.

[Note: there are various different senses of "complete"; this is one. On this sense, it is a LOGIC that is complete, sometimes called (strongly) *semantically complete*. We will see other senses in which a *theory* may be semantically or syntactically complete.]

Note: First-order predicate logic on any of its standard axiomatizations is sound and complete. Higher-order logics such as Montague's typed intensional logic are sound but often not complete. Soundness is an essential requirement; completeness isn't always possible. Completeness is one of the properties that makes first-order logic nice.

2. Consistency, completeness, independence, and other notions.

(PtMW 8.5.2, Landman Chapter 1)

Let *L* be a first order language. (I.e. we specify a particular set of individual constants and predicate constants (of various arities) for *L*; the rest of the specification of *L* follows from the definition of well-formed formulas of predicate logic.)

A *theory* in *L* is a set of sentences of *L*. [This is Landman's usage; PtMW is slightly different; below, discussing algebraic approach we also will use this notion in a slightly different way]
So a first order theory is a set of first order sentences.

2.1. Some syntactic concepts.

Let Δ be a theory.
Δ is *inconsistent* if $\Delta \vdash \bot$. Δ is *consistent* if $\Delta \nvdash \bot$ (i.e. NOT ($\Delta \vdash \bot$))

I.e. a theory is inconsistent if you can derive a contradiction from it, consistent if you can't. It's often hard to prove that you *can't* derive a contradiction – that requires a metalevel proof about possible proofs. We'll come to an easier semantic way of showing consistency in a minute.

Δ is *deductively closed* iff: if $\Delta \vdash \varphi$, then $\varphi \in \Delta$. Everything you can derive from Δ is already in Δ.

Δ is *maximally consistent* iff Δ is consistent and there is no Δ' such that $\Delta \subseteq \Delta'$ and $\Delta \neq \Delta'$ and Δ' is consistent.

Δ^c, the *deductive closure of* Δ, is the set $\{\varphi : \Delta \vdash \varphi\}$. So Δ^c is the result of adding to Δ every sentence that can be derived from Δ.

A theory Δ is (*formally*) *complete* iff Δ^c, the deductive closure of Δ, is maximally consistent. Equivalently, Δ is *formally complete* if for every sentence φ in the language, either $\Delta \vdash \varphi$ or $\Delta \vdash \neg \varphi$. (This is the syntactic notion of completeness alluded to above.)

<u>Axioms and theories in the syntactic sense</u>: Δ is a *set of axioms for* Γ iff $\Delta^c = \Gamma^c$.
Γ is *finitely axiomatizable* iff there is a finite set of axioms for Γ.

Some facts (Landman p.11)

FACT 1: If Δ is consistent, then Δ^c is consistent.

FACT 2: If Δ is maximally consistent then Δ is deductively closed.
(Optional exercise: prove FACT 2.)

FACT 3: (Deduction theorem) If $\Delta \cup \{\varphi\} \vdash \psi$, then $\Delta \vdash (\psi \rightarrow \varphi)$.

FACT 5: (Lindenbaum's Lemma) Any consistent theory can be extended to a maximally consistent theory.

2.2. Some semantic concepts.

Δ is *satisfiable*, equivalently Δ *has a model*, iff there is a model M such that for all $\delta \in \Delta$: $M \models \delta$.

Δ is *closed under logical consequence* (*under entailment*) iff: if $\Delta \models \varphi$ then $\varphi \in \Delta$.

If φ is a sentence (i.e. has no free variables) and $M \models \varphi$, then we say that M is a model for φ, or φ holds in (or on) M.
Similarly, we say that M is a model for theory Δ, and we write $M \models \Delta$, if M is a model for every $\delta \in \Delta$.
Landman writes MOD(φ) for the class of all models for φ, and MOD(Δ) for the class of all models for Δ.
A set of axioms Δ is *semantically complete with respect to a model M*, or *weakly semantically complete*, if every sentence which holds in M is derivable from Δ.

Three notions of completeness. We have now seen three notions of completeness: (i) a logic may be complete: everything which should be a theorem in the semantic sense, i.e. every sentence which is *valid* is indeed a theorem in the syntactic sense, i.e. is *derivable, provable.* (ii) Given a logic and a particular first-order language, a set of axioms Δ is *formally complete* if the deductive closure of Δ is maximally consistent: for every sentence, either it or its negation is provable from Δ. (iii) A set of axioms Δ is *(weakly) semantically complete with respect to model M* if every sentence which holds in M is derivable from Δ.

What do all these notions have in common? They all say that your logic or your axioms are sufficient to derive everything that meets a certain criterion; what varies is the criterion.

2.3. Soundness and completeness again.

Another formulation of soundness and completeness for a logic, provably equivalent to the earlier one, is the following.

Soundness (of a logic): If Δ has a model, then Δ is consistent.
Completeness (of a logic): If Δ is consistent, then Δ has a model.

Because first-order logic *is* sound and complete, we can freely choose whether to give a semantic or syntactic argument of consistency or inconsistency. Suppose you are asked to show whether some set of sentences Δ is consistent or not. Usually if the answer is YES, the easiest way to show it is to show that Δ has a model (by giving a model and showing, if it isn't obvious, that all of the sentences in Δ hold in the model.) And if the answer is NO, usually the easiest way to show it is by deriving a contradiction, i.e. by showing that $\Delta \vdash \bot$. See homework problems 5-8.

2.4. Independence.

The notion of independence is less crucial than some of the other notions we have studied; it concerns the question of whether a given axiom within an axiom system is superfluous. There is nothing logically wrong with having some superfluous axioms; in fact they may make the system easier to work with (just as we typically work with two quantifiers and five connectives even though we know we could define some in terms of others.) But it is often an interesting issue, as in the case of the discovery of non-Euclidean geometries, which resulted from the quest to derive the fifth Euclidean postulate from the other four. The fifth postulate turned out to be independent of the others, and that was demonstrated by producing models (non-Euclidean geometries) in which the first four postulates were true and the fifth one false.

In general: if an axiom is *not* independent, you can prove it from the remaining axioms, and that is the standard way to prove non-independence. If an axiom *is* independent, the easiest way to show it is to produce a model that satisfies the remaining axioms but does not satisfy the one in question. See homework questions 2,3,4,9.

3. An algebraic view on provability and on the notions discussed above

[Here we use *Universal Algebra* by P.M. Cohn (1965), Harper & Row, New York, Evanston, and London]

We will begin with extremely abstract (but simple) notions such as closure systems and Galois Connections. We will see how they clarify the notions we have just discussed.

3.1. Closure systems

Let A be any set and $\wp(A)$ its power set, i.e. set of all its subsets. We wish to consider certain subsets of $\wp(A)$, or as we shall say, *systems* of subsets of A. A system $\mathfrak{R}$ of subsets of A is said to be a *closure system* if $\mathfrak{R}$ is closed under intersections, i.e.

For any subsystem $\mathfrak{I} \subseteq \mathfrak{R}$, we have $\cap\, \mathfrak{I} \in \mathfrak{R}$

In particular, taking $\mathfrak{I} = \varnothing$, we see that A is always belong to $\mathfrak{R}$ [if you don't see it don't despair, it is not always easy to think about intersection of empty set of sets; just take it that A always belongs to any closure system $\mathfrak{R}$ of subsets of A by definition].

Note. (if you have heard of closure in topology): The algebraic notion of closure is similar to the topological one but weaker.

Examples

1. The old one. The system of all subalgebras of a given algebra is a closure system. [Think why, it's is a good exercise]

2. Let S be a set of all sentences of some first order language. Then

a) the set $\mathfrak{R}_D$ of all deductively closed theories is a closure system;

b) the set $\mathfrak{R}_E$ of all theories closed under logical consequence (under entailment) is a closure system;

c) $\mathfrak{R}_D = \mathfrak{R}_E$.

[Think why 2 is true]

Closure operator

A *closure operator* on a set A is a mapping J of $\wp(A)$ in itself with the properties:

J.1. If $X \subseteq Y$, then $J(X) \subseteq J(Y)$

J.2. $X \subseteq J(X)$

J.3. $JJ(X) = J(X)$

Theorem: *Every closure system $\Re$ on a set A defines a closure operator J on A by the rule*

$$J(X) = \cap \{ Y \in \Re \mid Y \supseteq X \}.$$

Conversely, every closure operator J on A defines a closure system by

$$\Re = \{ X \subseteq A \mid J(X) = X \},$$

and the correspondence $\Re \leftrightarrow J$ between closure systems and closure operators thus defined is bijective (*one to one*).

[Try to prove that Theorem. Make use of the definitions of closure system and closure operator.]

Examples

Consider a set S of all sentences of some first order language and define operators J_D and J_E on S such that for every $\Delta \subseteq S$

1. $J_D(\Delta) = \Delta^c$.
2. $J_E(\Delta) = \Delta^{**}$ where $\Delta^{**} = \{\phi \mid \Delta \models \phi\}$

It is easy to see that J_D and J_E are closure operators on S and $J_D = J_E$. In particular, if Δ is a set of axioms for Γ then $J_D(\Delta) = J_D(\Gamma) = \Delta^c = \Gamma^c$.

3.2. Galois connection

Let A and B be any sets and $R \subseteq A \times B$ a binary relation from A to B. For any subset X of A we define a subset X^* of B by the equation

$$X^* = \{y \in B \mid \langle x,y \rangle \in R \text{ for all } x \in X\}$$

And similary, for any subset Y of B we define a subset Y^* of A by

$$Y^* = \{x \in A \mid \langle x,y \rangle \in R \text{ for all } y \in Y\}$$

Thus we have the mappings

(1) $X \to X^*$, $Y \to Y^*$ of $\wp(A)$, $\wp(B)$ into each other, with the properties

(2) If $X_1 \subseteq X_2$, then $X_1^* \supseteq X_2^*$, If $Y_1 \subseteq Y_2$, then $Y_1^* \supseteq Y_2^*$

(3) $X \subseteq X^{**}$, $Y \subseteq Y^{**}$

(4) $X^{***} = X^*$, $Y^{***} = Y^*$

Conditions (2) and (3) follow immediately from the definitions. If (2) is applied to (3) we get $X^* \supseteq X^{***}$, while (3) applied to X^* gives the reverse inequality. Thus any mappings (1) which satisfy (2) and (3) also satisfy (4).

A pair of mappings (1) between $\wp(A)$ and $\wp(B)$ is called a *Galois connection* if it satisfies (2) and (3) (and hence (4).

To establish the link with closure systems we observe that in any Galois connection the mapping $X \to X^{**}$ is a closure operator in A and $Y \to Y^{**}$ is a closure operator in B (by (2)-(4)). Moreover, the mapping (1) gives a bijection between thes two closure systems.

Examples

1. Our main example is a Galois connection based on a relation $M \models \varphi$ (φ is valid in M) from class ***Mod***(L) of all models for first order language L to the set S of all sentences of L.

In this way we obtain a Galois connection between $\wp$(***Mod***(L)) and $\wp(S)$. By this connection,

(i) to any class MOD $\subseteq$ ***Mod***(L), there corresponds the set MOD* of all sentences which are true in each $M \in$ MOD, and

(ii) to any set $\Delta \subseteq S$ there corresponds the class Δ* of all those models in which all the sentences of Δ are true.

In a notation we used before, MOD* = $\{\varphi \mid M \models \varphi$ for every $M \in$ MOD $\}$ and Δ* = MOD(Δ).

So any model from the class Δ* = MOD(Δ) is called a model for Δ. Any sentence of MOD* is called a *theorem* in MOD.

A class of models which is of the form Δ* is said to be an *axiomatic class* and Δ a set of its axioms. Δ** is the set of all theorems in Δ* = MOD(Δ), a *theory* of Δ* = MOD(Δ).

So the semantic notion of axioms and theory given here corresponds to syntactic one given in 2.1 above.

2. Let ***Obj*** be a set of objects, ***Feat*** a set of their features, and R a relation from ***Obj*** to ***Feat*** „to have a feature“(„object O has a feature F“). The Galois connection based on this relation is similar to the previous one. It is used in *Formal concepts theory,* rather popular now.

4. Morphisms for models; categoricity

Earlier we defined *isomorphism*, *homomorphism*, and *automorphism* for Ω-*algebras*: systems with a carrier set and *operations* of various arities on this carrier corresponding to symbols from the signature. Similar notions (different kinds of morphisms) are also considered for models.

Describing Predicate Logic (the First Order Language) in the Lecture 6 we did not use the word *signature* but really used the notion. For every concrete language L we fix the set Const of constants and the set Pred of predicate symbols of different arities, Pred = Pred1 $\cup$ Pred 2 $\cup$ … We can define the signature Ω of this language as a the union of its constants and predicate symbols, Ω = Const $\cup$ Pred. Then we can call all the models of this language of the kind M = <D,I> as models over signature Ω or Ω-models.

Consider two Ω-models $\mathbf{M} = \langle D_M, I_M \rangle$ and $\mathbf{N} = \langle D_N, I_N \rangle$. A mapping f: $D_M \rightarrow D_N$ is called a *homomorphism* from M to N (and is denoted also as f: $\mathbf{M} \rightarrow \mathbf{N}$) if

(1) for every constant $\boldsymbol{c} \in$ Const, $f(I_M(c)) = I_N(\boldsymbol{c})$

(2) for every n-ary predicate symbol P $\in$ Pred and every n -tuple $\langle d_1, \ldots, d_n \rangle \in D_M^n$, if $\langle d_1, \ldots, d_n \rangle \in I_M(P)$ then $\langle f(d_1), \ldots, f(d_n) \rangle \in I_N(P)$

A homomorphism f: $\mathbf{M} \rightarrow \mathbf{N}$ is called an *isomorphism* between **M** and **N** if the inverse relation f^{-1}: $D_N \rightarrow D_M$ is a function and it is a homomorphism.

A homomorphism f: $\mathbf{M} \rightarrow \mathbf{M}$ is called an *automorphism.*

An axiom system is called *categorical* if all of its models are isomorphic. (See homework questions 3,6,8.)

Note. The notion of homomorphism for models is weaker than homomorphism for algebras. We have from the definition that for every *n*-tuple $<d_1, \ldots, d_n> \in I_M(P)$ its image $<f(d_1), \ldots, f(d_n)>$ belongs to $I_N(P)$. But the opposite need not be true: if some n-tuple $<d_1, \ldots, d_n>$ doesn't belong to $I_M(P)$, its image $<f(d_1), \ldots, f(d_n)>$ can be in $I_N(P)$. So for every relation $I_M(P)$ in the model **M,** its image $f(I_M(P))$ is a subset of corresponding relation $I_N(P)$ in the model **N**, i.e. we have $f(I_M(P)) \subseteq I_N(P)$, but we don't in general have the equality $f(I_M(P)) = I_N(P)$.

So in the case of models, in contrast to algebras, a bijective homomorphism f: $\mathbf{M} \rightarrow \mathbf{N}$ (a homomorphism such that the mapping f: $D_M \rightarrow D_N$ is a bijection (one-to-one and onto)) need not be an isomorphism between the models **M** and **N** because the inverse mapping f^{-1}: $D_N \rightarrow D_M$ is not obligatorily a homomorphism.

Homework 8.

I. All of the first set of problems are based on the elementary formal system *L* defined by Axioms A1-A6 in Section 8.5.4. of PtMW, though some of the questions, like 2 and 3, are fully general. [Most of these questions are from Partee (1978).]

1. Find three models for *L* other than the one given in the text.
2. If the deletion of a certain axiom from a formally complete system changes the system into one which is not formally complete, then that axiom is independent. Why?
3. If the deletion of a certain axiom changes a formal system from categorical to non-categorical, must that axiom be independent? Why?
4. Find two models for axioms A2-A6 which are not isomorphic to the models of *L* nor two each other. What does this tell you about axiom A1?
5. If A2 is replaced by A2': $\forall x\, Rxx$, is the resulting system consistent? If so, find a model for it. If not, deduce a contradiction from the new set of axioms.
6. If A2 is replaced by A2' as above and A3 and A4 are deleted, is the resulting system consistent? Justify as above. Is the resulting system categorical? If not, find two non-isomorphic models for it.
7. What happens if we replace A5 by A5': $(\exists y)(\forall x)\, Rxy$? Is the resulting axiom system consistent or inconsistent? Justify as above.

8. [means maybe harder] Let axioms A2 – A6 be replaced by the single axiom A2'': $\forall x \forall y \forall z((Rxy \;\&\; Rxz) \rightarrow y \neq z)$. Is this system consistent? Is it categorical? Justify your answers.

*9. Show that axiom A3 is not independent in the system *L*.

II. Return to the question about the axiom system *W* with "points" and "lines" from PtMW Ch8, p. 235, question 13. [This was Homework 7, Question 5.] If you already did it, try the questions below. If you didn't do it before, you might try it now together with a subset of the questions below.

10. Is the axiom system *W* consistent? *Is it categorical? If you already did that homework problem, you have the answer to the first question and most of the arguments that you need for the second question.

11. What do your answers to the questions 13a-e tell you about the independence of various of the axioms of *W*?