Lagrangian Formalism

Hamiltonian’s Least Action Principle

Let \( x(t) \) be some function of time, \( m \) be some positive number, and \( U(x) \) be some function of \( x \). Consider a functional

\[
S = \int_{t_a}^{t_b} L(x, \dot{x}) \, dt ,
\]

with

\[
L(x, \dot{x}) = \frac{m \dot{x}^2}{2} - U(x) ,
\]

and find the function \( x(t) \) that minimizes the functional \( S \) at given boundary conditions

\[
x(t_a) = x_a , \quad x(t_b) = x_b .
\]

Calculus of Variations says that the function in question is given by corresponding Euler’s equation

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} ,
\]

and with Eq. (2) we readily get... Yes, the Newton’s equation!

\[
m\ddot{x} = -U'(x) .
\]

This amazing circumstance generalizes to any number of particles in any dimensions. If we take

\[
S = \int_{t_a}^{t_b} L(\{r_i\}, \{\dot{r}_i\}) \, dt ,
\]

\[L = T - U , \quad T = \sum_{i=1}^{N} \frac{m_i \dot{r}_i^2}{2} , \quad U = \sum_{i<j} U(|r_i - r_j|) ,\]

and look for the functions \( \{r_i(t)\} (i = 1, 2, \ldots, N) \) minimizing the functional (6) under fixed boundary conditions

\[
r_i(t_a) = r_i^{(a)} , \quad r_i(t_b) = r_i^{(b)} ,
\]

we will see that the system of equations

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{r}_i} = \frac{\partial L}{\partial r_i} \quad (i = 1, 2, \ldots N)
\]

defining these functions is nothing but Newton’s equations of motion.

The importance of the fact that we have just established—known as Hamilton’s least action principle—can hardly be overestimated. The variational formulation of Mechanics has at least two major advantages. First, one can easily switch from Cartesian coordinates to any other coordinates, including collective variables, since the least action principle is explicitly invariant with respect to the choice of coordinates. One just have to re-write the functional \( S \), called action, in terms of new variables, \( \{q_\alpha\} \), called generalized coordinates. To this end one expresses \( r_i \) and \( \dot{r}_i \) in terms of \( \{q_\alpha\} \) and \( \{\dot{q}_\alpha\} \), plugs the result into the function \( L \), called Lagrangian function, or just the Lagrangian, and gets the action

\[
S = \int_{t_a}^{t_b} L(\{q_\alpha\}, \{\dot{q}_\alpha\}) \, dt .
\]
Minimization of the action leads to the equations of motion in terms of new variables:
\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} = \frac{\partial L}{\partial q_\alpha} .
\]  
(11)

Mathematically, these are the Euler’s equations of the Calculus of Variations. In Physics, Eqs. (11) are called Lagrange equations, and the whole formalism is called Lagrangian Mechanics. The second crucial advantage of the Lagrangian formalism is closely related to the first one. It concerns realistic macroscopic mechanical systems consisting of a huge number of particles and thus not amenable to direct microscopic treatment. In many cases, to a very good approximation, a macroscopic system features just a few relevant degrees of freedom. For example, an idealized pendulum—a mass constrained by a weightless, extensionless rod—is characterized by just two spherical angles, \((\theta, \varphi)\); or even just one polar angle, \(\theta\), if the motion is restricted to a plane. Let us see how Lagrangian formalism works for these two examples.

**Plane Pendulum.** Let the \(y\)-axis be vertical. We have
\[
L = T - U , \quad T = \frac{mv^2}{2} , \quad U = mgy ,
\]  
(12)
where \(m\) is the pendulum mass, \(v^2 = \dot{x}^2 + \dot{y}^2\) is the square of the velocity, \(g\) is the free-fall acceleration. Because of the constraint, the system is described by just one generalized coordinate, the angle \(\theta\) between the rod and the negative direction of the \(y\)-axis. We then notice that \(v^2 = l^2\dot{\theta}^2\), \(y = -l\cos \theta\), with \(l\) is the length of the rod, and thus
\[
L = ml^2(\dot{\theta}^2/2 + \omega^2 \cos \theta) , \quad \omega^2 = g/l .
\]  
(13)

We then write the Lagrange equation (note that the global factor \(ml^2\) drops out):
\[
\ddot{\theta} = -\omega^2 \sin \theta .
\]  
(14)

**Spherical Pendulum.** Let the \(z\)-axis be vertical and directed down. The Lagrangian
\[
L = T - U , \quad T = \frac{mv^2}{2} , \quad U = -mgz .
\]  
(15)
There are two degrees of freedom that are conveniently parameterized by two spherical angles $\theta$ and $\varphi$. The radius is equal to a fixed number $l$, the length of the rod. To express $L$ in terms of the generalized coordinates $\theta$ and $\varphi$, we need to relate $v$ to $\theta$ and $\varphi$:

\[
x = l \sin \theta \cos \varphi \quad \Rightarrow \quad \dot{x} = l \dot{\theta} \cos \theta \cos \varphi - l \dot{\varphi} \sin \theta \sin \varphi .
\]

\[
y = l \sin \theta \sin \varphi \quad \Rightarrow \quad \dot{y} = l \dot{\theta} \cos \theta \sin \varphi + l \dot{\varphi} \sin \theta \cos \varphi .
\]

\[
z = l \cos \theta \quad \Rightarrow \quad \dot{z} = -l \dot{\theta} \sin \theta .
\]

\[
v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) .
\]

Hence,

\[
L = ml^2 \left( \frac{\dot{\theta}^2}{2} + \frac{\dot{\varphi}^2}{2} \sin^2 \theta + \omega^2 \cos \theta \right) , \quad \omega^2 = g/l .
\]

The two Lagrange equations read:

\[
\ddot{\theta} = \dot{\varphi}^2 \sin \theta \cos \theta - \omega^2 \sin \theta ,
\]

\[
\frac{d}{dt} \left( \dot{\varphi} \sin^2 \theta \right) = 0 .
\]

**Problem 26.** Develop the Lagrangian and find the equations of motion for the system of two coupled plane pendulums, Fig. 1.

**Problem 27.** Develop the Lagrangian and find the equations of motion for the system shown in Fig. 2. The system consists of a pendulum with a bob of the mass $m_2$ attached by a rod of the length $l$ to a body of the mass $m_1$ that slides freely along the horizontal axis $x$. Use the pendulum angle $\theta$ and the coordinate $x_1$ of the sliding body as generalized coordinates.
Driven Motion

Yet another situation where the Lagrangian formalism proves very convenient for deriving the dynamic equations is the case of a driven motion, when one or more variables of a mechanical system are being changed in time by some given law, independently of the motion of the system. If we take a pendulum and will drive the point of attachment—irrespective to the motion of the bob—we will have a system known as a driven pendulum. In fact, there are many different ways to drive a pendulum. For example, a vertically driven pendulum is the pendulum which point of attachment performs some given motion (typically, harmonic $\propto \cos \Omega t$) in the vertical direction. Analogously, there is a horizontally driven pendulum when the point of attachment performs given horizontal motion. One can also have a pendulum with the point of attachment rotating along a circle with, say, constant angular velocity, etc.

Variable Length Plane Pendulum. As an example, let us derive the equation of motion for a plane pendulum driven by changing the length of the wire by some given law $l(t)$. The staring point is the same as in the case of the fixed-length pendulum ($y$ axis is vertical):

$$L = T - U, \quad T = \frac{mv^2}{2}, \quad U = mgy = -mgl \cos \theta.$$  \hspace{1cm} (23)

Now we should be accurate with $v$, because apart from the perpendicular to the wire component $v_\perp = l \dot{\theta}$ we also have the parallel component $v_\parallel = \dot{l}$, so that

$$v^2 = v_\perp^2 + v_\parallel^2 = l^2 \dot{\theta}^2 + \dot{l}^2.$$  \hspace{1cm} (24)

The Lagrangian then reads

$$L = m \left( l^2(t) \dot{\theta}^2 / 2 + gl(t) \cos \theta + \dot{l}^2(t)/2 \right).$$ \hspace{1cm} (25)

The global constant factor $m$ can be omitted since it does not effect the equations of motion. The same is true for any additive function of time. [More generally, this is true for any additive total derivative of a function of coordinates—but not their derivatives (!)—and time, since upon integration the total derivative reduces to some boundary terms which are supposed to be fixed during the variation.] Hence, we can safely omit the term $\dot{l}(t)$ and write the Lagrangian in the form

$$L \propto l^2(t) \dot{\theta}^2 / 2 + gl(t) \cos \theta.$$ \hspace{1cm} (26)

The Lagrange equation

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$ \hspace{1cm} (27)

yields the equation of motion

$$\frac{d}{dt} \left[ l^2(t) \dot{\theta} \right] = -g l(t) \sin \theta.$$ \hspace{1cm} (28)

Mathematically, this is a second-order ordinary differential equation with respect to unknown function $\theta(t)$, in which $l(t)$ is a certain known function.

Problem 28. Develop the Lagrangian and find the equations of motion for the vertically driven plane pendulum the point of attachment of which, $y_\ast$, performs vertical motion $y_\ast(t) = a \cos \Omega t$, where $a$ and $\Omega$ are some constants. The mass of the pendulum bob is $m$ and the length of the rod is $l$.  

Symmetries and Conservation Laws

With the Lagrangian formalism we can reveal a deep connection between symmetries and conservation laws. Let us start with a particular example of a spherical pendulum for which one of the Lagrange equations of motion, namely, Eq. (22), comes in the form of a conservation law saying that

\[ \dot{\varphi} \sin^2 \theta = \text{const}. \]  

(29)

The mathematical origin of this special form of the equation of motion is the absence of the explicit dependence of \( L \) on \( \varphi \). Indeed, whenever Lagrangian \( L \) is explicitly independent of a generalized variable \( q \), the corresponding Lagrange equation will simplify to

\[ \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0 \quad \iff \quad \frac{\partial L}{\partial \dot{q}} = \text{const}. \]  

(30)

Thus any generalized variable \( q \) which the Lagrangian is independent of is “responsible” for a constant of motion, the explicit expression for which is the partial derivative of the Lagrangian with respect to \( \dot{q} \).

The next step is to realize that the existence of the above-mentioned special variable(s) can be guaranteed by the symmetries of the problem. In the case of the spherical pendulum, there is a rotational symmetry with respect to the \( z \) axis. The absolute value of velocity does not change with respect to any rotation, the potential energy remains the same if the pendulum is rotated around the \( z \) axis. But in the spherical coordinates the rotation around the \( z \) axis is nothing but shifting the variable \( \varphi \). Hence, neither kinetic nor potential energy can explicitly depend on \( \varphi \).

The rotational and translational symmetries take place for any system of interacting particles in the absence of an external potential. Corresponding conserving quantities are the angular and linear momenta, respectively. We establish this fact below. It will turn out that there will be no need to explicitly introduce corresponding generalized variables \( q \).

**Conservation of Linear Momentum.** Consider the Lagrangian (7) of the system of interacting particles. It is invariant with respect to translation of all radius-vectors by one and the same vector. Let us take an infinitesimally small vector \( \vec{\epsilon} \) and perform the transformation

\[ \mathbf{r}_j \rightarrow \mathbf{r}_j + \vec{\epsilon} \quad (j = 1, 2, \ldots, N). \]  

(31)

Would \( L \) be sensitive to this infinitesimal translation, its variation would be given by the standard formula of multi-variable calculus

\[ \delta L = \sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j} \cdot \vec{\epsilon} = \vec{\epsilon} \cdot \sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j}. \]  

(32)

But our \( L \) is independent of \( \vec{\epsilon} \), and thus for any direction of \( \vec{\epsilon} \) we are supposed to have

\[ \vec{\epsilon} \cdot \sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j} = 0, \]  

(33)

which is possible if and only if

\[ \sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j} = 0. \]  

(34)
Lagrange equations read
\[ \frac{\partial L}{\partial \dot{r}_j} = \frac{d}{dt} \frac{\partial L}{\partial \dot{r}_j}, \] (35)
and thus
\[ \sum_{j=1}^{N} \frac{d}{dt} \frac{\partial L}{\partial \dot{r}_j} = 0 \quad \Leftrightarrow \quad \frac{d}{dt} \sum_{j=1}^{N} \frac{\partial L}{\partial \dot{r}_j} = 0 \quad \Leftrightarrow \quad \sum_{j=1}^{N} \frac{\partial L}{\partial \dot{r}_j} = \text{const}. \] (36)

It is easy to see that the conserving quantity is the linear momentum of the system:

\[ \sum_{j=1}^{N} \frac{\partial L}{\partial \dot{r}_j} = \sum_{j=1}^{N} m_j \dot{r}_j = \mathbf{P}. \] (37)

We established the relationship of the momentum conservation to the translation invariance of the Lagrangian.

Conservation of Angular Momentum. Lagrangian (7) of the system of interacting particles is also invariant with respect to a rotation of the system by any angle around any axis. Indeed, the Lagrangian depends only on the absolute values of velocities and relative inter-particle distances, both being preserved by a rotation. Let us take some axis \( z \) and consider a rotation around this axis by an infinitesimal angle \( \delta \varphi \). Since each particular \( r_j \) and each particular \( v_j \equiv \dot{r}_j \) are changed by rotation:

\[ r_j \rightarrow r_j + \delta r_j, \quad v_j \rightarrow v_j + \delta v_j, \] (38)

the requirement that the variation of \( L \) be identically equal to zero leads to the relation

\[ 0 \equiv \delta L = \sum_{j=1}^{N} \left( \frac{\partial L}{\partial r_j} \cdot \delta r_j + \frac{\partial L}{\partial v_j} \cdot \delta v_j \right). \] (39)

To further proceed, we need explicit expressions for \( \delta r_j \) and \( \delta v_j \). These are obtained by noticing that for any vector \( \mathbf{A} \) the rotation around the axis \( z \) by an infinitesimal angle \( \delta \varphi \) corresponds to the transformation

\[ \mathbf{A} \rightarrow \mathbf{A} + \delta \mathbf{A} \] (40)
with

\[ \delta \mathbf{A} = \delta \varphi (\hat{e}_z \times \mathbf{A}). \] (41)

To prove (41) we note that geometrically it is quite clear that (i) \( \delta \mathbf{A} \) should be perpendicular to both \( \hat{e}_z \) and \( \mathbf{A} \) and (ii) \( |\delta \mathbf{A}| = \delta \varphi \sin \theta \), where \( \theta \) is the angle between \( \mathbf{A} \) and \( \hat{e}_z \). Eq. (41) is conveniently written as

\[ \delta \mathbf{A} = \tilde{\delta} \varphi \times \mathbf{A}, \quad \tilde{\delta} \varphi \equiv \delta \varphi \hat{e}_z. \] (42)

Eq. (39) now becomes

\[ \sum_{j=1}^{N} \left( \frac{\partial L}{\partial r_j} \cdot (\tilde{\delta} \varphi \times r_j) + \frac{\partial L}{\partial v_j} \cdot (\tilde{\delta} \varphi \times v_j) \right) = 0, \] (43)
and can be re-written as

\[ \tilde{\delta} \varphi \cdot \sum_{j=1}^{N} \left( r_j \times \frac{\partial L}{\partial r_j} + v_j \times \frac{\partial L}{\partial v_j} \right) = 0. \] (44)
Since the orientation of the $z$ axis is arbitrary, the equality (44) implies
\[ \sum_{j=1}^{N} \left( \mathbf{r}_j \times \frac{\partial L}{\partial \mathbf{r}_j} + \mathbf{v}_j \times \frac{\partial L}{\partial \mathbf{v}_j} \right) = 0 . \] 
(45)

Now we use (35) to get
\[ \sum_{j=1}^{N} \left( \mathbf{r}_j \times \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_j} + \mathbf{v}_j \times \frac{\partial L}{\partial \mathbf{v}_j} \right) = 0 \iff \frac{d}{dt} \sum_{j=1}^{N} \mathbf{r}_j \times \frac{\partial L}{\partial \mathbf{v}_j} = 0 \iff \sum_{j=1}^{N} \mathbf{r}_j \times \frac{\partial L}{\partial \mathbf{v}_j} = \text{const} . \]
(46)

We just need to make sure that the conserved quantity is the angular momentum:
\[ \sum_{j=1}^{N} \mathbf{r}_j \times \frac{\partial L}{\partial \mathbf{v}_j} = \sum_{j=1}^{N} m_j \mathbf{v}_j . \]
(47)

Hence, we have demonstrated that the conservation of the angular momentum is a consequence of rotational invariance of the Lagrangian.

**Galilean Invariance.** In Lagrangian Mechanics, there is a subtle difference between the notions of *symmetry* and *invariance* if applied to the Lagrangian rather than equations of motion. A symmetry means invariance of the equations of motion with respect to this or that transformation of variables. An important question then is whether the invariance of the equations of motion *necessarily* implies corresponding invariance of the Lagrangian. An intuitive answer would be ‘yes’, especially in view of the above examples of translational and rotational symmetry. The correct answer, however, is ‘no’, because two formally different Lagrangians can actually result in identically the same equations of motion, provided the two differ by a total derivative of some function of coordinates and time:
\[ L_1 = L_2 + \frac{d}{dt} Q(q,t) . \]
(48)

This fact is of simple mathematical nature. The functionals of action corresponding to these two Lagrangians will differ only by boundary terms:
\[ S_2 - S_1 = \int_{t_a}^{t_b} \left[ \frac{d}{dt} Q(q,t) \right] dt = Q(q_b,t_b) - Q(q_a,t_a) . \]
(49)

But the Least Action Principle requires that $q_a$ and $q_b$ be fixed when looking for the extremum of the action. Since the two actions differ by just a fixed constant, the two extrema are identically the same. Note that any constant or a function of time can be trivially represented as a total derivative. Hence, additive constants and additive functions of time can be always excluded from the Lagrangian. It is a good manner to always do that to simplify expressions for Lagrangians.

The above remark is directly relevant to the Galilean symmetry. Performing Galilean transformation
\[ \mathbf{r}_j \rightarrow \mathbf{r}_j + \mathbf{v}_0 t , \quad \mathbf{v}_j \rightarrow \mathbf{v}_j + \mathbf{v}_0 \quad (j = 1, 2, \ldots, N) \]
(50)
in the Lagrangian (7), we get
\[ L \rightarrow L + \sum_{j=1}^{N} \left( \frac{m_j}{2} \mathbf{v}_0 \cdot (2\mathbf{v}_j + \mathbf{v}_0) \right) \equiv L + \frac{d}{dt} \sum_{j=1}^{N} \left( \frac{m_j}{2} \right) \mathbf{v}_0 \cdot (2\mathbf{r}_j + \mathbf{v}_0 t) . \]
(51)
Hence, we do have the symmetry of the equations of motions since the new Lagrangian differs from
the old one by the total derivative of the function

\[ Q(\{r_j\}, t) = \sum_{j=1}^{N} (m_j/2) v_0 \cdot (2r_j + v_0 t) , \quad (52) \]

but we do not have literal invariance of the Lagrangian with respect to the Galilean transformation.
This subtlety is very important for establishing the constant of motion associated with the Galilean
symmetry. The procedure is essentially similar to what we did in the cases of translational and
rotational symmetries. We consider an infinitesimal Galilean transformation \(|⃗ ν| → 0) \]

\[ r_j \rightarrow r_j + ⃗ ν t , \quad v_j \rightarrow v_j + ⃗ ν \quad (j = 1, 2, \ldots , N) \quad (53) \]

and analyze the structure of \(δL\) in terms of all its variables. Instead of the former requirement \(δL = 0\)
now we have to deal with

\[ δL = \frac{d}{dt} \sum_{j=1}^{N} m_j r_j \cdot ⃗ ν . \quad (54) \]

Note that in the limit \(|⃗ ν| → 0 we omit the terms \(∼⃗ ν^2\). The l.h.s. of Eq. (54) can be represented as

\[ δL = \sum_{j=1}^{N} \left( \frac{∂L}{∂r_j} \cdot δr_j + \frac{∂L}{∂v_j} \cdot δv_j \right) = \sum_{j=1}^{N} \left( \frac{∂L}{∂r_j} \cdot ⃗ ν t + \frac{∂L}{∂v_j} \cdot ⃗ ν \right) = \vec{v} \cdot \sum_{j=1}^{N} \left( \frac{∂L}{∂r_j} + \frac{∂L}{∂v_j} \right) . \quad (55) \]

With the Lagrange equations of motion we transform this to

\[ δL = \vec{v} \cdot \sum_{j=1}^{N} \left( t \frac{∂L}{∂v_j} + \frac{∂L}{∂v_j} \right) = \vec{v} \cdot \frac{d}{dt} \sum_{j=1}^{N} t \frac{∂L}{∂v_j} . \quad (56) \]

Combining this with (54) and taking into account the arbitrariness of the direction of \(⃗ ν\), we find

\[ \vec{v} \cdot \frac{d}{dt} \sum_{j=1}^{N} \left( t \frac{∂L}{∂v_j} - m_j r_j \right) = 0 \iff \frac{d}{dt} \sum_{j=1}^{N} \left( t \frac{∂L}{∂v_j} - m_j r_j \right) = 0 \iff \]

\[ \sum_{j=1}^{N} \left( t \frac{∂L}{∂v_j} - m_j r_j \right) = \text{const} . \quad (57) \]

Recalling Eq. (37), as well as the definition of the center of mass coordinate

\[ R = \frac{\sum_{j=1}^{N} m_j r_j}{\sum_{j=1}^{N} m_j} , \quad (58) \]

we see that the conservation law we have revealed implies

\[ R(t) = R(t = 0) + \frac{\mathbf{P}}{\sum_{j=1}^{N} m_j} t . \quad (59) \]

This is nothing but the law of the motion of the center of mass. We remember that this law easily
follows from the conservation of momentum, so that Galilean symmetry does not lead to a non-trivial
constant of motion. Nevertheless, as a mater of principle, we did demonstrate that Galilean symmetry
is directly associated with a certain conserving quantity. Namely,

\[ R(t) - \frac{\mathbf{P}}{\sum_{j=1}^{N} m_j} t = \text{const} \equiv R(t = 0) . \quad (60) \]
Problem 29. On the basis of translational and Galilean symmetry of the system of Fig. 2 along the $x$ axis, argue that there is a much better choice of generalized variables than that you were asked to use in Problem 27. Namely, instead of the variable $x_1$ it is reasonable to use the variable

$$X = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2},$$  \hspace{1cm} (61)$$

where $x_2$ is the $x$ coordinate of the pendulum bob. Develop the Lagrangian in the variables $(X, \theta)$, and then derive corresponding equations of motion.

Noether’s Theorem. Amazingly, all the above-discussed conservation laws are just particular cases of a general theorem established by Emmy Noether. The theorem states that any differentiable symmetry of a system implies corresponding constant of motion, and allows one to explicitly construct this constant. The proof of the Noether’s Theorem is as follows.

By a differentiable symmetry one understands a symmetry transformation of generalized variables controlled by a parameter $\lambda$ in such a way that the functions describing the transformations are differentiable with respect to $\lambda$. Without loss of generality, we can assume that $\lambda = 0$ corresponds to the identity transformation. Differentiability of the transformation means that, in the limit $\lambda \to 0$, the transformation of each generalized coordinate $q_\alpha$ can be written as

$$q_\alpha \to q_\alpha + \lambda A_\alpha(\{q\}, t),$$  \hspace{1cm} (62)$$

where $A_\alpha$ is a certain function of all the generalized coordinates and time. That is we simply Taylor-expand the transformation with respect to $\lambda$ and keep only the leading term. In the case when there is more than one parameter controlling the transformation—like in the above-discussed cases, when we had transformations parameterized by vectors rather than scalars, we simply treat each of them independently (componentwise).

Equation (62) implies

$$\delta q_\alpha = \lambda A_\alpha(\{q\}, t), \hspace{1cm} \delta \dot{q}_\alpha = \lambda \frac{d}{dt} A_\alpha(\{q\}, t),$$  \hspace{1cm} (63)$$

which can be used to write the transformation of the Lagrangian as

$$\delta L = \sum_\alpha \left[ \frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha \right] = \lambda \sum_\alpha \left[ \frac{\partial L}{\partial q_\alpha} A_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} A_\alpha \right].$$  \hspace{1cm} (64)$$

Now it is time to recall that we are talking of a symmetry transformation, meaning that the only possible way the Lagrangian can be changed, if at all, is by adding a total time derivative of some function of coordinates and time. We thus have

$$\sum_\alpha \left[ \frac{\partial L}{\partial q_\alpha} A_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} A_\alpha \right] = \frac{d}{dt} Q(\{q\}, t),$$  \hspace{1cm} (65)$$

where $Q$ is a certain function of generalized coordinates and time, specific for the given symmetry transformation. Equation (65) is valid for any formal trajectory of the system, no matter physical or non-physical. Meanwhile, we are interested in physical trajectories only. Hence, we can use Euler-Lagrange equations to modify (65):

$$\sum_\alpha \left[ A_\alpha \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} A_\alpha \right] = \frac{d}{dt} Q(\{q\}, t).$$  \hspace{1cm} (66)$$

And this brings us to the conclusion that

$$\sum_\alpha A_\alpha \frac{\partial L}{\partial q_\alpha} - Q = \text{const}.$$  \hspace{1cm} (67)$$
**Conservation of Energy.** Conservation of energy is a consequence of the homogeneity of time, that is independence of the laws of Nature on time. The homogeneity of time is clearly seen in the Lagrangian (7) which does not depend explicitly on the variable $t$. The known general theorem of Calculus of Variations—see Eq. (50) of corresponding section—states

$$ \frac{d}{dt} \left( \sum_{j=1}^{N} \dot{r}_j \cdot \frac{\partial L}{\partial \dot{r}_j} - L \right) = - \frac{\partial L}{\partial t} , \quad (68) $$

and since

$$ \frac{\partial L}{\partial t} = 0 , \quad (69) $$

we conclude that

$$ \sum_{j=1}^{N} \dot{r}_j \cdot \frac{\partial L}{\partial \dot{r}_j} - L = \text{const} . \quad (70) $$

As is seen from (7), this conserved quantity is the energy:

$$ \sum_{j=1}^{N} \dot{r}_j \cdot \frac{\partial L}{\partial \dot{r}_j} - L = 2T - L = T + U \equiv E . \quad (71) $$

In a general case, for any set of generalized coordinates $\{q_s\}$ and time-independent $L \equiv L(\{q_s\}, \{\dot{q}_s\})$ we have

$$ E = \sum_s \dot{q}_s \frac{\partial L}{\partial \dot{q}_s} - L = \text{const} . \quad (72) $$

**Motion in a Uniform Magnetic Field**

The problem of one or more charged particles moving in a uniform static magnetic field is extremely instructive for illustrating some subtle features of Lagrange formalism. We start with just one particle. The Lagrangian reads

$$ L = \frac{mv^2}{2} + \frac{e}{2} \mathbf{v} \cdot (\mathbf{B} \times \mathbf{r}) , \quad (73) $$

where $e$ is the electric charge of the particle and $\mathbf{B}$ is the magnetic field; as usual, $m$ is the mass, $\mathbf{r}$ is the radius vector, and $\mathbf{v}$ is the velocity. (In the real World, Lagrangian (73) corresponds to the non-relativistic regime, when the velocity of the particle is much smaller than the speed of light. This fact, however, is not relevant for our discussion of the general aspects of the theory.) Two aspects of Eq. (73) are worth special noting. First, the explicit coordinate dependence of the Lagrangian rendering the issue of translational invariance rather non-trivial. Second, the structure of $L$ does not seem to be simply $L = T - U$, since the second term has little to do with either kinetic or potential energy.

Let us derive the equations of motion. To this end we parameterize the radius vector as follows. Selecting the $z$ axis in the direction of the magnetic field,

$$ \mathbf{B} = B\hat{e}_z , \quad (74) $$

we write

$$ \mathbf{r} = z\hat{e}_z + \mathbf{r}_\perp , \quad \mathbf{r}_\perp = x\hat{e}_x + y\hat{e}_y . \quad (75) $$
The Lagrangian then decouples into two independent terms:

\[ L = L^{(z)} + L^{(\perp)} , \]

(76)

\[ L^{(z)} = \frac{m v_z^2}{2}, \]

(77)

\[ L^{(\perp)} = \frac{m v_{\perp}^2}{2} + \frac{e}{2} v_{\perp} \cdot (B \times r_{\perp}), \]

(78)

where \( v_z = \dot{z} \) and \( v_{\perp} = \dot{r}_{\perp} \). The decoupling of the Lagrangian into independent terms means that corresponding degrees of freedom do not interact with each other and can be treated as if they represent just different systems. In our case, \( L^{(z)} \) describes a free one-dimensional particle moving along the \( z \) axis while \( L^{(\perp)} \) is a Lagrangian of a two-dimensional particle moving in the \( xy \) plane and interacting with the magnetic field perpendicular to this plane. The motion of the coordinate \( z \) is trivial. Let us concentrate on the motion of \( r_{\perp} \). To simplify the notation, we will be omitting the subscript ’\( \perp \)’, assuming that \( r \) is a two-dimensional vector in the \( xy \) plane.

With Eq. (78) we find

\[ \frac{\partial L}{\partial v} = m v + \frac{e}{2} (B \times r), \]

(79)

\[ \frac{\partial L}{\partial r} = \frac{e}{2} (v \times B), \]

(80)

and plugging this into the Lagrange equation

\[ \frac{d}{dt} \frac{\partial L}{\partial v} = \frac{\partial L}{\partial r} , \]

(81)

we get

\[ m \ddot{v} = e (v \times B). \]

(82)

The equation of motion (82) is translationally invariant: it does not contain \( r \). We now want to establish the translational invariance of the problem at the level of Lagrangian and derive corresponding constant of motion. The transformation

\[ r \rightarrow r + \varepsilon \]

(83)

leads to

\[ L \rightarrow L + \frac{e}{2} \varepsilon \cdot (v \times B) = L + \frac{d}{dt} \frac{e}{2} \varepsilon \cdot (r \times B). \]

(84)

We see that the difference between the new and old Lagrangians is a total derivative and this explains why the equation of motion is translationally invariant. With the explicit form of the translational transformation (84), we are in a position to derive corresponding conservation law, an analog of the momentum conservation. We take \( |\varepsilon| \rightarrow 0 \) and write

\[ \delta L = \frac{d}{dt} \frac{e}{2} \varepsilon \cdot (r \times B). \]

(85)

We then proceed similarly to the previous derivation of the momentum conservation:

\[ \delta L = \varepsilon \cdot \frac{\partial L}{\partial r} = \varepsilon \cdot \frac{d}{dt} \frac{\partial L}{\partial v}. \]

(86)

Combining this with (85), we get (for all the orientations of \( \varepsilon \))

\[ \varepsilon \cdot \frac{d}{dt} \frac{\partial L}{\partial v} = \frac{d}{dt} \frac{e}{2} \varepsilon \cdot (r \times B) \quad \Rightarrow \quad \frac{\partial L}{\partial v} - \frac{e}{2} (r \times B) = \text{const}. \]

(87)
With Eq. (79) we arrive at
\[ m \mathbf{v} - e (\mathbf{r} \times \mathbf{B}) = \text{const}, \] (88)
which is conveniently written as
\[ m \mathbf{v} = e (\mathbf{r} - \mathbf{r}_0) \times \mathbf{B}, \] (89)
where \( \mathbf{r}_0 \) is some constant vector which actually can be set equal to zero by shifting the origin.
The obtained conservation law coincides with the first integral of the equation of motion. This is not surprising. For one particle in the absence of magnetic field momentum conservation is also equivalent to the first integral of the equation of motion. However, it is quite instructive that the ‘magnetic counterpart’ of the momentum conservation leads to a non-trivial character of motion. Eq. (89) implies that the particle performs a uniform rotation in a circle centered at the point \( \mathbf{r}_0 \), the radius \( R_0 \) of the circle being related to \( \mathbf{v} \) by
\[ R_0 = \frac{mv}{eB}. \] (90)
The conservation law we have established is straightforwardly generalized to the case of \( N \) particles (with different masses and charges) moving in the uniform static magnetic field and interacting via pair potentials:
\[ \sum_{j=1}^{N} m_j \mathbf{v}_j + e \sum_{j=1}^{N} e_j \mathbf{r}_j = \text{const}. \] (91)

Energy. With the general relation (72) and our particular Lagrangian (78) we have
\[ E = \mathbf{v} \cdot \frac{\partial L}{\partial \mathbf{v}} - L = \frac{mv^2}{2}. \] (92)
Amazingly enough, the second term in Eq. (78) turns out to be irrelevant for energy.

Galilean transformation. Now we want to understand what happens to the Lagrangian (78) after Galilean transformation
\[ \mathbf{r} \rightarrow \mathbf{r} + \mathbf{v}_0 t, \quad \mathbf{v} \rightarrow \mathbf{v} + \mathbf{v}_0. \] (93)
We see that
\[ L \rightarrow L + \frac{e}{2} \mathbf{v}_0 \cdot (\mathbf{B} \times \mathbf{r}) + \frac{et}{2} \mathbf{v}_0 \cdot (\mathbf{v} \times \mathbf{B}) + mv_0 \cdot \mathbf{v} + \frac{m}{2} \mathbf{v}_0^2. \] (94)
We already know that the two last terms coming from the kinetic energy are a total derivative and thus can be omitted:
\[ L \rightarrow L + \frac{e}{2} \mathbf{v}_0 \cdot (\mathbf{B} \times \mathbf{r}) + \frac{et}{2} \mathbf{v}_0 \cdot (\mathbf{v} \times \mathbf{B}). \] (95)
Next we note that
\[ tv_0 \cdot (\mathbf{v} \times \mathbf{B}) = \frac{d}{dt} [tv_0 \cdot (\mathbf{r} \times \mathbf{B})] - \mathbf{v}_0 \cdot (\mathbf{r} \times \mathbf{B}). \] (96)
Hence, in the Lagrangian we can replace
\[ \frac{et}{2} \mathbf{v}_0 \cdot (\mathbf{v} \times \mathbf{B}) \rightarrow - \frac{e}{2} \mathbf{v}_0 \cdot (\mathbf{r} \times \mathbf{B}), \] (97)
since the difference is a total derivative. The final expression is really elegant:
\[ L \rightarrow L + e \mathbf{\hat{E}} \cdot \mathbf{r}, \quad \mathbf{\hat{E}} = \mathbf{v}_0 \times \mathbf{B}. \] (98)
The system is \textit{not} Galilean invariant, since we get a term which is not a total derivative. However, the structure of the new term is really interesting. It is indistinguishable from the potential corresponding to a uniform electric field \( \mathbf{\hat{E}} \) perpendicular to the original magnetic field. This means that for an
observer in a moving reference frame the motion of a particle will look like a motion in a superposition of the magnetic and electrical field. By the way, this remarkable transformational property of our Lagrangian immediately suggests a solution to the problem of a motion in a superposition of uniform and orthogonal to each other electric and magnetic fields: just change the reference frame by Eq. (93) with the velocity \( v_0 \) chosen in such a way that the emerging ‘fake’ electric field completely compensates the original ‘physical’ one. Solve the problem in the moving frame—it is just a circular orbit—and return back to the original frame. In fact, there is no ‘fake’ and ‘physical’ electric fields. It is the fundamental property of the electromagnetic field that what appears to be a pure magnetic field in one reference frame turns out to be a superposition of both magnetic and electric fields in a moving frame.

**Problem 30.** Use the above-discussed trick of removing electric field by Galilean transformation (in the presence of uniform magnetic field) to solve the problem of motion of a particle of the charge \( e \) and mass \( m \) in the crossed uniform magnetic and electric fields, with the initial condition that at \( t = 0 \) the particle is at rest (in the laboratory frame). The term ‘crossed’ means that the electric and magnetic fields are orthogonal. Also, find the initial condition at which the trajectory of the particle is just a straight line.

**Problem 31.** The Lagrangian of a system of \( N \) non-relativist particles moving in uniform static electric and magnetic fields (the two are not necessarily orthogonal to each other) reads

\[
L = \sum_{i=1}^{N} \frac{m_i v_i^2}{2} + \sum_{i=1}^{N} \frac{e_i}{2} \mathbf{v}_i \cdot (\mathbf{B} \times \mathbf{r}_i) + \sum_{i=1}^{N} e_i \mathbf{E} \cdot \mathbf{r}_i - \sum_{i<j} \frac{e_i e_j}{|\mathbf{r}_i - \mathbf{r}_j|},
\]

where \( m_i \)'s are masses, \( e_i \)'s are charges, \( \mathbf{B} \) is the magnetic field, and \( \mathbf{E} \) is the electric fields. On the basis of this Lagrangian, show that

(i) the energy of the system is given by the expression

\[
E = \sum_{i=1}^{N} \frac{m_i v_i^2}{2} - \sum_{i=1}^{N} e_i \mathbf{E} \cdot \mathbf{r}_i + \sum_{i<j} \frac{e_i e_j}{|\mathbf{r}_i - \mathbf{r}_j|}.
\]

(ii) the problem is translationally invariant, and the translational symmetry implies the following conservation law:

\[
\sum_{i=1}^{N} m_i \mathbf{v}_i + \mathbf{B} \times \sum_{i=1}^{N} e_i \mathbf{r}_i - t \mathbf{E} \sum_{i=1}^{N} e_i = \text{const}.
\]

Reveal a simple physical meaning of the \( z \)-projection of this conservation law, provided the \( z \) axis is along \( \mathbf{B} \).

**Lagrange Multipliers**

Consider a body sliding down along a smooth spherical/cylindrical surface, see Fig. 3. We are interested in finding the critical angle \( \theta_0 \) at which the body leaves the surface. The simplest version of the Lagrange formalism does not yield an easy solution to this problem, since we would have to change the number of generalized coordinates during the motion. Before the body leaves the surface, there is only one independent coordinate, say, the angle \( \theta \). After leaving the surface, the body is described by two independent coordinates, say, the angle \( \theta \) and radius \( r \). The trick is to use both \( \theta \) and \( r \) from the very beginning, but minimize the action under the constraint

\[
r = a,
\]

as long as the constraint is relevant. From the Calculus of Variations we know that minimization of functionals under constraints is done with the technique of undetermined Lagrange multipliers.
Suppose we need to minimize the action

$$ S = \int_{t_a}^{t_b} L(\{q_s\}, \{\dot{q}_s\}) \, dt $$

under the constraint

$$ f(\{q_s\}, \{\dot{q}_s\}) = C , $$

meaning that a certain function of coordinates and their derivatives has to be equal to a certain constant $C$. If instead of the original functional $S$ we will be minimizing the functional

$$ S' = \int_{t_a}^{t_b} L'(\{q_s\}, \{\dot{q}_s\}) \, dt , $$

with

$$ L'(\{q_s\}, \{\dot{q}_s\}) = L(\{q_s\}, \{\dot{q}_s\}) + \lambda(t) f(\{q_s\}, \{\dot{q}_s\}) , $$

where $\lambda(t)$ is an arbitrary function of $t$, we will get absolutely the same result, because as long as the constraint (104) is respected, the two actions differ by just a constant number,

$$ S' - S = C \int_{t_a}^{t_b} \lambda(t) \, dt , $$

that drops out from the variations:

$$ \delta S' = \delta S \quad \text{(as long as the constraint is respected)} . $$

Now if there exists some special choice of the function $\lambda(t)$ such that the minima of the functional $S'$ with and without the constraint just coincide, then, by definition, the solution for the unconstraint minimum of the functional $S'$ yields the minimum of the original functional $S$ under the constraint (104). The utility of this observation is that we do not need to know this special $\lambda(t)$ before looking for the minimum of the functional $S'$. Rather, we write $\lambda(t)$ as an undetermined function, find the unconstraint minimum of the functional $S'$, and then find such function $\lambda(t)$ for which the constraint is satisfied.

Let us see how this technique works for our particular system, Fig. 3. The Lagrangian reads:

$$ L = \frac{m}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - m g r \cos \theta . $$

Our constraint is $r = a$, and we thus write

$$ L' \propto \frac{1}{2} (\dot{r}^2 + r^2 \dot{\theta}^2) - g r \cos \theta + \lambda(t) r . $$

For the sake of convenience, we replaced $\lambda(t) \rightarrow m \lambda(t)$ so that it is seen from the very outset that the mass enters the problem as a global factor in front of the action and therefore drops out from the answers for $r(t)$ and $\theta(t)$. The solutions for $r(t)$, $\theta(t)$, and $\lambda(t)$ come from the system of three equations

$$ \frac{d}{dt} \frac{\partial L'}{\partial \dot{r}} = \frac{\partial L'}{\partial r} , $$

$$ \frac{d}{dt} \frac{\partial L'}{\partial \dot{\theta}} = \frac{\partial L'}{\partial \theta} , $$

$$ r(t) \equiv a . $$
If we solve these equations and find \( r(t) \) and \( \theta(t) \), we will naturally get \( r(t) = a \), and the result for \( \theta(t) \) will be precisely the same as if we fixed \( r(t) = a \) in the original Lagrangian \( L \). The question now is why do we use this trick with \( \lambda(t) \) if it just yields identically the same \( r(t) = a \) and \( \theta(t) \)? The crucial point is that the function \( \lambda(t) \) that comes as a ‘by product’ of the calculation, actually contains an explicit information about the moment \( t = t_0 \) when the particle leaves the surface. Indeed, at \( t = t_0 \), the constraint should become irrelevant. That is \( \lambda(t > t_0) \equiv 0 \). By continuity this implies that

\[
\lambda(t = t_0) = 0. \tag{114}
\]

Hence, the essence of constraint trick is to find the explicit form of the function \( \lambda(t) \) and see when it hits zero.

To finish our calculation, we need to solve the system (111)-(113). With the constraint (113), equation (111) reduces to

\[
\lambda = g \cos \theta - a \dot{\theta}^2. \tag{115}
\]

Instead of writing and then solving Eq. (112), we utilize the energy conservation law:

\[
\frac{1}{2} a^2 \dot{\theta}^2 + ga \cos \theta = E/m. \tag{116}
\]

Using (116) to exclude \( \dot{\theta}^2 \) from (115), we find

\[
\lambda = 3g \cos \theta - \frac{2E}{ma}. \tag{117}
\]

For the critical angle \( \theta_0 \) at which the body leaves the surface we get

\[
\theta_0 = \cos^{-1} \left( \frac{2E}{3mga} \right). \tag{118}
\]

In particular, if the motion starts at \( \theta = 0 \) with an infinitesimally small velocity, then \( E = mga \) and \( \theta_0 = \cos^{-1}(2/3) \).

**Problem 32.** Consider the system (shown in Fig. 4) of a mass \( m \) performing a plane motion within a fixed
Figure 4: A body moving inside a fixed spherical/cylindrical surface of the radius $a$.

sphere/cylinder in the gravitational field. The motion starts from the lowest point $\theta = 0$ with some horizontal velocity $v$. For a certain range of values of $v$, the body will leave the surface at some critical angle $\theta_0(v)$. Find this range of $v$’s, as well as the function $\theta_0(v)$. 
