Lagrangian Formalism

Hamilton's Least Action Principle

Let $x(t)$ be some function of time, m be some positive number, and $U(x)$ be some function of x. Consider a functional

$$S = \int_{t_a}^{t_b} L(x, \dot{x})\, dt \;, \tag{1}$$

with

$$L(x, \dot{x}) = \frac{m\dot{x}^2}{2} - U(x) \;, \tag{2}$$

and find the function $x(t)$ that minimizes[1] the functional S at given boundary conditions

$$x(t_a) = x_a \;, \qquad\qquad x(t_b) = x_b \;. \tag{3}$$

The Calculus of Variations tells us that the function in question satisfies the Euler's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \;. \tag{4}$$

With L of Eq. (2), this means

$$m\ddot{x} = -U'(x) \;, \tag{5}$$

which happens to be the Newton's Second Law for a one-dimensional particle of the mass m moving in the external potential $U(x)$. This amazing circumstance generalizes to any number of particles in any dimensions. Indeed, introduce

$$S = \int_{t_a}^{t_b} L(\{\mathbf{r}_i\}, \{\dot{\mathbf{r}}_i\})\, dt \;, \tag{6}$$

$$L = T - U \;, \qquad T = \sum_{i=1}^{N} \frac{m_i\, \dot{\mathbf{r}}_i^2}{2} \;, \qquad U = \sum_{i<j} U(|\mathbf{r}_i - \mathbf{r}_j|) \;, \tag{7}$$

and look for the functions $\{\mathbf{r}_i(t)\}$ $(i = 1, 2, \ldots, N)$ minimizing (extremizing) the functional (6) under fixed boundary conditions

$$\mathbf{r}_i(t_a) = \mathbf{r}_i^{(a)} \;, \qquad\qquad \mathbf{r}_i(t_b) = \mathbf{r}_i^{(b)} \;. \tag{8}$$

Observe that the system of Euler's equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}_i} = \frac{\partial L}{\partial \mathbf{r}_i} \qquad\qquad (i = 1, 2, \ldots N) \;, \tag{9}$$

defining these functions is nothing but Newton's equations of motion.

The importance of the fact that we have established—known as *Hamilton's least action principle*[2]—can hardly be overestimated. The variational formulation of Mechanics has at least two major advantages. First, we can now easily switch from Cartesian coordinates to any other coordinates, including collective variables, since the least action principle is explicitly invariant with respect to

[1] Or just extremizes, which would be absolutely sufficient for our purposes.

[2] We use the historic term "least action principle" while keeping in mind that all we actually need is the *extremal action* condition, by which we mean the condition of vanishing all the variational derivatives of the functional S with respect to all the functions $\mathbf{r}_i(t)$.

the choice of coordinates. We simply need to rewrite the functional S, called *action*, in terms of new variables, $\{q_\alpha\}$, called generalized coordinates. To this end we express $\mathbf{r}_i$ and $\dot{\mathbf{r}}_i$ in terms of $\{q_\alpha\}$ and $\{\dot{q}_\alpha\}$, substitute the result into the function L, called *Lagrangian function*, or simply the *Lagrangian*, and get the action of the following form

$$S \;=\; \int_{t_a}^{t_b} L(\{q_\alpha\},\{\dot{q}_\alpha\})\,dt\;. \tag{10}$$

Extrimization of the action under the condition of fixed values of the generalized coordinates at times t_a and t_b, leads to the equations of motion in terms of new variables:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_\alpha} \;=\; \frac{\partial L}{\partial q_\alpha}\;. \tag{11}$$

Mathematically, these are the Euler's equations of the Calculus of Variations. In the context of Mechanics, Equations (11) are called *Lagrange equations*, or *Euler-Lagrange equations*, and the whole formalism is called *Lagrangian Mechanics*.

A remark is in order here. The condition of fixed values of the (generalized) coordinates at times t_a and t_b is an important part of the formulation of Hamilton's principle, but nothing beyond that. The condition is necessary to unambiguously define the variational derivative leading to the Euler-Lagrange equations. The two times, t_a and t_b, are *any* two time moments; they are not supposed to have anything to do with some special times, and, in particular, should not be interpreted as the "initial" and "final" times.

The second crucial advantage of the Lagrangian formalism is closely related to the first one. Think of a realistic macroscopic mechanical system consisting of a huge number of particles and thus not amenable to direct microscopic treatment. In many cases, to a very good approximation, a macroscopic system features a very limited number of relevant degrees of freedom. For example, the position of an idealized pendulum—a mass constrained by a weightless, extensionless rod—is characterized by just two spherical angles, (θ,φ); or even one polar angle, θ, if the motion is restricted to a plane. Let us see how Lagrangian formalism works for these two examples.

Plane Pendulum. Let the y-axis be vertical. We have

$$L \;=\; T - U\;, \qquad T \;=\; \frac{mv^2}{2}\;, \qquad U \;=\; mgy\;, \tag{12}$$

where m is the pendulum mass, $v^2 \;=\; \dot{x}^2 + \dot{y}^2$ is the square of the velocity, g is the free-fall acceleration. Because of the constraint, the system is described by one generalized coordinate, the angle θ between the rod and the negative direction of the y-axis. We then notice that $v^2 \;=\; l^2\dot{\theta}^2$, $y \;=\; -l\cos\theta$, with l is the length of the rod, and thus

$$L \;=\; ml^2(\dot{\theta}^2/2 + \omega^2\cos\theta)\;, \qquad \omega^2 \;=\; g/l\;. \tag{13}$$

The Lagrange equation is:

$$\ddot{\theta} \;=\; -\omega^2\sin\theta\;. \tag{14}$$

Note that the global factor ml^2 drops out. This is a good example of how basic similarity analysis can be performed at the level of Lagrangian, before resorting to the equations of motion.

Spherical Pendulum. Let the z-axis be vertical and directed downwards. The Lagrangian is

$$L \;=\; T - U\;, \qquad T \;=\; \frac{mv^2}{2}\;, \qquad U \;=\; -mgz\;. \tag{15}$$

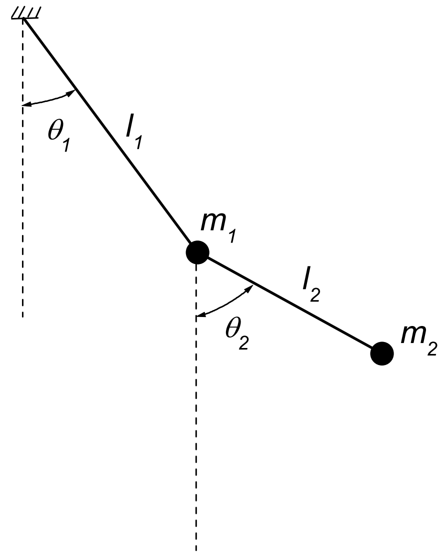

Figure 1: Coupled plane pendulums.

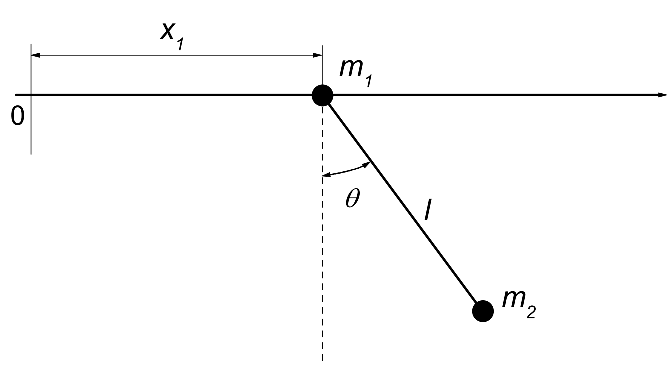

Figure 2: A sliding body with an attached pendulum.

There are two degrees of freedom that are conveniently parameterized by two spherical angles θ and φ. The radius is equal to a fixed number l, the length of the rod. To express L in terms of the generalized coordinates θ and φ, we need to relate v to θ and φ:

$$x = l\sin\theta\cos\varphi \qquad \Rightarrow \qquad \dot{x} = l\,\dot{\theta}\cos\theta\cos\varphi - l\,\dot{\varphi}\sin\theta\sin\varphi\,. \tag{16}$$

$$y = l\sin\theta\sin\varphi \qquad \Rightarrow \qquad \dot{y} = l\,\dot{\theta}\cos\theta\sin\varphi + l\,\dot{\varphi}\sin\theta\cos\varphi\,. \tag{17}$$

$$z = l\cos\theta \qquad \Rightarrow \qquad \dot{z} = -l\,\dot{\theta}\sin\theta\,. \tag{18}$$

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = l^2(\dot{\theta}^2 + \dot{\varphi}^2\sin^2\theta)\,. \tag{19}$$

Hence,

$$L = ml^2\left(\frac{\dot{\theta}^2}{2} + \frac{\dot{\varphi}^2}{2}\sin^2\theta + \omega^2\cos\theta\right)\,, \qquad \omega^2 = g/l\,. \tag{20}$$

The two Lagrange equations read:

$$\ddot{\theta} = \dot{\varphi}^2\sin\theta\cos\theta - \omega^2\sin\theta\,, \tag{21}$$

$$\frac{d}{dt}\left(\dot{\varphi}\sin^2\theta\right) = 0\,. \tag{22}$$

Equation (22) has the form of a conservation law. Up to a constant factor, the conserved quantity is the z-projection of the angular momentum. And there is a reason for that. The crucial circumstance here is the so-called *cyclic* coordinate. The cyclic coordinate is a special (generalized) coordinate that Lagrangian does not explicitly depend on, depending only on its time derivative. As a result, the partial derivative of the Lagrangian with respect to this coordinate is zero, and the Lagrange equation acquires the form of conservation law. In our case, the cyclic coordinate is φ. Cyclic coordinates reflect this or that continuous symmetry of the problem: The symmetry transformation associated with a cyclic coordinate corresponds to shifting this coordinate by a constant. In our case, the symmetry in question is the rotational symmetry with respect to the z-axis. We thus see how continuous symmetries—if expressed in terms of cyclic coordinates—immediately imply conservation laws. Later we will prove Noether's theorem that further generalizes the remarkable connection between continuous symmetries and conservation laws.

Problem 17. A rigid body of the mass M can freely rotate around—but not slide along—the fixed axis z. The system is placed in the uniform gravitational field characterized by the free fall acceleration $\mathbf{g}$. Let $\alpha \in [0,\, \pi/2]$ be the (fixed) angle between the direction the axis z and the vector $\mathbf{g}$. The moment of inertia of the body with respect to the axis z is I; the distance between the center of mass of the body and the axis z is l. Develop the Lagrangian of the system. Provide a thorough comparison with the case of plane pendulum. What is the role of the angle α? Are there special values of α?

Hints/Reminders. The z-component of the angular momentum and kinetic energy of a rigid body—rotating around the axis z with the (instant) angular velocity Ω—are given by the relations

$$L_z = I\Omega\,, \qquad T = \frac{I\Omega^2}{2}\,. \tag{23}$$

These formulas—along with explicit microscopic definition of I—are established by representing the rigid body as a set of elementary plane pendulums sharing one and the same instant angular velocity. For the bob of the j-th elementary pendulum, introduce the mass m_j, the linear velocity v_j, and the distance l_j form the rotation axis. Then

$$L_z = \sum_j l_j(m_j v_j) = \sum_j l_j(m_j l_j \Omega) = \Omega \sum_j m_j l_j^2 = I\Omega\,, \qquad I = \sum_j m_j l_j^2,$$

$$T = \sum_j \frac{m_j v_j^2}{2} = \sum_j \frac{m_j (l_j \Omega)^2}{2} = \frac{\Omega^2}{2} \sum_j m_j l_j^2 = \frac{I\Omega^2}{2},$$

For any system of particles (not necessarily rigid), the potential energy of interaction with a uniform gravitational field can be expressed as

$$U = -M\,\mathbf{g}\cdot\mathbf{R},$$

where M is the total mass, $\mathbf{R}$ is the center of mass position, and $\mathbf{g}$ is the free fall acceleration.

Problem 18. Develop the Lagrangian and find the equations of motion for the system of two coupled plane pendulums, Fig. 1.

Problem 19. Develop the Lagrangian and find the equations of motion for the system shown in Fig. 2. The system consists of a pendulum with a bob of the mass m_2 attached by a rod of the length l to a body of the mass m_1 that slides freely along the horizontal axis x. Use the pendulum angle θ and the coordinate x_1 of the sliding body as generalized coordinates.

Driven Motion

Yet another situation where the Lagrangian formalism proves very convenient for deriving the dynamic equations is the case of a *driven motion*, when one or more generalized coordinates of a mechanical system are being changed in time by some given law, *independently of the motion of the rest of the system.* If we take a pendulum and drive the point of attachment—irrespectively to the motion of the bob—we get a system known as driven pendulum. There are different ways to drive a pendulum. For example, a vertically driven pendulum is the pendulum the point of attachment of which performs some given motion (typically, harmonic $\propto \cos\Omega t$) in the vertical direction. Analogously, there is a horizontally driven pendulum when the point of attachment performs given horizontal motion. One can also have a pendulum with the point of attachment rotating along a circle with, say, constant angular velocity, etc.

To understand how to use Lagrangian formalism in the case of driven motion, we start with a simple observation that the formalism still applies if the external potential depends on time. Then, for a generalized coordinate q_0, we can introduce a generalized time-dependent external potential $U_0(q_0,t)$. Now if the amplitude of the potential U_0 is very large, then, up to higher-order corrections, the Lagrange equation for the coordinate q_0 simplifies to one term:

$$\frac{\partial U_0(q_0,t)}{\partial q_0} = 0\,. \tag{24}$$

Equation (24) completely defines the function $q_0(t)$, which strictly follows the (time-dependent) position of the minimum of the potential U_0. Substituting the function $q_0(t)$ into the Lagrangian, we get the (time-dependent) action describing the evolution of the system in terms of remaining degrees of freedom. Upon the substitution of $q_0(t)$ into the action, the term $U(q_0(t),t)$ becomes trivial. Being a function of time only, the term will drop from all the Lagrange equations. Hence, we can safely eliminate this term from the action. As a result, the action depends only on the form of the driven trajectory $q_0(t)$, but not on the driving potential. Therefore for describing the driven motion we can use the known function $q_0(t)$ as the prime quantity. Similarly, one can drive any number of generalized coordinates.

The concept of driven motion provides us with an instructive perspective on the motion with constraints. Indeed, this or that constraint can be interpreted as originating from *time-independent* driving potential fixing the value of corresponding generalized coordinate. The plane and spherical pendulums nicely illustrate this idea. We just need to think of the length of the rod as the generalized coordinate l whose value is fixed by a strong generalized potential $U(l)$ having a minimum at the desired value of l.

Variable Length Plane Pendulum. As an example, let us derive the equation of motion for a plane pendulum driven by changing the length of the wire by some given law $l = l(t)$. The staring point is the same as in the case of the fixed-length pendulum (y axis is vertical):

$$L \,=\, T - U \,, \qquad T \,=\, \frac{mv^2}{2} \,, \qquad U \,=\, mgy \,=\, -mgl\cos\theta \,. \tag{25}$$

Now we should be careful with v: Apart from the perpendicular to the wire component $v_\perp \,=\, l\dot{\theta}$ we also have the parallel component $v_\parallel \,=\, \dot{l}$, so that

$$v^2 \,=\, v_\perp^2 + v_\parallel^2 \,=\, l^2\dot{\theta}^2 + \dot{l}^2 \,. \tag{26}$$

The Lagrangian then reads

$$L \,=\, m\left[\frac{l^2(t)\,\dot{\theta}^2}{2} + gl(t)\cos\theta + \frac{\dot{l}^2(t)}{2}\right] . \tag{27}$$

The global constant factor m can be omitted since it does not effect the equations of motion. The same is true for any additive function of time.[3] Hence, we can safely omit the term $\dot{l}(t)$ and write the Lagrangian in the form

$$L \,\propto\, l^2(t)\,\dot{\theta}^2/2 + gl(t)\cos\theta \,. \tag{28}$$

The Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} \,=\, \frac{\partial L}{\partial\theta} \tag{29}$$

yields the equation of motion

$$\frac{d}{dt}\left[l^2(t)\,\dot{\theta}\right] \,=\, -g\,l(t)\,\sin\theta \,. \tag{30}$$

Mathematically, this is a second-order ordinary differential equation with respect to unknown function $\theta(t)$, in which $l(t)$ is a certain known function.

Problem 20. Develop the Lagrangian and find the equations of motion for the vertically driven plane pendulum the point of attachment of which, y_*, performs vertical motion $y_*(t) = a\cos\Omega t$, where a and Ω are some constants. The mass of the pendulum bob is m and the length of the rod is l.

Symmetries and Conservation Laws

Lagrangian formalism allows one to establish a deep connection between symmetries and conservation laws. Recall our discussion of cyclic coordinates in the context of Eq. (22). Whenever Lagrangian L

[3]More generally, this is true for any additive *total derivative* of a function of coordinates (but not their derivatives) and time. We will discuss this important circumstance below; see Eqs. (49)–(50) and corresponding text.

is explicitly independent of a generalized coordinate q (in which case q is called cyclic coordinate), the corresponding Lagrange equation simplifies to the conservation law:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0 \qquad \Leftrightarrow \qquad \frac{\partial L}{\partial \dot{q}} = \text{const} \,. \tag{31}$$

Hence, any cyclic coordinate q is responsible for a certain constant of motion, the explicit expression for which is the partial derivative of the Lagrangian with respect to $\dot{q}$.

The next step is to realize that the existence of the above-mentioned special variable(s) can be guaranteed by the symmetries of the problem. Recall that in the case of spherical pendulum, the cyclic coordinate φ was all about the rotational symmetry with respect to the z axis.

The rotational and translational symmetries take place for any system of interacting particles in the absence of external potentials. Corresponding conserved quantities are the angular and linear momenta, respectively. We establish this fact below. Furthermore—and crucially important from the practical point of view, we will see that we can work with the original set of (generalized) coordinates. The cyclic generalized coordinates enter the treatment *implicitly*, as parameters of infinitesimal transformations leaving the theory invariant.

Conservation of Linear Momentum. Consider the Lagrangian (7) of the system of interacting particles. It is invariant with respect to translation of all radius-vectors by one and the same vector. Let us take an infinitesimally small vector $\vec{\epsilon}$ and perform the transformation

$$\mathbf{r}_j \to \mathbf{r}_j + \vec{\epsilon} \qquad (j = 1, 2, \ldots, N) \,. \tag{32}$$

Would L be sensitive to this infinitesimal translation, its variation would be given by the standard formula of multi-variable calculus

$$\delta L = \sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j} \cdot \vec{\epsilon} = \vec{\epsilon} \cdot \sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j} \,. \tag{33}$$

But our L is independent of $\vec{\epsilon}$, and thus for any direction of $\vec{\epsilon}$ we are supposed to have

$$\vec{\epsilon} \cdot \sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j} = 0 \,, \tag{34}$$

which is possible if and only if

$$\sum_{j=1}^{N} \frac{\partial L}{\partial \mathbf{r}_j} = 0 \,. \tag{35}$$

Now recall that Lagrange equations read

$$\frac{\partial L}{\partial \mathbf{r}_j} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}_j} \,, \tag{36}$$

and thus (summing up all the Lagrange equations)

$$\sum_{j=1}^{N} \frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{r}}_j} = 0 \qquad \Leftrightarrow \qquad \frac{d}{dt}\sum_{j=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_j} = 0 \qquad \Leftrightarrow \qquad \sum_{j=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_j} = \text{const} \,. \tag{37}$$

With the explicit expression for the Lagrangian, Eq. (7), we see that the conserved quantity is the linear momentum of the system:

$$\sum_{j=1}^{N} \frac{\partial L}{\partial \dot{\mathbf{r}}_j} \;=\; \sum_{j=1}^{N} m_j\, \dot{\mathbf{r}}_j \;=\; \mathbf{P}\,. \tag{38}$$

We established the relationship between the conservation of the linear momentum and the translation invariance of the Lagrangian.

Conservation of Angular Momentum. Lagrangian (7) of the system of interacting particles is also invariant with respect to rotation of the system by any angle around any axis. Indeed, the Lagrangian depends only on the absolute values of velocities and relative inter-particle distances, both being preserved by any rotation. Let us take some axis z and and consider a rotation around this axis by an infinitesimal angle $\delta\varphi$. Since each particular $\mathbf{r}_j$ and each particular $\mathbf{v}_j \equiv \dot{\mathbf{r}}_j$ are changed by rotation:

$$\mathbf{r}_j \;\to\; \mathbf{r}_j + \delta\mathbf{r}_j\,, \qquad\qquad \mathbf{v}_j \;\to\; \mathbf{v}_j + \delta\mathbf{v}_j\,, \tag{39}$$

the requirement that the variation of L be identically equal to zero leads to the relation

$$0 \;\equiv\; \delta L \;=\; \sum_{j=1}^{N} \left(\frac{\partial L}{\partial \mathbf{r}_j} \cdot \delta\mathbf{r}_j + \frac{\partial L}{\partial \mathbf{v}_j} \cdot \delta\mathbf{v}_j \right)\,. \tag{40}$$

To further proceed, we need explicit expressions for $\delta\mathbf{r}_j$ and $\delta\mathbf{v}_j$. These are obtained by noticing that for any vector $\mathbf{A}$ the rotation around the axis z by an infinitesimal angle $\delta\varphi$ corresponds to the transformation

$$\mathbf{A} \;\to\; \mathbf{A} + \delta\mathbf{A} \tag{41}$$

with

$$\delta\mathbf{A} \;=\; \delta\varphi\,(\hat{\mathrm{e}}_z \times \mathbf{A})\,. \tag{42}$$

To prove (42) we note that geometrically it is quite clear that (i) $\delta\mathbf{A}$ should be perpendicular to both $\hat{\mathrm{e}}_z$ and $\mathbf{A}$ and (ii) $|\delta\mathbf{A}| \;=\; \delta\varphi\,|\mathbf{A}|\sin\theta$, where θ is the angle between $\mathbf{A}$ and $\hat{\mathrm{e}}_z$. Equation (42) is conveniently written as

$$\delta\mathbf{A} \;=\; \vec{\delta\varphi} \times \mathbf{A}\,, \qquad\qquad \vec{\delta\varphi} \;\equiv\; \delta\varphi\,\hat{\mathrm{e}}_z\,. \tag{43}$$

Equation (40) now becomes

$$\sum_{j=1}^{N} \left[\frac{\partial L}{\partial \mathbf{r}_j} \cdot (\vec{\delta\varphi} \times \mathbf{r}_j) + \frac{\partial L}{\partial \mathbf{v}_j} \cdot (\vec{\delta\varphi} \times \mathbf{v}_j) \right] \;=\; 0\,, \tag{44}$$

and can be re-written as

$$\vec{\delta\varphi} \cdot \sum_{j=1}^{N} \left(\mathbf{r}_j \times \frac{\partial L}{\partial \mathbf{r}_j} + \mathbf{v}_j \times \frac{\partial L}{\partial \mathbf{v}_j} \right) \;=\; 0\,. \tag{45}$$

Since the orientation of the z axis is arbitrary, the equality (45) implies

$$\sum_{j=1}^{N} \left(\mathbf{r}_j \times \frac{\partial L}{\partial \mathbf{r}_j} + \mathbf{v}_j \times \frac{\partial L}{\partial \mathbf{v}_j} \right) \;=\; 0\,. \tag{46}$$

Combining (46) with Lagrange equations (36), we get

$$\sum_{j=1}^{N}\left(\mathbf{r}_j\times\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}_j}+\mathbf{v}_j\times\frac{\partial L}{\partial \mathbf{v}_j}\right)=0 \quad\Leftrightarrow\quad \frac{d}{dt}\sum_{j=1}^{N}\mathbf{r}_j\times\frac{\partial L}{\partial \mathbf{v}_j}=0 \quad\Leftrightarrow\quad \sum_{j=1}^{N}\mathbf{r}_j\times\frac{\partial L}{\partial \mathbf{v}_j}=\text{const}\,. \tag{47}$$

And with the explicit expression for the Lagrangian, Eq. (7), we see that the conserved quantity is the angular momentum:

$$\sum_{j=1}^{N}\mathbf{r}_j\times\frac{\partial L}{\partial \mathbf{v}_j}=\sum_{j=1}^{N}\mathbf{r}_j\times m_j\,\mathbf{v}_j\,. \tag{48}$$

Hence, we have demonstrated that the conservation of the angular momentum is a consequence of rotational invariance of the Lagrangian.

Galilean Invariance. In Lagrangian Mechanics, there is a subtle difference between the notions of *symmetry* and *invariance* if applied to the Lagrangian rather than equations of motion. A symmetry means invariance of the equations of motion with respect to this or that transformation of variables. An important question then is whether the invariance of the equations of motion *necessarily* implies corresponding invariance of the Lagrangian. An intuitive answer would be "yes," especially in view of the above-discussed examples of translational and rotational symmetry. The correct answer, however, is "no," because two formally different Lagrangians, L_1 and L_2, would result in identically the same equations of motion, provided the two differ by a total derivative of some function of coordinates and time:

$$L_1 = L_2 + \frac{d}{dt}Q(q,t)\,. \tag{49}$$

This fact is of simple mathematical nature. The functionals of action corresponding to these two Lagrangians will differ only by boundary terms:

$$S_2 - S_1 = \int_{t_a}^{t_b}\left[\frac{d}{dt}Q(q,t)\right]dt = Q(q_b,t_b) - Q(q_a,t_a)\,. \tag{50}$$

But the Least Action Principle requires that q_a and q_b be fixed when looking for the extremum of the action. Since the two actions differ by just a fixed constant, the two extrema are identically the same. Note that any constant or a function of time can be trivially represented as a total derivative. Hence, additive constants and additive functions of time can be always excluded from the Lagrangian. It is a good manner to always do that to simplify expressions for Lagrangians.

The above remark is directly relevant to the Galilean symmetry. Performing Galilean transformation

$$\mathbf{r}_j \to \mathbf{r}_j + \mathbf{v}_0\,t\,, \qquad \mathbf{v}_j \to \mathbf{v}_j + \mathbf{v}_0 \qquad (j = 1,2,\ldots,N) \tag{51}$$

in the Lagrangian (7), we get

$$L \to L + \sum_{j=1}^{N}(m_j/2)\,\mathbf{v}_0\cdot(2\mathbf{v}_j+\mathbf{v}_0) \equiv L + \frac{d}{dt}\sum_{j=1}^{N}(m_j/2)\,\mathbf{v}_0\cdot(2\mathbf{r}_j+\mathbf{v}_0 t)\,. \tag{52}$$

We do have the symmetry of the equations of motions since the new Lagrangian differs from the old one by the total derivative of the function

$$Q(\{\mathbf{r}_j\},t) = \sum_{j=1}^{N}(m_j/2)\,\mathbf{v}_0\cdot(2\mathbf{r}_j+\mathbf{v}_0 t)\,. \tag{53}$$

But we do not have literal invariance of the Lagrangian with respect to the Galilean transformation. This subtlety is very important for establishing the constant of motion associated with the Galilean symmetry. The procedure is very similar to what we did in the cases of translational and rotational symmetries. We consider an infinitesimal Galilean transformation ($|\vec{\nu}| \to 0$)

$$\mathbf{r}_j \to \mathbf{r}_j + \vec{\nu}\, t\,, \qquad \mathbf{v}_j \to \mathbf{v}_j + \vec{\nu} \qquad (j = 1, 2, \ldots, N) \tag{54}$$

and analyze the structure of δL in terms of all its variables. Instead of the former requirement $\delta L = 0$ now we have to deal with

$$\delta L = \frac{d}{dt} \sum_{j=1}^{N} m_j\, \mathbf{r}_j \cdot \vec{\nu}\,. \tag{55}$$

Note that we work in the limit $|\vec{\nu}| \to 0$ and thus omit the terms $\sim \vec{\nu}^2$. The l.h.s. of Eq. (55) can be represented as

$$\delta L = \sum_{j=1}^{N} \left(\frac{\partial L}{\partial \mathbf{r}_j} \cdot \delta \mathbf{r}_j + \frac{\partial L}{\partial \mathbf{v}_j} \cdot \delta \mathbf{v}_j \right) = \sum_{j=1}^{N} \left(\frac{\partial L}{\partial \mathbf{r}_j} \cdot \vec{\nu}\, t + \frac{\partial L}{\partial \mathbf{v}_j} \cdot \vec{\nu} \right) = \vec{\nu} \cdot \sum_{j=1}^{N} \left(t\, \frac{\partial L}{\partial \mathbf{r}_j} + \frac{\partial L}{\partial \mathbf{v}_j} \right). \tag{56}$$

With the Lagrange equations (36), we transform this to

$$\delta L = \vec{\nu} \cdot \sum_{j=1}^{N} \left(t\, \frac{d}{dt} \frac{\partial L}{\partial \mathbf{v}_j} + \frac{\partial L}{\partial \mathbf{v}_j} \right) = \vec{\nu} \cdot \frac{d}{dt} \sum_{j=1}^{N} t\, \frac{\partial L}{\partial \mathbf{v}_j}\,. \tag{57}$$

Combining this with (55) and taking into account the arbitrariness of the direction of $\vec{\nu}$, we find

$$\vec{\nu} \cdot \frac{d}{dt} \sum_{j=1}^{N} \left(t\, \frac{\partial L}{\partial \mathbf{v}_j} - m_j\, \mathbf{r}_j \right) = 0 \quad \Leftrightarrow \quad \frac{d}{dt} \sum_{j=1}^{N} \left(t\, \frac{\partial L}{\partial \mathbf{v}_j} - m_j\, \mathbf{r}_j \right) = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \sum_{j=1}^{N} \left(t\, \frac{\partial L}{\partial \mathbf{v}_j} - m_j\, \mathbf{r}_j \right) = \text{const}\,. \tag{58}$$

Recalling Eq. (38), as well as the definition of the center-of-mass coordinate,

$$\mathbf{R} = \frac{\sum_{j=1}^{N} m_j\, \mathbf{r}_j}{M}\,, \qquad M = \sum_{j=1}^{N} m_j\,, \tag{59}$$

we see that the conservation law we have revealed states that

$$\mathbf{R}(t) = \mathbf{R}(t=0) + \frac{\mathbf{P}}{M}\, t\,. \tag{60}$$

It turns out that our quite lengthy derivation results in nothing but the law of the motion of the center of mass. Nevertheless, as a mater of principle, we did demonstrate that Galilean symmetry is directly associated with a certain conserved quantity. Namely,

$$\mathbf{R}(t) - \frac{\mathbf{P}}{M}\, t = \text{const} \equiv \mathbf{R}(t=0)\,. \tag{61}$$

Even more importantly, the way we were treating Galilean transformation takes us very close to the idea of Noether's theorem, which we prove below.

Problem 21. On the basis of translational and Galilean symmetry of the system of Fig. 2 along the x axis, argue that there is a much better choice of generalized coordinates than that you were asked to use in Problem 19. Namely, instead of the variable x_1 it is reasonable to use the variable

$$X \;=\; \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}\,, \tag{62}$$

where x_2 is the x coordinate of the pendulum bob. Develop the Lagrangian in the variables (X, θ), and then derive corresponding equations of motion.

Semi-cyclic coordinates.[4] Consider Lagrangian that has the following peculiar form with respect to a certain generalized coordinate q:

$$L(q, \dot q, t) \;=\; L_0(\dot q, t) \;+\; f(t)\, q\,. \tag{63}$$

The part $L_0(\dot q, t)$ of Lagrangian (63) may—and normally does—depend on other generalized coordinates, while $f(t)$ is a certain function of time only. A special and quite typical case is when f is simply a time-independent constant.

The Lagrange equation for q,

$$\frac{d}{dt}\frac{\partial L_0}{\partial \dot q} \;=\; f(t)\,, \tag{64}$$

can be written in the form of the conservation law:

$$\frac{d}{dt}\left[\frac{\partial L_0}{\partial \dot q} \;-\; F(t)\right] \;=\; 0\,, \qquad\quad \Leftrightarrow \qquad\quad \frac{\partial L_0}{\partial \dot q} \;-\; F(t) \;=\; \text{const}\,, \tag{65}$$

where $F(t)$ is an antiderivative of $f(t)$:

$$F(t) \;=\; \int f(t)\, dt\,. \tag{66}$$

Equation (65) is quite reminiscent of Eq. (31). There is a deep reason behind this fact. In both cases, the Lagrange equation for q is invariant with respect to translations $q \to q + q_0$. And it is precisely this continuous symmetry that, in both cases, guarantees the existence of corresponding constant of motion. This justifies the term "semi-cyclic coordinate" for q. Note also that $f(t)$ has the meaning of generalized time-dependent or time-independent—but necessarily q-independent—force acting on the coordinate q.

In some special cases, Lagrangian L_0 happens to be independent of the rest of generalized coordinates and their derivatives. In such a situation, the semi-cyclic coordinate is *separable* and its motion is completely described by Eq. (65). Quite often this happens for one or more components of the center of mass vector, provided the problem at hand features corresponding translational symmetry.

The notion of semi-cyclic coordinate will be further generalized in the context of Noether's theorem that we discuss below.

Problem 22. Consider a generalized version of Problem 21: Now the fixed sliding axis x can have an arbitrary angle $\alpha \in [0, \pi/2]$ with the vector of free fall acceleration $\mathbf{g}$.

Problem 23. Consider a generalized version of the system discussed in Problem 17: Now in addition to rotation around the axis z, the rigid body has one more degree of freedom—it can slide freely along the axis z.

[4] The term "semi-cyclic coordinate" is my own invention, which by no means excludes a natural possibility that precisely this or similar notion already exists in literature and I simply failed to find it.

Noether's Theorem

All the above-discussed various conservation laws are particular cases of a beautiful general theorem established by Emmy Noether. The theorem states that any differentiable symmetry of a system implies the existence of associated constant of motion, and provides an explicit expression for the conserved quantity.

By a differentiable symmetry we understand a symmetry transformation of generalized coordinates controlled by a certain parameter λ in such a way that the functions describing the transformations are differentiable with respect to λ. Without loss of generality, we assume that $\lambda = 0$ corresponds to the identity transformation. Differentiability of the transformation means that, in the limit $\lambda \to 0$, the transformation of each generalized coordinate q_α can be written as

$$q_\alpha \;\to\; q_\alpha + \lambda A_\alpha(\{q\}, t)\,, \tag{67}$$

where A_α is a certain function of all the generalized coordinates and time. That is, we Taylor-expand the transformation with respect to λ and keep only the leading term (below $\{\tilde{q}\}$ is the set of new variables):

$$q_\alpha \;=\; f_\alpha(\lambda, \{\tilde{q}\}, t) \;=\; f_\alpha(\lambda = 0, \{\tilde{q}\}, t) \;+\; \lambda\, \frac{f_\alpha(\lambda, \{\tilde{q}\}, t)}{\partial \lambda}\bigg|_{\lambda=0}\,,$$

$$f_\alpha(\lambda = 0, \{\tilde{q}\}, t) \;=\; \tilde{q}_\alpha\,, \qquad\qquad A_\alpha(\{\tilde{q}\}, t) \equiv \frac{f_\alpha(\lambda, \{\tilde{q}\}, t)}{\partial \lambda}\bigg|_{\lambda=0}\,.$$

In the case when there is more than one parameter controlling the transformation—like in the above-discussed cases, when we had transformations parameterized by vectors rather than scalars, we simply treat each of them independently (componentwise).[5]

For an infinitesimal λ, Eq. (67) implies

$$\delta q_\alpha \;=\; \lambda A_\alpha(\{q\}, t)\,, \qquad\qquad \delta \dot{q}_\alpha \;=\; \lambda \frac{d}{dt} A_\alpha(\{q\}, t)\,, \tag{68}$$

which can be used to write the transformation of the Lagrangian as

$$\delta L \;\equiv\; \sum_\alpha \left[\frac{\partial L}{\partial q_\alpha} \delta q_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \delta \dot{q}_\alpha \right] \;=\; \lambda \sum_\alpha \left[\frac{\partial L}{\partial q_\alpha} A_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} A_\alpha \right]\,. \tag{69}$$

Now we take into account that we are talking of a *symmetry* transformation, meaning that the only possible way the Lagrangian can be changed, if at all, is by adding a total time derivative of some function of coordinates and time. We thus have

$$\sum_\alpha \left[\frac{\partial L}{\partial q_\alpha} A_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} A_\alpha \right] \;=\; \frac{d}{dt} Q(\{q\}, t)\,, \tag{70}$$

where Q is a certain function of generalized coordinates and time, specific for the given symmetry transformation. Equation (70) is valid for any formal trajectory of the system, no matter physical or non-physical. Meanwhile, we are interested in physical trajectories only. Hence, we can use Lagrange equations to modify the l.h.s. of (70):

$$\sum_\alpha \left[\frac{\partial L}{\partial q_\alpha} A_\alpha + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} A_\alpha \right] \;=\; \sum_\alpha \left[A_\alpha \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_\alpha} + \frac{\partial L}{\partial \dot{q}_\alpha} \frac{d}{dt} A_\alpha \right] \;=\; \frac{d}{dt} \sum_\alpha A_\alpha \frac{\partial L}{\partial \dot{q}_\alpha}\,. \tag{71}$$

[5]Infinitesimal Galilean transformation, Eq. (54), is a perfect illustration of transformation (67), with the role of parameter λ played by a component of the vector $\vec{\nu}$.

And this brings us to the desired conclusion:

$$\sum_\alpha A_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - Q = \text{const}\,. \tag{72}$$

Noether's theorem suggests *further generalization of the notion of semi-cyclic coordinate*. Suppose that for a certain generalized coordinate q, our Lagrangian has the following property:

$$\frac{\partial L}{\partial q} = \frac{d}{dt} Q(\{q\}, t)\,. \tag{73}$$

Then the Lagrange equation for q acquires the form of the conservation law:

$$\frac{\partial L}{\partial \dot{q}} - Q(\{q\}, t) = \text{const}\,. \tag{74}$$

In this language, the symmetry transformation (67) can be interpreted as an implicit infinitesimal shift of a certain semi-cyclic generalized coordinate. Later we will see that such a peculiar situation takes place for the radius vector of a particle moving in a uniform magnetic field.

Conservation of Energy. Conservation of energy is a consequence of the *homogeneity of time*—the invariance of the laws of Nature with respect to time-translations. This invariance is clearly seen in the Lagrangian (7) which does not depend explicitly on the variable t. With the general mathematical theorem—Eq. (50) of the Chapter "Calculus of Variations"—applied to Lagrangian mechanics, we have

$$\frac{d}{dt}\left(\sum_{j=1}^N \dot{\mathbf{r}}_j \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}_j} - L \right) = -\frac{\partial L}{\partial t}\ , \tag{75}$$

and since

$$\frac{\partial L}{\partial t} = 0\ , \tag{76}$$

we conclude that

$$\sum_{j=1}^N \dot{\mathbf{r}}_j \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}_j} - L = \text{const}\ . \tag{77}$$

As is seen from Eq. (7), this conserved quantity is the total energy, which we already introduced in Newtonian mechanics:

$$\sum_{j=1}^N \dot{\mathbf{r}}_j \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}_j} - L = 2T - L = T + U \equiv E\ . \tag{78}$$

In a general case, for any set of generalized coordinates $\{q_\alpha\}$ and time-independent $L \equiv L(\{q_\alpha\}, \{\dot{q}_\alpha\})$, we have

$$E = \sum_\alpha \dot{q}_\alpha \frac{\partial L}{\partial \dot{q}_\alpha} - L = \text{const}\ . \tag{79}$$

In the axiomatic Lagrangian mechanics, the first equality in (79) is the definition of the system's energy, while the second equality in (79) is the theorem of the energy conservation.

Problem 24. Consider the following system of two coupled rotors. Each rotor has one degree of freedom—the polar angle θ_α ($\alpha = 1, 2$). The kinetic energy consists of two independent terms:

$$T = \frac{I_1 \dot{\theta}_1^2}{2} + \frac{I_2 \dot{\theta}_2^2}{2} \, . \tag{80}$$

Here I_α is the moment of inertia of the rotor α ($\alpha = 1, 2$). The coupling is due to the potential energy:

$$U = -J \cos(\theta_1 - \theta_2) \, , \tag{81}$$

where $J > 0$ is the coupling constant.

(i) Derive Lagrange equations for this system.

(ii) This problem features a certain symmetry that can be utilized for a clever choice of new generalized coordinates—such that one of the two coordinates is cyclic thus revealing a constant of motion. Find those coordinates and write corresponding Lagrange equations.

(iii) Now consider the driven motion of this system in the case when J is a periodic function of time: $J = J_0 \sin(\Omega t)$. What are the Lagrange equations? What happens to the above-discussed constant of motion?

It might seem that the law of the energy conservation (79), while being unquestionably associated with a continuous symmetry—invariance with respect to transformation

$$q_\alpha(t) \to q_\alpha(t + t_0) \, ,$$

controlled by the parameter $\lambda \equiv t_0$—does not fit into the paradigm (72) of the Noether's theorem. Indeed, the analog of Eq. (67) now becomes

$$q_\alpha(t) \to \lambda \dot{q}_\alpha \, ,$$

bringing the *third-order* time derivatives to the transformed Lagrangian and, apparently, fundamentally changing the whole structure of the Lagrange equations. However, if we explicitly check what happens to the transformed Lagrangian, we readily see that

$$L \to L + \lambda \frac{d}{dt} L \, ,$$

meaning that the third-order derivatives are fully absorbed into the "innocent" total-derivative term. We know that such term does not affect Lagrange equations. We thus readily generalize our proof to the case where the infinitesimal transformation

$$q_\alpha \to q_\alpha + \lambda A_\alpha(\{q\}, \{\dot{q}\}, t)$$

leads to

$$L \to L + \lambda \frac{d}{dt} Q(\{q\}, \{\dot{q}\}, t) \, .$$

The result is the very same expression (72), in which A and Q can now also depend on $\{\dot{q}\}$. With $A_\alpha = \dot{q}_\alpha$ and $Q = L$, equation (72) yields (79).

Motion in a Uniform Magnetic Field

The problem of one or more charged particles moving in a uniform static magnetic field is extremely instructive for illustrating certain subtle features of Lagrangian formalism.[6] We start with just one particle. The Lagrangian reads[7]

$$L \; = \; \frac{mv^2}{2} \; + \; \frac{e}{2}\, \mathbf{v}\cdot(\mathbf{B}\times\mathbf{r}) \; , \tag{82}$$

where e is the electric charge of the particle and $\mathbf{B}$ is the magnetic field; as usual, m is the mass, $\mathbf{r}$ is the radius vector, and $\mathbf{v}$ is the velocity. Two features of Eq. (82) are worth noting. The first one is the explicit coordinate dependence of the Lagrangian rendering the issue of translational invariance rather non-trivial: Now it is all about most general type of semi-cyclic variables defined by Eq. (73). The second feature is the peculiar structure of L, which now does not reduce to $L = T - U$: The second term has little to do with either kinetic or potential energy.

Let us derive the equations of motion. To this end we parameterize the radius vector as follows. Selecting the z axis in the direction of the magnetic field,

$$\mathbf{B} \; = \; B\,\hat{\mathrm{e}}_z \; , \tag{83}$$

we write

$$\mathbf{r} \; = \; z\,\hat{\mathrm{e}}_z + \mathbf{r}_\perp \; , \qquad\qquad \mathbf{r}_\perp \; = \; x\,\hat{\mathrm{e}}_x + y\,\hat{\mathrm{e}}_y \; . \tag{84}$$

The Lagrangian then decouples into two independent terms:

$$L \; = \; L^{(z)} + L^{(\perp)} \; , \tag{85}$$

$$L^{(z)} \; = \; \frac{mv_z^2}{2} \; , \tag{86}$$

$$L^{(\perp)} \; = \; \frac{mv_\perp^2}{2} \; + \; \frac{e}{2}\, \mathbf{v}_\perp\cdot(\mathbf{B}\times\mathbf{r}_\perp) \; , \tag{87}$$

where $v_z = \dot{z}$ and $\mathbf{v}_\perp = \dot{\mathbf{r}}_\perp$. The decoupling of the Lagrangian into independent terms means that corresponding degrees of freedom do not interact with each other and can be treated as if they represent different systems: $L^{(z)}$ describes a free one-dimensional particle moving along the z axis, while $L^{(\perp)}$ is a Lagrangian of a two-dimensional particle moving in the xy plane and interacting with the magnetic field perpendicular to this plane. The motion of the coordinate z is trivial. Let us concentrate on the motion of $\mathbf{r}_\perp$. To simplify the notation, we will be omitting the subscript '$\perp$', assuming that $\mathbf{r}$ is a two-dimensional vector in the xy plane.

With Eq. (87) we find

$$\frac{\partial L}{\partial \mathbf{v}} \; = \; m\mathbf{v} + \frac{e}{2}\,(\mathbf{B}\times\mathbf{r})\, , \qquad\qquad \frac{\partial L}{\partial \mathbf{r}} \; = \; \frac{e}{2}\,(\mathbf{v}\times\mathbf{B}) \; . \tag{88}$$

Substituting this into the Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} \; = \; \frac{\partial L}{\partial \mathbf{r}} \; , \tag{89}$$

we get the Newton's Second Law with the Lorentz force in the right-hand side:

$$m\dot{\mathbf{v}} \; = \; e\,(\mathbf{v}\times\mathbf{B}) \; . \tag{90}$$

[6]Note also that going to the rotating frame amounts to introducing a fictitious uniform magnetic field responsible for the Coriolis force—in addition to the centrifugal potential responsible for the centrifugal force.

[7]In this course, we confine ourselves with the case of non-relativistic particles.

The equation of motion (90) is translationally invariant: it does not contain $\mathbf{r}$. This brings us to the conclusion that (each component of) $\mathbf{r}$ is a *semi*-cyclic variable in the sense of the definition (73), implying that we have an associated constant of motion in the form (74).

Indeed,

$$\frac{\partial L}{\partial \mathbf{r}} \;=\; \frac{d}{dt}\,\frac{e}{2}\,(\mathbf{r}\times\mathbf{B})\;, \tag{91}$$

meaning that we are dealing with Eq. (73) where

$$\mathbf{Q} \;=\; \frac{e}{2}\,(\mathbf{r}\times\mathbf{B})\,. \tag{92}$$

Note that the function Q of Eq. (73) is now a vector, because $\dot{\mathbf{r}} \equiv \mathbf{v}$ is a vector. Hence, the conservation law (74) in our case reads

$$\frac{\partial L}{\partial \mathbf{v}} \;-\; \frac{e}{2}\,(\mathbf{r}\times\mathbf{B}) \;=\; \vec{\text{const}}\;. \tag{93}$$

With Eq. (88), we finally arrive at

$$m\mathbf{v} \;-\; e\,(\mathbf{r}\times\mathbf{B}) \;=\; \vec{\text{const}}\;. \tag{94}$$

This relation is conveniently written as

$$\mathbf{v} \;=\; \omega\,(\mathbf{r}-\mathbf{r}_0)\times\hat{e}_z\;, \qquad\qquad \omega \;=\; \frac{eB}{m}\,, \tag{95}$$

where $\mathbf{r}_0$ is a certain constant vector. Without loss of generality, $\mathbf{r}_0$ can be set equal to zero by shifting the origin of coordinates. The conservation law we found provides us with the first integral of the equation of motion. This is not surprising. For a free particle, the momentum conservation also yields the first integral of the equation of motion. However, it is quite instructive that the magnetic counterpart of the momentum conservation leads to a non-trivial character of motion. Equation (95) means that the particle performs a uniform rotation—with the angular velocity ω—in a circle centered at the point $\mathbf{r}_0$, the radius R_0 of the circle being related to v by

$$R_0 \;=\; \frac{v}{\omega} \;=\; \frac{mv}{eB}\;. \tag{96}$$

The conservation law we have established is straightforwardly generalized to the case of N particles (with different masses and charges) moving in the uniform static magnetic field and interacting via pair potentials. The result,

$$\sum_{j=1}^{N} m_j\,\mathbf{v}_j \;+\; \mathbf{B}\times\sum_{j=1}^{N} e_j\,\mathbf{r}_j \;=\; \vec{\text{const}}\;, \tag{97}$$

is a nice and instructive example of the general Noether's relation (72). We simply need to observe that in our case, the transformation (67) is $\mathbf{r}_j \to \mathbf{r}_j + \vec{\lambda}$, meaning that the function A reduces to the unity matrix,[8] and then apply (72) to our many-particle Lagrangian.

Energy. With the general relation (79) and our Lagrangian (87), we have

$$E \;=\; \mathbf{v}\cdot\frac{\partial L}{\partial \mathbf{v}} \;-\; L \;=\; \frac{mv^2}{2}\;. \tag{98}$$

Remarkably, the second term in Eq. (87) turns out to be totally irrelevant for the energy.

[8] Since we are dealing with vectors, A has to be a matrix.

Galilean transformation. Let us see what happens to the Lagrangian (87) if we perform the Galilean transformation

$$\mathbf{r} \;\rightarrow\; \mathbf{r} + \mathbf{v}_0\, t\,, \qquad\qquad \mathbf{v} \;\rightarrow\; \mathbf{v} + \mathbf{v}_0\,. \tag{99}$$

Here we get

$$L \;\rightarrow\; L + \frac{e}{2}\,\mathbf{v}_0\cdot(\mathbf{B}\times\mathbf{r}) + \frac{et}{2}\,\mathbf{v}_0\cdot(\mathbf{v}\times\mathbf{B}) + m\mathbf{v}_0\cdot\mathbf{v} + \frac{m}{2}\,v_0^2\,. \tag{100}$$

We already know that the two last terms coming from the kinetic energy form a total derivative and thus can be safely omitted:

$$L \;\rightarrow\; L + \frac{e}{2}\,\mathbf{v}_0\cdot(\mathbf{B}\times\mathbf{r}) + \frac{et}{2}\,\mathbf{v}_0\cdot(\mathbf{v}\times\mathbf{B})\,. \tag{101}$$

Next, note that the equality

$$t\,\mathbf{v}_0\cdot(\mathbf{v}\times\mathbf{B}) \;=\; \frac{d}{dt}\,[t\,\mathbf{v}_0\cdot(\mathbf{r}\times\mathbf{B})] \;-\; \mathbf{v}_0\cdot(\mathbf{r}\times\mathbf{B}) \tag{102}$$

allows us to make the following replacement in the Lagrangian:

$$\frac{et}{2}\,\mathbf{v}_0\cdot(\mathbf{v}\times\mathbf{B}) \;\rightarrow\; -\frac{e}{2}\,\mathbf{v}_0\cdot(\mathbf{r}\times\mathbf{B})\,. \tag{103}$$

The final result is really elegant:

$$L \;\rightarrow\; L + e\,\vec{\mathcal{E}}\cdot\mathbf{r}\,, \qquad\qquad \vec{\mathcal{E}} \;=\; \mathbf{v}_0\times\mathbf{B}\,. \tag{104}$$

The system is *not* Galilean invariant, since we get a term which is not a total derivative. However, the structure of the new term is quite interesting. It is indistinguishable from the potential corresponding to a uniform *electric field* $\vec{\mathcal{E}}$ perpendicular to the original magnetic field. This means that for an observer in a moving reference frame the motion of a particle will look like a motion in a superposition of the magnetic and electrical field. Incidentally, this remarkable transformational property of our Lagrangian immediately suggests a solution to the problem of a motion in a superposition of uniform and orthogonal to each other electric and magnetic fields: Change the reference frame by Eq. (99) with the velocity $\mathbf{v}_0$ chosen in such a way that the emerging "fake" electric field completely compensates the original "physical" one. Solve the problem in the moving frame—it is just a circular orbit—and return back to the original frame. In fact, there is no "fake" and "physical" electric fields. It is the fundamental property of the electromagnetic field that what appears to be a pure magnetic field in one reference frame turns out to be a superposition of both magnetic and electric fields in a moving frame.

Problem 25. Use the trick of removing electric field by Galilean transformation (in the presence of uniform magnetic field) to solve the problem of motion of a particle of the charge e and mass m in the crossed uniform magnetic and electric fields, with the initial condition that at $t = 0$ the particle is at rest (in the laboratory frame). The term "crossed" means that the electric and magnetic fields are orthogonal. Also, find the initial condition at which the trajectory of the particle is just a straight line.

Problem 26. The Lagrangian of a system of N non-relativistic particles moving in uniform static electric and magnetic fields (the two are not necessarily orthogonal to each other) reads

$$L = \sum_{i=1}^{N} \frac{m_i v_i^2}{2} + \sum_{i=1}^{N} \frac{e_i}{2}\,\mathbf{v}_i\cdot(\mathbf{B}\times\mathbf{r}_i) + \sum_{i=1}^{N} e_i\,\vec{\mathcal{E}}\cdot\mathbf{r}_i - \sum_{i<j} \frac{e_i\,e_j}{|\mathbf{r}_i-\mathbf{r}_j|}\,, \tag{105}$$

where m's are masses, e's are charges, $\mathbf{B}$ is the magnetic field, and $\vec{\mathcal{E}}$ is the electric fields. On the basis of this Lagrangian, show that
(i) the energy of the system is given by the expression

$$E \;=\; \sum_{i=1}^{N} \frac{m_i v_i^2}{2} \;-\; \sum_{i=1}^{N} e_i\, \vec{\mathcal{E}} \cdot \mathbf{r}_i \;+\; \sum_{i<j} \frac{e_i\, e_j}{|\mathbf{r}_i - \mathbf{r}_j|} \;. \tag{106}$$

(ii) the problem is translationally invariant, and the translational symmetry implies the following conservation law:

$$\sum_{i=1}^{N} m_i\, \mathbf{v}_i \;+\; \mathbf{B} \times \sum_{i=1}^{N} e_i\, \mathbf{r}_i \;-\; t\, \vec{\mathcal{E}} \sum_{i=1}^{N} e_i \;=\; \vec{\text{const}} \;. \tag{107}$$

Reveal a simple physical meaning of the z-projection of this conservation law, provided the z axis is along $\mathbf{B}$.

Lagrange Multipliers

Consider a body[9] sliding down along a smooth spherical/cylindrical surface, see Fig. 3. We are interested in finding the critical angle θ_0 at which the body leaves the surface. The simplest version of the Lagrange formalism does not yield an easy solution to this problem, since we would have to *change the number of generalized coordinates during the motion.* Before the body leaves the surface, there is only one independent coordinate, say, the angle θ. After leaving the surface, the body is described by two independent coordinates, say, the angle θ and radius r.

The trick is to use both θ and r from the very beginning, but minimize the action under the constraint

$$r \;=\; a \;, \tag{108}$$

as long as the constraint is relevant. From the Calculus of Variations we know that minimization of functionals under constraints is done with the technique of undetermined Lagrange multipliers. Suppose we need to minimize the action

$$S \;=\; \int_{t_a}^{t_b} L(\{q_\alpha\}, \{\dot{q}_\alpha\})\, dt \tag{109}$$

under the constraint

$$f(\{q_\alpha\}, \{\dot{q}_\alpha\}) \;=\; C \;, \tag{110}$$

meaning that a certain function of coordinates and their derivatives has to be equal to a certain constant C. If instead of the original functional S we will be minimizing the functional

$$S' \;=\; \int_{t_a}^{t_b} L'(\{q_\alpha\}, \{\dot{q}_\alpha\})\, dt \;, \tag{111}$$

with

$$L'(\{q_\alpha\}, \{\dot{q}_\alpha\}) \;=\; L(\{q_\alpha\}, \{\dot{q}_\alpha\}) \;+\; \lambda(t)\, f(\{q_\alpha\}, \{\dot{q}_\alpha\}) \;, \tag{112}$$

where $\lambda(t)$ is an arbitrary function of t, we will get absolutely the same result, because as long as the constraint (110) is respected, the two actions differ by a constant number,

$$S' \;-\; S \;=\; C \int_{t_a}^{t_b} \lambda(t)\, dt \;, \tag{113}$$

[9]Treated as a particle having no size and thus no rotational inertia.

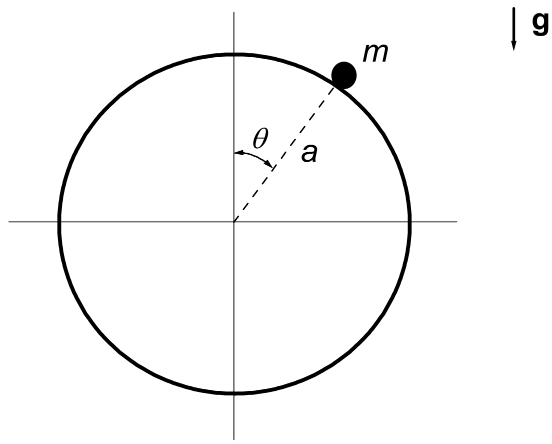

Figure 3: A body sliding from a fixed spherical/cylindrical surface of the radius a.

which drops out from the variations:

$$\delta S' \ = \delta S \qquad\qquad \text{(as long as the constraint is respected)} \ . \tag{114}$$

Now if there exists a special choice of the function $\lambda(t)$, such that the minima of the functional S' with and without the constraint coincide, then, by definition, the solution for the unconstraint minimum of the functional S' yields the minimum of the original functional S under the constraint (110). The utility of this observation is that we do not need to know this special $\lambda(t)$ *before* looking for the minimum of the functional S'. Rather, we write $\lambda(t)$ as an undetermined function, find the unconstraint minimum of the functional S', and then adjust the form of the function $\lambda(t)$ to satisfy the constraint.

Let us see how this technique works for our particular system, Fig. 3. In the absence of the constraint, the Lagrangian reads:

$$L \ = \ \frac{m}{2}\,(\dot{r}^2 + r^2\dot{\theta}^2) \ - \ mgr\cos\theta \ . \tag{115}$$

The constraint is $r = a$, and we thus write

$$L' \ \propto \ \frac{1}{2}\,(\dot{r}^2 + r^2\dot{\theta}^2) \ - \ gr\cos\theta \ + \ \lambda(t)\,r \ . \tag{116}$$

Here we introduced convenient rescaling $\lambda(t) \to m\,\lambda(t)$, so that it is seen from the very outset that the mass enters the problem as a global factor in front of the action and therefore drops out from the answers for $r(t)$ and $\theta(t)$. The solutions for $r(t)$, $\theta(t)$, and $\lambda(t)$ come from the system of three equations

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{r}} \ = \ \frac{\partial L'}{\partial r} \qquad \Rightarrow \qquad \ddot{r} \ = \ r\dot{\theta}^2 - g\cos\theta + \lambda \ , \tag{117}$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot\theta} = \frac{\partial L'}{\partial \theta} \qquad \Rightarrow \qquad \frac{d}{dt}(r^2\dot\theta) = gr\sin\theta \,, \tag{118}$$

$$r(t) \equiv a \,. \tag{119}$$

If we solve these equations and find $r(t)$ and $\theta(t)$, we will naturally get $r(t) = a$, and the result for $\theta(t)$ will be precisely the same as if we fixed $r(t) = a$ in the original Lagrangian L. The question now is why do we use this trick with $\lambda(t)$ if it simply yields identically the same $r(t) = a$ and $\theta(t)$? The key point is that the function $\lambda(t)$ that comes as a "by-product" of the calculation, actually contains an explicit information about the moment $t = t_0$ when the particle leaves the surface. Indeed, at $t = t_0$ the constraint should become irrelevant. That is $\lambda(t > t_0) \equiv 0$. By continuity this implies

$$\lambda(t = t_0) = 0 \,. \tag{120}$$

Hence, the essence of the constraint trick is to find the explicit form of the function $\lambda(t)$ and see when it hits zero.

To finish our calculation, we need to solve the system (117)–(119). With the constraint (119), trivially implying $\dot r \equiv 0$, equation (117) reduces to

$$\lambda = g\cos\theta - a\dot\theta^2 \,. \tag{121}$$

Instead of directly solving Eq. (118) [with $r \equiv a$], we utilize the energy conservation law:

$$\frac{1}{2}a^2\dot\theta^2 + ga\cos\theta = E/m \,. \tag{122}$$

We note in passing that as long as the constraint $r = a$ is satisfied, our system is equivalent to the plane pendulum, and the conservation law (122)—having the form of the first-order differential equation for the function $\theta(t)$—provides us with a simple way of finding the solution $\theta(t)$: Recall our discussion of one-dimensional motion and treat the angle θ as if it is a Cartesian coordinate of a particle in 1D. With such an interpretation, Eq. (122) corresponds to a 1D particle in a periodic external potential.

Using (122) to exclude $\dot\theta^2$ from (121), we find

$$\lambda = 3g\cos\theta - \frac{2E}{ma} \,. \tag{123}$$

For the critical angle θ_0 at which the body leaves the surface we get

$$\theta_0 = \cos^{-1}\left(\frac{2E}{3mga}\right) \,. \tag{124}$$

Pay attention to the similarity. Four different parameters get absorbed into one dimensionless group E/mga. In particular, if the motion starts at $\theta = 0$ with an infinitesimally small velocity, then $E = mga$ and $\theta_0 = \cos^{-1}(2/3)$.

Problem 27. Consider the system of a mass m performing a plane motion within a fixed sphere/cylinder in the gravitational field; see Fig. 4. The motion starts from the lowest point $\theta = 0$ with some horizontal velocity v. For a certain range of values of v, the body will leave the surface at some critical angle $\theta_0(v)$. Find this range of v's, as well as the function $\theta_0(v)$.

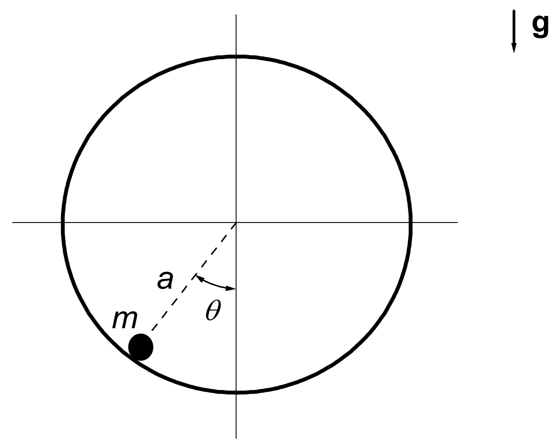

Figure 4: A body moving inside a fixed spherical/cylindrical surface of the radius a.

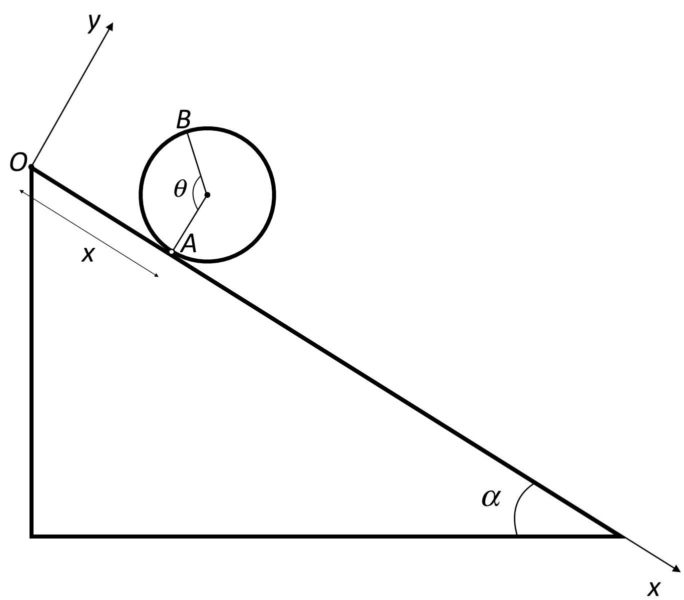

Figure 5: A hoop rolling down an incline plane. Point A is the point of contact between the hoop and the plane. Point B is a fixed reference point on the hoop. The points A and B define the angle θ.

Lagrange multipliers as generalized forces

Lagrangian formalism allows us to describe the motion of a system without dealing with forces. However, sometimes we do care about forces. Think of a realistic plane pendulum with a thin rod and/or very heavy bob. The rod is able to support the circular motion of the bob only if the stress in it is lower than a certain critical value; otherwise it breaks. Along similar lines, think of the system of Fig. 2 with an extra feature: Allow for the static-friction force that can immobilizes mass m_1 converting the system in a plane pendulum with the rod l and the bob m_2. The system will behave as a plane pendulum only up to the point when the ratio of the static-friction force to the normal force is less than the static friction coefficient. In both examples, we care about the forces, while the Lagrangian formalism—if applied directly—hides this information from us.

And here is how Lagrange multiplies come to rescue allowing us to reveal all the forces we care about. It is important that—in a close analogy with the above-discussed example—the information about the forces is extracted by *a posteriori* analysis of the equations of motion and/or conservation laws, without modifying the equations themselves.

The crucial observation is that the λ-term in the Lagrangian (116) has the form of the (generalized) force $\lambda(t)$ acting on the (generalized) coordinate r. [Up to a choice of the sign and units in which we prefer to measure $\lambda(t)$.] This provides us with the desired generic tool. Whenever we are interested in one or more forces applied to this or that part of our system by the constraining object (rod, surface, etc.), we introduce corresponding fictitious degrees of freedom (described by either Cartesian or generalized coordinates), and compensate each extra degree of freedom by corresponding Lagrange multiplier. The actual value of this multiplier, necessary to keep corresponding coordinate constant, then gives us the associated force.[10] For example, in the setup of Fig. 2 with the static friction keeping the position of mass m_1 fixed, we want to know both the horizontal and normal components of the force exerted on mass m_1 by the rod.[11] Hence, in addition to coordinate x_1 of the mass m_1 we also need to introduce the vertical degree of freedom y_1.

In a more general case, when the constraint on generalized coordinates comes in the form

$$G(\{q\}) \,=\, \mathrm{const}\,,$$

and we have

$$L' \,=\, L + \lambda(t) G(\{q\})\,,$$

it is absolutely appropriate and quite useful to interpret corresponding term $\lambda(t)G(\{q\})$ as representing a certain fictitious generalized (negative) time-dependent external potential,

$$\tilde{U}(\{q\},t) \,=\, -\lambda(t) G(\{q\})\,,$$

responsible for generalized forces acting on corresponding generalized coordinates:

$$F_\alpha(\{q\},t) \,=\, -\frac{\partial \tilde{U}(\{q\},t)}{\partial q_\alpha} \,=\, \lambda(t)\frac{\partial G(\{q\})}{\partial q_\alpha}\,.$$

While the potential $\tilde{U}$ is fictitious, the force $F_\alpha(\{q\},t)$ is real, provided $\lambda(t)$ satisfies the desired constraint.

[10]With this interpretation, the condition $\lambda(t) = 0$ means zero force.

[11]Because we care about the critical ratio of the two given by the static friction coefficient.

Hoop rolling down an incline plane. This problem—see Fig. 5—is very illustrative of the usage of Lagrange multipliers for extracting information about the forces. As long as the traction between the hoop and the incline plane is not lost, there is only one degree of freedom. The motion can be naturally described by the coordinate x of the point A at which the hoop touches the incline plane, or by the angle θ between the point A and some reference point B on the hoop. The two coordinates are related to each other by

$$x = R\theta \,, \tag{125}$$

where R is the radius of the hoop.

The kinetic energy of the hoop is (m is the mass of the hoop and I is the moment of inertia with respect to the center of mass)

$$T = \frac{m\dot{x}^2}{2} + \frac{I\dot{\theta}^2}{2} \,, \qquad I = mR^2 \,. \tag{126}$$

Given the constrain (125), both terms in (126) look similar and are actually equal to each other.[12] The expression for the potential energy is naturally written in terms of the coordinate x:

$$U = -mg_x x \,, \qquad g_x = g\sin\alpha \,, \tag{127}$$

rendering this coordinate more convenient than θ. Excluding θ by (125), we get the Lagrangian

$$L = m\dot{x}^2 + mg_x x \tag{128}$$

and a very simple equation of motion with a constant acceleration

$$\ddot{x} = \frac{g\sin\alpha}{2} \,. \tag{129}$$

This result tells us nothing about the traction, force the upper limit of which is controlled by the static friction coefficient κ, which is the maximal possible *ratio* of the traction force to the normal force. Hence, we need to know *both* the normal and traction (friction) forces. To this end, we formally allow x and θ to be independent, and also allow the center of mass to move in the y direction. The two constraints—Eq. (125) and $y = R$ (for y-coordinate of the center of mass—are now implemented by two Lagrange multipliers:

$$L' = \frac{m(\dot{x}^2 + \dot{y}^2)}{2} + \frac{I\dot{\theta}^2}{2} + mg_x x + mg_y y + \lambda_1(x - R\theta) + \lambda_2 y. \tag{130}$$

Here λ_1 is the friction force (because it is coupled to the x coordinate, parallel to the surface), while λ_2 is the normal force (because it is coupled to the y coordinate, perpendicular to the surface). From the equations of motion and the constraints we find:

$$|\lambda_1| = \frac{m|g_x|}{2} \,, \qquad |\lambda_2| = m|g_y| \,. \tag{131}$$

We conclude that the critical condition $|\lambda_1/\lambda_2| = \kappa$ defines the critical value α_* of the angle α beyond which the traction will be lost:

$$\tan\alpha_* = 2\kappa \,. \tag{132}$$

The fact that the critical angle α_* depends only on the coefficient κ means that the result (132) is more general. It applies to the slope of any shape, provided α is understood as the local angle of the slope.

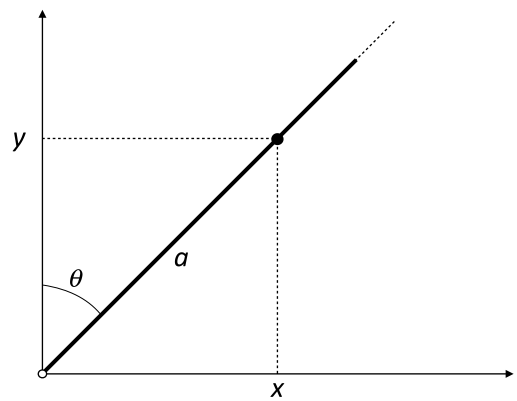

Figure 6: Falling rod problem(s). The open circle at the coordinate origin indicates the position of the (semi-)pivot, while the full circle on the rod indicates the position of the center of mass. The mass distribution along the rod is not necessarily uniform. In particular, all the mass can be concentrated at the center-of-mass point. The kinematic properties of the rod are fully characterized by three parameters: the total mass m, the distance a between the center of mass and the pivot, and the moment of inertia with respect to a certain point. Usually, the moment of inertia is specified/calculated with respect to either the center of mass, $I_{\text{c.m.}}$, or the pivot, I_{pivot}; the two are related to each other by $I_{\text{pivot}} = ma^2 + I_{\text{c.m.}}$.

Falling rod problem(s). This problem—in fact, a class of similar problems; see Fig. 6—is prototypical for the usage of Lagrange formalism in a setup involving a *solid body with one rotational degree of freedom.* In all such cases, it makes perfect sense to start with two very simple relations (cf. the problem of the rolling hoop, Fig. 5):

$$T \;=\; \frac{m\dot{\mathbf{R}}^2}{2} + \frac{I\dot{\theta}^2}{2} \qquad\qquad \text{(kinetic energy)}, \tag{133}$$

$$U \;=\; -m\,\mathbf{g}\cdot\mathbf{R} \qquad\qquad \text{(potential energy)}, \tag{134}$$

where m is the total mass of the rotating body, $\mathbf{R}$ is the position of the center of mass, I is the moment of inertia with respect to the center of mass, $\dot{\theta}$ is the angular velocity of the rotation,[13] and $\mathbf{g}$ is the vector of free fall acceleration.

Already at this very general stage, we can fully non-dimensionalize the problem. To this end we need to introduce some characteristic length. For example, in the problem of rolling hoop (or disk, or ball, or cylinder, or any other circular object) the relevant length would be the radius. In our case, the natural length is a. The natural unit of energy then is mga, the natural unit of moment of inertia is ma^2, the natural unit of time is $\sqrt{a/g}$, etc. The rescaled (non-dimensional) quantities are given by the left-hand sides of the following relations:

$$\mathbf{R} \leftrightarrow \frac{\mathbf{R}}{a}, \qquad t \leftrightarrow t\sqrt{\frac{g}{a}}, \qquad I \leftrightarrow \frac{I}{ma^2}, \qquad U \leftrightarrow \frac{U}{mga}, \qquad T \leftrightarrow \frac{T}{mga}, \quad \text{etc.} \tag{135}$$

The rescaled Eqs. (133)–(134) are

$$T \;=\; \frac{\dot{\mathbf{R}}^2}{2} + \frac{I\dot{\theta}^2}{2} \qquad\qquad \text{(kinetic energy)}, \tag{136}$$

$$U \;=\; -\hat{e}_g\cdot\mathbf{R} \qquad\qquad \text{(potential energy)}. \tag{137}$$

Here $\hat{e}_g = \mathbf{g}/g$ is the unit vector in the direction of $\mathbf{g}$. We see that the kinematics of the problem is fully controlled by just one dimensionless parameter I. The rest is all about the constraints and initial condition.

As long as our semi-pivot behaves as a regular pivot, we have two constraints on $\mathbf{R} \equiv (x, y)$ and θ

$$x = \sin\theta\,, \qquad\qquad y = \cos\theta\,, \tag{138}$$

meaning that there is only one degree of freedom and suggesting to use θ as the generalized coordinate. This brings us to the most simple plane pendulum Lagrangian:

$$L \;=\; \frac{I+1}{2}\,\dot{\theta}^2 \;-\; \cos\theta\,. \tag{139}$$

Now we see that we can further improve the choice of our units by absorbing the factor $(I+1)$ into the unit of time:

$$t \leftrightarrow t\sqrt{g/a(I+1)}\,. \tag{140}$$

[12]The exact equality is a special property of the hoop. This fact, however, does not play any qualitative role. For example, in the case of a disk, the two terms are also similar—differ only by a numeric coefficient.

[13]It is always good to remember that the angular velocity is independent of the choice of the origin for measuring the angle θ, and, more generally, the choice of the reference frame, as long as the frame is *inertial.* This freedom buys us some extra convenience/flexibility: Compare the choices of θ in Figs. 5 and 6. Beware of a typical mistake of using relation Eq. (133) with a "wrong" θ that does not correspond to an inertial frame (for example, in the case of a hoop rolling over a *curved* surface)!

This choice becomes really important at $I \gg 1$, where it reveals the $\propto \sqrt{I}$ scaling of the characteristic time.[14] The Lagrangian is now free of any parameters

$$L = \frac{1}{2}\dot{\theta}^2 - \cos\theta \qquad \left[t \leftrightarrow t\sqrt{g/a(I+1)}\right]. \tag{141}$$

It yields the one-dimensional equation of motion (fully non-dimensionalized equation of motion of a plane pendulum):

$$\ddot{\theta} = \sin\theta \qquad \left[t \leftrightarrow t\sqrt{g/a(I+1)}\right], \tag{142}$$

and its first integral in the form of the (fully non-dimensionalized) energy conservation law:

$$\frac{\dot{\theta}^2}{2} + \cos\theta = E \qquad \left[t \leftrightarrow t\sqrt{g/a(I+1)}, \qquad E \leftrightarrow \frac{E}{mga}\right]. \tag{143}$$

The problem becomes less trivial with a *conditional pivot* (a semi-pivot). Here the lower tip of the rod behaves as a pivot only if a certain condition is met. For example, if the tip is kept in place by the static friction of the floor, then the ratio of the tangential and normal forces acting on the tip should be less than the static friction coefficient. Another characteristic case is when the tip is kept in place by the vertical wall perpendicular to the x-direction. Here we have a condition on the sign of the x-component of the force acting on the tip. The sign cannot be negative.

Let us solve a general problem of finding x- and y-components of the force acting on the tip. Our general solution then can be used to address any specific condition under which the semi-pivot works as a pivot. We start with Eq. (135) that provides us with the natural choice of dimensionless variables.[15] Now we write our Lagrangian. In addition to the kinetic and potential terms, Eqs. (136)–(137), it has two Lagrange multipliers:

$$L' = \frac{\dot{\mathbf{R}}^2}{2} + \frac{I\dot{\theta}^2}{2} + \hat{e}_g \cdot \mathbf{R} + \lambda_1 x_0 + \lambda_2 y_0 , \tag{144}$$

where (x_0, y_0) are the Cartesian coordinates of the lower tip of the rod. By construction, λ_1 and λ_2 are, respectively, the x- and y-components of the force[16] acting on the tip. This force is supposed to keep the tip at the origin:

$$x_0 = 0 , \qquad y_0 = 0 . \tag{145}$$

The Lagrangian (144) contains 5 variables: (x, y), (x_0, y_0), and θ, but only 3 degrees of freedom. Indeed, there are two purely geometric conditions:

$$x = x_0 + \sin\theta , \qquad y = y_0 + \cos\theta . \tag{146}$$

Important observation 1. The fact that we are solving for the force exerted on the point (x_0, y_0) by no means implies that we have to use (x_0, y_0) as generalized coordinates!

[14] Clearly, the case $I \gg 1$ corresponds to a certain "exotic" distribution of the mass along the rod. Can you give an example of such a distribution?

[15] Using (140) from the very outset is also possible, but not recommended: It renders the formulas less intuitive, because the characteristic time $\sqrt{a(I+1)/g}$ emerges only on top of the constrained motion, which is *implicit* in our treatment.

[16] With our dimensionless variables, the forces are measured in the (very natural) units of mg.

We will use (x, y) and θ as our coordinates, excluding (x_0, y_0) by the geometric constraint (146). The expression for our Lagrangian becomes

$$L' = \frac{\dot{\mathbf{R}}^2}{2} + \frac{I\dot{\theta}^2}{2} + \hat{e}_g \cdot \mathbf{R} + \lambda_1(x - \sin\theta) + \lambda_2(y - \cos\theta)\,. \tag{147}$$

In components (x, y), we have

$$L' = \frac{\dot{x}^2 + \dot{y}^2}{2} + \frac{I\dot{\theta}^2}{2} - y + \lambda_1(x - \sin\theta) + \lambda_2(y - \cos\theta)\,. \tag{148}$$

The three equations of motion are (we specify the units of time because later we will change them)

$$\ddot{x} = \lambda_1 \qquad \left[t \leftrightarrow t\sqrt{g/a}\right]\,, \tag{149}$$

$$\ddot{y} = \lambda_2 - 1 \qquad \left[t \leftrightarrow t\sqrt{g/a}\right]\,, \tag{150}$$

$$I\ddot{\theta} = -\lambda_1 \cos\theta + \lambda_2 \sin\theta \qquad \left[t \leftrightarrow t\sqrt{g/a}\right]\,. \tag{151}$$

Now we need to use these equations in combination with the constraints[17] (138) to conveniently express λ_1 and λ_2 in terms of θ and E. By the statement of the problem, the system of Eqs. (149)–(151) and (146) naturally implies Eqs. (142)–(143), meaning that Eqs. (142)–(143) can also be used whenever it is convenient.

Important observation 2. The above comment actually means that deriving (151) was almost a waste of our time.[18]

Indeed, Eqs. (149) and (150) directly relate λ_1 and λ_2 to $\ddot{x}$ and $\ddot{y}$, respectively. In their turn, $\ddot{x}$ and $\ddot{y}$ are readily calculated in terms of θ and E by taking into account the constraints (138) and the equation of motion (142) with its first integral (143).

Important observation 3. Since now we are dealing with the explicitly constrained motion, it is good to switch to the proper scaling of time, Eq. (140).

We have[19]

$$\lambda_1 = \frac{\ddot{x}}{I+1} \qquad \left[t \leftrightarrow t\sqrt{g/a(I+1)}\right]\,, \tag{152}$$

$$\lambda_2 = 1 + \frac{\ddot{y}}{I+1} \qquad \left[t \leftrightarrow t\sqrt{g/a(I+1)}\right]\,. \tag{153}$$

Now we use (138), (142), and (143) to express $\ddot{x}$ and $\ddot{y}$ in terms of θ and E. This brings us to the final answers:

$$\lambda_1 = \frac{3\cos\theta - 2E}{I+1}\sin\theta\,, \tag{154}$$

$$\lambda_2 = \frac{I}{I+1} + \frac{3\cos\theta - 2E}{I+1}\cos\theta\,. \tag{155}$$

[17]Constraints (145) and (138) are equivalent in view of the geometric conditions (146).
[18]See, however, important general remark below.
[19]The proper scaling becomes revealing already at this stage: We see that at $I \gg 1$, λ_1 gets suppressed while $\lambda_2 \to 1$.

Further analysis depends on the type of the semi-pivot. Consider, for example, the hard-wall case (no static friction). Here the critical condition $\lambda_1 = 0$ takes place at the critical angle

$$\cos\theta_0 \;=\; (2/3)E \qquad\qquad \text{(hard-wall semi-pivot).} \tag{156}$$

This expression for θ_0 is formally the same as in the sliding body problem of Fig. 3; see Eq. (124). However, what happens at $\theta = \theta_0$ is very different from that problem. At any $I > 0$, the rod detaches from the wall, but not from the floor, because λ_2 remains finite. (In the limit $I \to 0$—that is, when all the mass is at the center-of-mass point, the two problems become equivalent; for an obvious reason.)

At $E > 3/2$, equation (156) has no solutions, meaning that at such energies the hard wall cannot support the pivot-like behavior of the tip of the rod. (The rod bounces off the wall the moment it starts its motion.)

Important general remark. Looking attentively at Eqs. (149)–(150) we realize that they express the well-known general theorem of Newtonian mechanics stating that the acceleration of the center of mass of any (sub)system equals to the net force acting on the (sub)system over the total mass of the (sub)system. Similarly, Eq. (151) expresses yet another general theorem stating that the angular acceleration of a solid body—with respect to *any* selected axis—equals to the net torque of the forces with respect to this axis over the moment of inertia of the body with respect to the same axis. Especially simple is the case—nicely illustrated by Eq. (151)—when the selected axis contains the center of mass. Here the net torque of gravitational forces vanishes.

We can now observe that all the problems we solved so far with Lagrange multipliers could be alternatively solved by resorting to the two above-mentioned general theorems, whichever of the two is more convenient for treating this or that case. On the other hand, the machinery of Lagrange multipliers is simple and universal, and as such is strongly recommended for employing in all the cases when the answer is not immediately clear. Speaking practically, it is a good idea to use *both* approaches for cross-validation.