Unit 9 – ONE Sample Inference
Solutions

1. **This exercise gives you practice calculating a confidence interval for the mean of a Normal distribution in the setting where the variance parameter is known.**

Suppose we assume that the results of IQ tests are distributed normal. Suppose that in 2016, the distribution of IQ test scores for persons aged 18-35 years has a variance $\sigma^2 = 225$. A simple random sample of 9 persons take the IQ test. The sample mean score is 115. Calculate the 50%, 75%, 90% and 95% confidence interval estimates of the unknown population mean IQ score.

Answer:

50% CI	**(111.6 , 118.4)**
75% CI	**(109.2, , 120.8)**
90% CI	**(106.8 , 123.2)**
95% CI	**(105.2 , 124.8)**

Solution:
Let the random variable X = IQ test result assumed normal with:

μ unknown
$\sigma^2 = 225$, known
$\sigma = 15$, known

A confidence interval estimate of the unknown mean is given by:

Point estimate $\pm$ { confidence coefficient} { se of point estimate}

Solution for Point Estimate (one time) = **115**
Point estimate = observed sample mean = $\bar{X}$ =115

Solution for SE [Point Estimate] (one time) = **5**
se of estimate = standard error of sample mean = $SE(\bar{X}) = \sqrt{\sigma^2/n}$

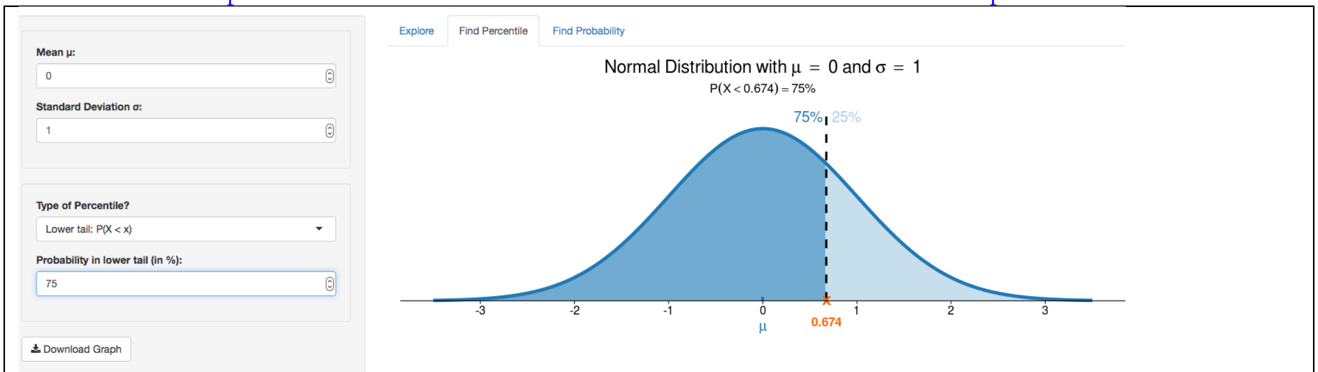
$= \sqrt{225/9}$
$= 15 / 3$
$= 5$

Solution for Confidence Coefficient Values

Desired Confidence $(1 - \alpha)100\%$	α	$\alpha/2$	Desired Percentile $(1 - \alpha/2)100\%$	Confidence Coefficient
50%	.50	.25	75^{th}	**.674**
75%	.25	.125	87.5^{th}	**1.15**
90%	.10	.05	95^{th}	**1.645**
95%	.05	.025	97.5^{th}	**1.96**

Solution for Confidence Coefficient Values (Percentiles) Using Art of Stat Online Calculator for the Standard Normal(0,1)

Note- full screen capture is shown for the 50% confidence interval and associated 75th percentile ONLY.



Normal Distribution with $\mu = 0$ and $\sigma = 1$

$P(X < 1.15) = 87.5\%$

Normal Distribution with $\mu = 0$ and $\sigma = 1$

$P(X < 1.645) = 95\%$

Normal Distribution with $\mu = 0$ and $\sigma = 1$

$P(X < 1.96) = 97.5\%$

Using R

```
> paste("75th Percentile of Standard Normal(0,1) = ",qnorm(.75))
[1] "75th Percentile of Standard Normal(0,1) =  0.674489750196082"

> paste("87.5th Percentile of Standard Normal(0,1) = ",qnorm(.875))
[1] "87.5th Percentile of Standard Normal(0,1) =  1.15034938037601"

> paste("95th Percentile of Standard Normal(0,1) = ",qnorm(.95))
[1] "95th Percentile of Standard Normal(0,1) =  1.64485362695147"

> paste("97.5th Percentile of Standard Normal(0,1) = ",qnorm(.975))
[1] "97.5th Percentile of Standard Normal(0,1) =  1.95996398454005"
```

Final Solution for Confidence Interval Lower and Upper Limit Values

Solutions

Desired Confidence $(1 - \alpha)100\%$	Point Estimate	SE of Point Estimate	$\alpha/2$	Confidence Coefficient	Lower Limit Estimate – SE*conf coeff	Upper Limit Estimate + SE*conf coeff
50%	115	5	.25	.674	111.6	118.4
75%	115	5	.125	1.15	109.2	120.8
90%	115	5	.05	1.645	106.8	123.2
95%	115	5	.025	1.96	105.2	124.8

2. **This exercise is asking you to think about, and compare, two aspects of the concept of a confidence interval: (1) its width, and (2) the level of confidence that we attach to the interval we are reporting.** ***Hint*** **– precision versus confidence…**

What trade-offs are involved in reporting one interval estimate over another?

Answer:
For a given probability distribution with a known variance and a fixed sample size,

(i) Increasing the confidence coefficient is at the price of a wider confidence interval.

(ii) Decreasing the width of a confidence interval estimate is at the price of a lower confidence coefficient.

This is apparent in the following summary of the solution to Exercise #1.

Desired Confidence	Lower Limit	Upper Limit	Width = [Upper limit – Lower limit]
.50	111.6	118.4	6.8
.75	109.2	120.8	11.6
.90	106.8	123.2	16.4
.95	105.2	124.8	19.6

3. **This exercise gives you practice calculating a confidence interval for the variance of a Normal distribution in.**

The objectives of a study by Kennedy and Bhambhani (1991) were to use physiological measurements to determine the test-retest reliability of the Baltimore Therapeutic Equipment Work Simulator during three simulated tasks performed at light, medium, and heavy work intensities, and to examine the criterion validity of these tasks by comparing them to real tasks performed in a controlled laboratory setting. Subjects were 30 healthy men between the ages of 18 and 35. The investigators reported a standard deviation of s=0.57 for the variable peak oxygen consumption (1/min) during one of the procedures. Assuming normality, compute a 95% confidence interval for the population variance for the oxygen consumption variable.

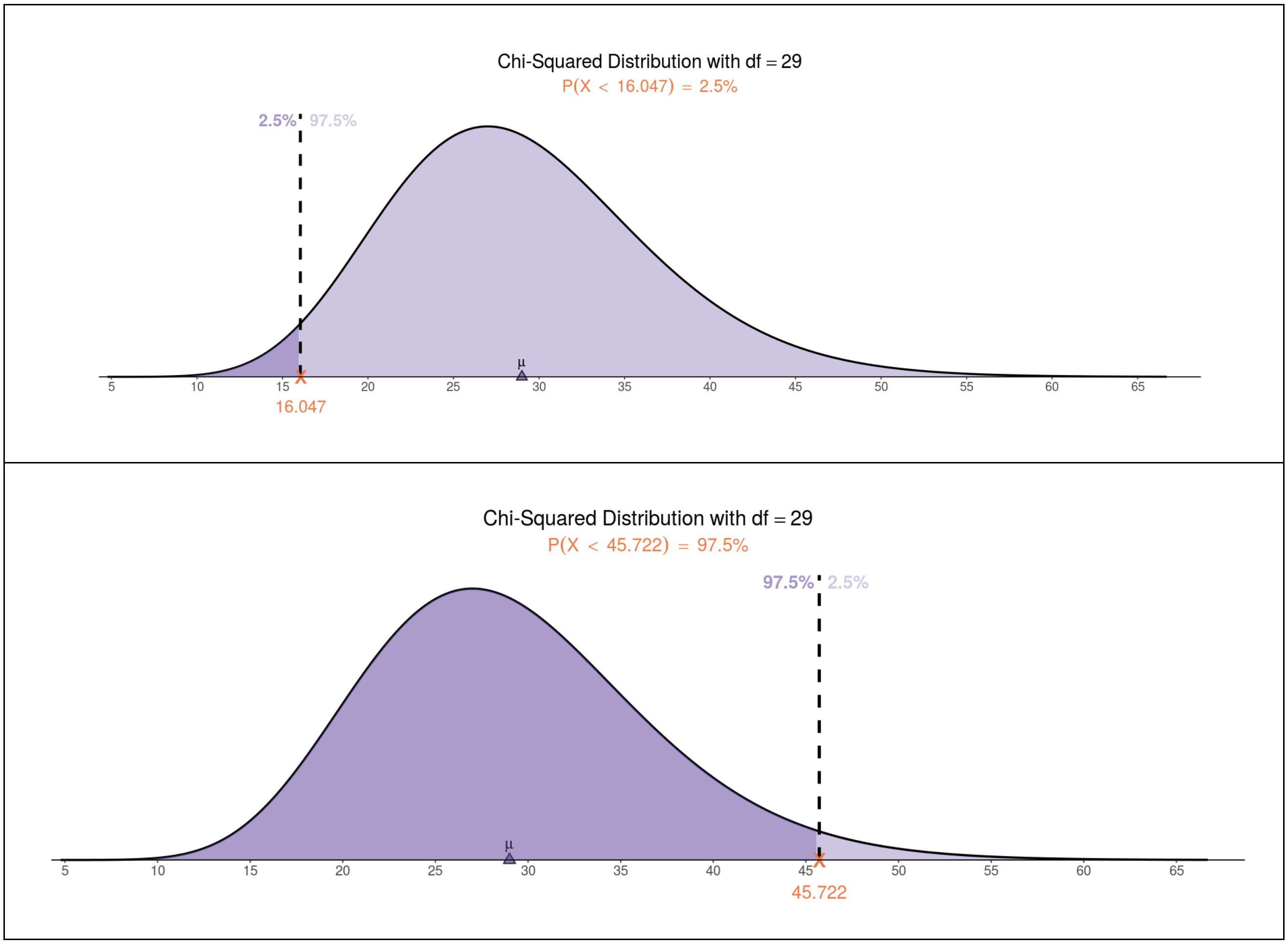
Answer: (.21, .59)
Solution:

(n-1) = 29 $S^2 = 0.57^2$ $\chi^2_{1-\alpha/2} = \chi^2_{.975;\ DF=29} = 45.722$ $\chi^2_{\alpha/2} = \chi^2_{.025;\ DF=29} = 16.047$

$$\text{Lower limit} = \frac{(n-1)S^2}{\chi^2_{1-\alpha/2;df=(n-1)}} = \frac{(29)(0.57^2)}{45.722} = .2061$$

$$\text{Upper limit} = \frac{(n-1)S^2}{\chi^2_{\alpha/2;df=(n-1)}} = \frac{(29)(0.57^2)}{16.047} = .5872$$

Solution for Confidence Coefficient Values (percentiles) Using Art of Stat Calculator for the Chi Square (df = 29)



Using R

```
> paste("2.5th Percentile of Chi Square (df=29) = ",qchisq(.025,df=29))
[1] "2.5th Percentile of Chi Square (df=29) =  16.0470716953649"

> paste("97.5th Percentile of Chi Square (df=29) = ",qchisq(.975,df=29))
[1] "97.5th Percentile of Chi Square (df=29) =  45.7222858041745"
```

4. This exercise gives you practice calculating a confidence interval for the mean difference of paired data that is distributed Normal.

The purpose of an investigation by Alahuhta et al (1991) was to evaluate the influence of extradural block for elective caesarian section simultaneously on several maternal and fetal hemodynamic variables and to determine if the block modified fetal myocardial function. The study subjects were eight healthy parturient in gestational weeks 38-42 with uncomplicated singleton pregnancies undergoing elective caesarian section under extradural anesthesia. Among the measurements taken were maternal diastolic arterial pressure during two stages of the study. The following are the lowest values of this variable at the two stages. Compute a 95% confidence interval

for the difference in diastolic blood pressure between the two stages.

Patient ID	1	2	3	4	5	6	7	8
Stage 1	70	87	72	70	73	66	63	57
Stage 2	79	87	73	77	80	64	64	60

Source: Alahuhta S., Rasanen J, Jouppila R, Jouppila P., and Kangas-Saarela T., and Hoomen AI (1991)"Uteroplacental and fetal hemoydynamics during extradural anesthesia for caesarian section", British Journal of Anesthesia, 66: 319-323. Cited in Daniel (p 248, 7.4.3) Copyright 1999 by John Wiley & Sons, Inc. By permission of John Wiley.

Answer: (-0.06, +6.6)

Solution:

Because two measurements are made on each patient, at stages 1 and 2, these data fit the definition of "paired". The analysis focuses on the differences, per the table below:

Patient ID, i	1	2	3	4	5	6	7	8
d_i = Stage 2-1	**9**	**0**	**1**	**7**	**7**	**-2**	**1**	**3**

For this exercise, we have

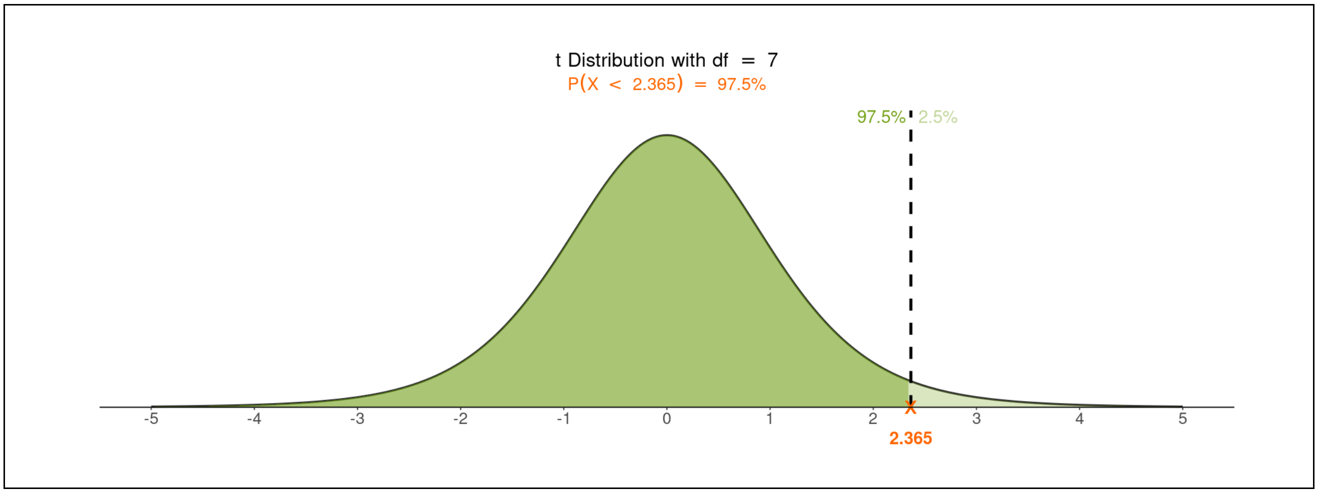
$$\bar{d}=3.25 \qquad S_d^2 = 15.643 \qquad S_d = 3.9551 \qquad SE(\bar{d})=\frac{S_d}{\sqrt{n}} = \frac{3.9551}{\sqrt{8}} = 1.3983$$

$$df=(n-1)=7 \qquad t_{1-\alpha/2;df} = t_{.975;7} = 2.365$$

$$95\% \text{ CI for } \mu_d = \bar{d} \pm (t_{.975;DF=7})\, SE(\bar{d})$$

$$= (3.25) \pm (2.365)(1.3983) = (-0.057, 6.557)$$

Solution for Confidence Coefficient (Percentile) Value Using Art of Stat Calculator for the Student-t (df = 7)



Using R

```
> paste("97.5th Percentile of Student-t (df=7) = ",qt(.975,df=7))
[1] "97.5th Percentile of Student-t (df=7) =  2.36462425159278"
```

5. This exercise gives you practice performing a statistical hypothesis test for paired data that is distributed Normal.

Halcion is a sleeping pill that is relatively rapidly metabolized by the body and therefore having fewer hangover effects the next morning, compared to other sleeping pills. Opponents of Halcion argue that, because this agent is so rapidly metabolized by the body, patients do not sleep as long with this drug as with Dalmane. Data on 10 insomniacs, each of whom took Dalmane on one occasion and Halcion on a second, is collected. The variable measured is number of hours of sleep:

	Number of Hours Sleep with	
Patient	**Dalmane**	**Halcion**
1	4.58	3.97
2	5.19	4.88
3	3.94	4.09
4	6.32	5.87
5	7.68	6.93
6	3.48	4.00
7	5.72	5.08
8	7.04	6.95
9	5.27	4.96
10	5.84	5.13

Do these data suggest that Halcion is not as effective as Dalmane with respect to number of hours of sleep? Carry out an appropriate statistical test and interpret your findings. You may assume that the measurements of sleep are continuous, distributed normal.

ANSWER

These data are paired measurements of an outcome measured on a continuum in a single sample. It is of interest to compare the responses of the paired measurements. The correct test is therefore a paired t-test.

For these data, a paired t-test suggests that the average difference in hours slept (Dalmane – Halcion) = 0.32 is statistically significant (one sided p-value = .018).

SOLUTION

This question is asking for a hypothesis test of the equality of two means in the setting of paired data. The data are paired because each participant was measured on two occasions, once on Dalmane and once on Halcion.

Research Question. Are sleep durations shorter on Dalmane than on Halcion?

Assumptions.

$\bar{d}$ is distributed Normal (μ_d , $\sigma_d^2/10$)

Differences are calculated as d = (Dalmane – Halcion)

Obs	dalmane	halcion	d
1	4.58	3.97	0.61
2	5.19	4.88	0.31
3	3.94	4.09	-0.15
4	6.32	5.87	0.45
5	7.68	6.93	0.75
6	3.48	4.00	-0.52
7	5.72	5.08	0.64
8	7.04	6.95	0.09
9	5.27	4.96	0.31
10	5.84	5.13	0.71

H_O and H_A.

$H_O : \mu_d = 0$

$H_A : \mu_d > 0$ ("Dalmane is better than Halcion") – one sided

Test statistic is a t-score.

$$t_{score} = \left[\frac{(\bar{d}) - E[\bar{d})|H_O \text{true}]}{\hat{SE}[(\bar{d})|H_O \text{true}]} \right]$$

Obtain sample mean of the differences, $\bar{d}$

$$\bar{d} = \frac{\sum_{i=1}^{10} d_i}{10} = \left[\frac{0.61 + 0.31 + ... + 0.71}{10} \right] = 0.32$$

Preliminary – Obtain sample variance of the differences, S_d^2

$$S_d^2 = \frac{\sum_{i=1}^{10} \left(d_i - \bar{d}\right)^2}{(n\text{-}1)} = \frac{\sum_{i=1}^{10} \left(d_i - 0.32\right)^2}{9} = \frac{(0.61 - 0.32)^2 + ... + (0.71 - 0.32)^2}{9} = 0.1688889$$

Obtain the estimated standard error, $\hat{SE}\,[\,\bar{d}\,|\,H_O \text{ true}]$

$$\hat{SE}\,[\,\bar{d}\,|\,H_O \text{ true}] = \sqrt{\frac{S_d^2}{n}} = \sqrt{\frac{0.1688889}{10}} = 0.1299573$$

Putting these all together, the solution for the test statistic is

$$t_{score} = \left[\frac{(\bar{d}) - E[\bar{d})|H_O \text{true}]}{\hat{SE}[(\bar{d})|H_O \text{true}]} \right] = \left[\frac{0.32\ -\ 0}{0.1299573} \right] = 2.4623$$

Degrees of freedom = (n-1) = (10-1) = 9.

"Evaluation" rule.

The likelihood of these findings or ones more extreme if H_O is true is

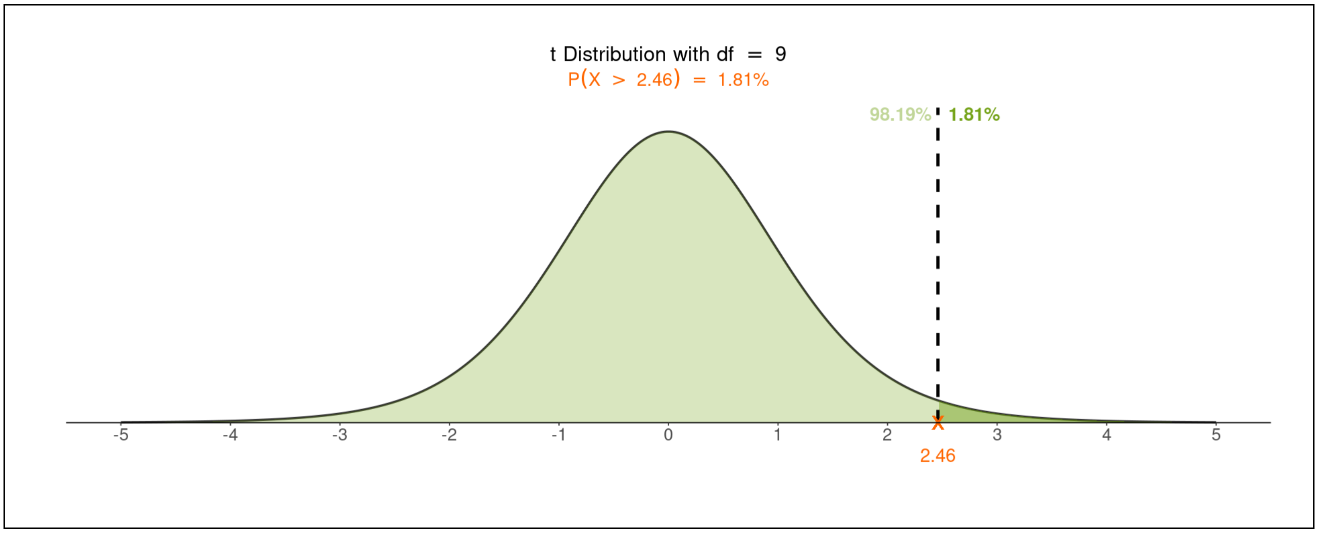
$$\text{p-value} = \Pr\left[\left(\bar{d}\right) \geq 0.32 \mid H_o \text{ true}\right].$$

Calculations.

$$\text{p-value} = \Pr\left[t_{score} \geq 2.46\right] \text{ where degrees of freedom } = 9$$

$$=.018$$

Solution for p-value (probability) Using Online Calculator for the Student-t Distribution (df = 9)
https://istats.shinyapps.io/tdist/



Using R

```
> paste("Pr[t-score (df=9) > 2.46] = ",pt(2.46,df=9,lower.tail=FALSE))
[1] "Pr[t-score (df=9) > 2.46] =  0.0180793234596174"
```

"Evaluate".

The assumption of the null hypothesis H_O (duration of sleep is the same with both drugs) has led to an unlikely result. Under the null hypothesis the chance that the difference in average hours slept is as great or greater than 0.32 hours is about 2 in 100. "2 chances in 100" is "unlikely" enough that I warrants rejecting the null hypothesis. Put another way, in this sample, the average difference of 0.32 hours is statistically significant.

6. For the Halcion versus Dalmane data in Exercise 5, construct a 99% confidence interval estimate of discrepancy in the efficacies of the two drugs. Compare this to the acceptance region that would have been obtained had you constructed a statistical test with type I error pre-specified at 0.01.

ANSWER The 99% confidence interval is (-0.10, 0.74).
The acceptance region is $\bar{d} < 0.3666$.

SOLUTION

Solution for the 99% CI is as follows

$$\bar{d} = 0.32 \qquad \hat{SE}(\bar{d}) = \frac{S_d}{\sqrt{n}} = 0.1299573$$

$df=(n-1)=9$ $\qquad t_{1-\alpha/2;df} = t_{.995;9} = 3.25$ from the calculator on the web above.

$$99\% \text{ CI for } \mu_d = \bar{d} \pm (t_{.995;DF=9})\ \hat{SE}(\bar{d})$$

$$= (0.32) \pm (3.25)(0.1299573) = (-0.1024, +0.7424)$$

Solution for the acceptance region of a one sided test with alpha = .01 is obtained by reasoning as follows

Rejection occurs for t-score $\geq$ the 99th percentile of a student's t on df=9 →

Rejection occurs for t-score $\geq$ $t_{.99;\, df=9}$ →

Acceptance occurs for t-score $<$ $t_{.99;\, df=9}$ →

Substituting in the definition of a t-score allows us to write this equivalently as

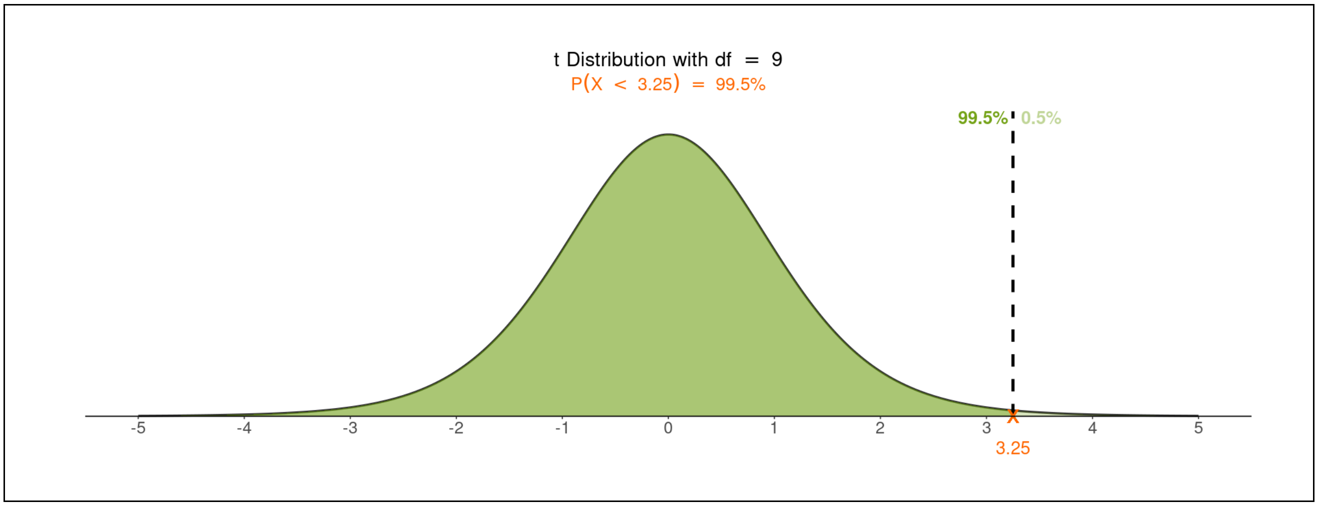
Acceptance occurs for $\dfrac{\bar{d}-0}{\hat{SE}(\bar{d})} < t_{.99;df=9}$ By plugging in numbers for the SE and the percentile →

Acceptance occurs for $\dfrac{\bar{d}}{0.1299573} < 2.821$ where I used the calculator on the web as before →

Acceptance occurs for $\bar{d} < 0.3666$

Comparison
These two regions overlap but are not identical. They are not identical because the confidence interval is two sided whereas the acceptance region is one sided.

Solution for Confidence Coefficient (Percentile) Value Using Online Calculator for the Student-t (df = 9)



Using R

```
> paste("99.5th Percentile of Student-t (df=9) = ",qt(.995,df=9))
[1] "99.5th Percentile of Student-t (df=9) =  3.24983554159213"
```

7. **This exercise is a straightforward confidence interval calculation for a binomial proportion.**

An entomologist samples a field for egg masses of a harmful insect by placing a yard-square frame at random locations and carefully examining the ground within the frame. A simple random sample of 75 locations selected from a county's pasture land found egg masses in 13 locations. Compute a 95 confidence interval estimate of all possible locations that are infested.

Answer: (.0876, .2590)

Solution:

The setting is estimation of a binomial proportion π. In this exercise, the number of trials is N=75. Since this is sufficiently large, we can obtain a confidence interval using $\hat{\pi} \pm \left(z_{1-\alpha/2}\right)\hat{SE}\left(\hat{\pi}\right)$ using the standard error formula "(3)" that appears on page 60. Thus, the calculations are

$$\bar{X} = \frac{X}{N} = \frac{13}{75} = .1733$$

$$\hat{\pi} = \bar{X} = .1733$$

$$\hat{SE} = \sqrt{\frac{\bar{X}\left(1-\bar{X}\right)}{N}} = \sqrt{\frac{(.1733)(.8267)}{75}} = .0437$$

$$z_{1-\alpha/2} = z_{.975} = 1.96$$

$$\hat{\pi} \pm \left(z_{1-\alpha/2}\right)\hat{SE}\left(\hat{\pi}\right) = .1733 \pm (1.96)(.0437) = \underline{\textbf{(.0876, .2590)}}$$

8. ***Optional (for the brave)*** **This exercise is NOT a mimicking of the lecture notes. It is asking you to start your thinking from the width of a confidence interval and then reason your solution from there.**

Alzheimers' disease has a poorer prognosis when it is diagnosed at a relatively young age. Suppose we want to estimate the age at which the disease was first diagnosed using a 90% confidence interval. Under the assumption that the distribution of age at diagnosis is normal, if the population variance is $\sigma^2=85$, how large a sample size is required **if we want a confidence interval that is 10 years wide**?

Hint. **Confidence Interval Width = [Upper Limit] - [Lower Limit]**

Answer: n=10

Solution:

Recall: The 90% confidence interval is given by $\bar{X} \pm (z_{.95})SE(\bar{X})$, →

Confidence interval width = [upper limit of CI] - [lower limit of CI]

$$= \left[\bar{X}+(z_{.95})SE(\bar{X})\right]-\left[\bar{X}-(z_{.95})SE(\bar{X})\right]$$
$$= \bar{X}+(z_{.95})SE(\bar{X}) - \bar{X} + (z_{.95})SE(\bar{X})$$
$$= (2)(z_{.95})SE(\bar{X})$$
$$= (2)(z_{.95})\frac{\sigma}{\sqrt{n}}$$

Setting confidence interval width on the left hand side to width = 10 allows us to write

$$10 = (2)(z_{.95})\frac{\sigma}{\sqrt{n}} \rightarrow$$

$$5 = (z_{.95})\frac{\sigma}{\sqrt{n}} \rightarrow$$

$$\sqrt{n} = \left[\frac{(z_{.95})\sigma}{5}\right] \rightarrow$$

$$n = \left[\frac{(z_{.95})\sigma}{5}\right]^2 \rightarrow$$

$$n = \left[\frac{(1.645)^2(85)}{25}\right] = 9.2005$$

Rounding up yields the required sample size of 10.