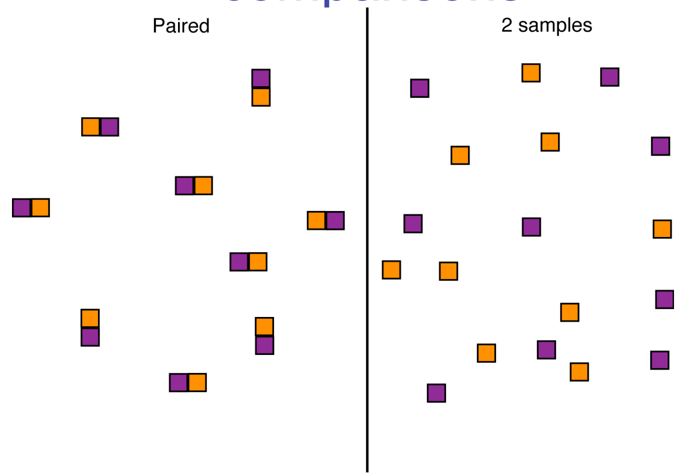
# Comparing means

- Tests with one categorical and one numerical variable
- Goal: to compare the mean of a numerical variable for different groups.

# Paired vs. 2 sample comparisons




# Paired comparisons allow us to account for a lot of extraneous variation

# 2-sample methods are sometimes easier to collect data

# Paired designs

- Data from the two groups are paired
- Each member of the pair shares much in common with the other, *except* for the tested categorical variable
- There is a one-to-one correspondence between the individuals in the two groups

## Paired design: Examples

- Before and after treatment
- Upstream and downstream of a power plant
- Identical twins: one with a treatment and one without
- Earwigs in each ear: how to get them out? Compare tweezers to hot oil

## Paired comparisons

- We have many pairs
- In each pair, there is one member that has one treatment and another who has another treatment

(“Treatment” can mean “group”)

## Paired comparisons

- To compare two groups, we use the mean of the *difference* between the two members of each pair

## Paired *t* test

- Compares the mean of the differences to a value given in the null hypothesis
- For each pair, calculate the difference. The paired *t*-test is simply a one-sample *t*-test on the differences.

# Example: National No Smoking Day

- Data compares injuries at work on National No Smoking Day (in Britain) to the same day the week before
- Each data point is a year

Data from Waters et al. (1998) Nicotine withdrawal and accident rates. *Nature* 394: 137.

# data

| Year | Injuries before No Smoking Day | Injuries on No Smoking Day |
|---|---|---|
| 1987 | 516 | 540 |
| 1988 | 610 | 620 |
| 1989 | 581 | 599 |
| 1990 | 586 | 639 |
| 1991 | 554 | 607 |
| 1992 | 632 | 603 |
| 1993 | 479 | 519 |
| 1994 | 583 | 560 |
| 1995 | 445 | 515 |
| 1996 | 522 | 556 |

# Hypotheses

$H_0$: Work related injuries do not change during No Smoking Days. ($\mu_d = 0$)

$H_A$: Work related injuries change during No Smoking Days. ($\mu_d \neq 0$)

# Calculate differences

| Injuries before No Smoking Day | Injuries on No Smoking Day | Difference ($d$) |
|---|---|---|
| 516 | 540 | 24 |
| 610 | 620 | 10 |
| 581 | 599 | 18 |
| 586 | 639 | 53 |
| 554 | 607 | 53 |
| 632 | 603 | -29 |
| 479 | 519 | 40 |
| 583 | 560 | -23 |
| 445 | 515 | 70 |
| 522 | 556 | 34 |

## Calculate $t$ using $d$'s

$$\bar{d} = 25$$
$$s_d^2 = 1043.78$$
$$n = 10$$

$$t = \frac{25 - 0}{\sqrt{1043.78/10}} = 2.45$$

## CAUTION!

- The number of data points in a paired *t* test is the number of *pairs*. -- *Not* the number of individuals
- Degrees of freedom = Number of pairs - 1

## Critical value of $t$

$$t_{0.05(2),9} = 2.26$$
$$t = 2.45 > 2.26$$

So we can reject the null hypothesis. Stopping smoking increases job-related accidents in the short term.

## Assumptions of paired $t$ test

- Pairs are chosen at random
- The differences have a normal distribution

It does *not* assume that the individual values are normally distributed, only the differences.

## Comparing the means of two groups

Hypothesis test: 2-sample *t* test

## Estimation: Difference between two means

$$\bar{Y}_1 - \bar{Y}_2$$

Confidence interval: $\left(\bar{Y}_1 - \bar{Y}_2\right) \pm SE_{\bar{Y}_1 - \bar{Y}_2}\, t_{\alpha(2),df}$

## Standard error of difference in means

$$SE_{\bar{Y}_1 - \bar{Y}_2} = \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}$$

Pooled variance: $s_p^2 = \dfrac{df_1 s_1^2 + df_2 s_2^2}{df_1 + df_2}$

$df_1 = n_1 - 1$; $df_2 = n_2 - 1$

## Costs of resistance to aphids

2 genotypes of lettuce: *Susceptible* and *Resistant*

Do these genotypes differ in fitness in the absence of aphids?

## Data, summarized

| | Susceptible | Resistant |
|---|---|---|
| Mean number of buds | 720 | 582 |
| SD of number of buds | 223.6 | 277.3 |
| Sample size | 15 | 16 |

Both distributions are approximately normal.

## Calculating the standard error

$df_1$ =15 -1=14; $df_2$ = 16-1=15

$$s_p^2 = \frac{df_1 s_1^2 + df_2 s_2^2}{df_1 + df_2} = \frac{14(223.6)^2 + 15(277.3)^2}{14+15} = 63909.9$$

$$SE_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}} = \sqrt{\frac{63909.9}{15} + \frac{63909.9}{16}} = 90.86$$

## Finding *t*

$df = df_1 + df_2 = n_1 + n_2 - 2$

$= 15+16-2$
$=29$

$$t_{0.05(2),29} = 2.05$$

## The 95% confidence interval of the difference in the means

$$\left(\bar{Y}_1 - \bar{Y}_2\right) \pm SE_{\bar{Y}_1 - \bar{Y}_2}\, t_{\alpha(2),df} = (720 - 582) \pm 90.86(2.05)$$

$$= 138 \pm 186$$

# Testing hypotheses about the difference in two means

2-sample *t*-test

The *two sample* t-*test* compares the means of a numerical variable between two populations.

# 2-sample t-test

$$t = \frac{\bar{Y}_1 - \bar{Y}_2}{SE_{\bar{Y}_1 - \bar{Y}_2}}$$

# Hypotheses

$H_0$: There is no difference between the number of buds in the susceptible and resistant plants. ($\mu_1 = \mu_2$)

$H_A$: The resistant and the susceptible plants differ in their mean number of buds. ($\mu_1 \neq \mu_2$)

# Calculating *t*

$$t = \frac{(\bar{Y}_1 - \bar{Y}_2)}{SE_{\bar{Y}_1 - \bar{Y}_2}} = \frac{(720 - 582)}{90.86} = 1.52$$

# Drawing conclusions...

$t_{0.05(2),29}$=2.05

*t* <2.05, so we cannot reject the null hypothesis.

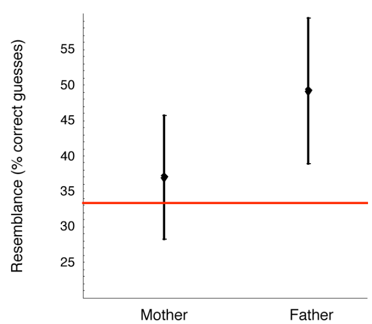
These data are not sufficient to say that there is a cost of resistance.

# Assumptions of two-sample *t* -tests

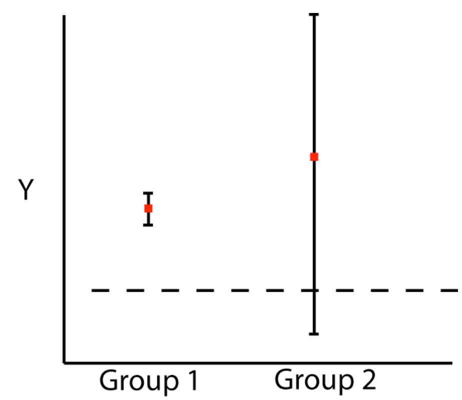
- Both samples are random samples.
- Both populations have normal distributions
- The variance of both populations is equal.

# The wrong way to make a comparison of two groups

“Group 1 is significantly different from a constant, but Group 2 is not. Therefore Group 1 and Group 2 are different from each other.”




# A more extreme case...



## Comparing means when variances are not equal

Welch's *t* test

*Welch's approximate t-test* compares the means of two normally distributed populations that have unequal variances.

## Burrowing owls and dung traps

## Dung beetles

## Experimental design

- 20 randomly chosen burrowing owl nests
- Randomly divided into two groups of 10 nests
- One group was given extra dung; the other not
- Measured the number of dung beetles on the owls' diets

## Number of beetles caught

- Dung added: $\bar{Y} = 4.8$

  $s = 3.26$

- No dung added: $\bar{Y} = 0.51$

  $s = 0.89$

## Hypotheses

$H_0$: Owls catch the same number of dung beetles with or without extra dung ($\mu_1 = \mu_2$)

$H_A$: Owls do not catch the same number of dung beetles with or without extra dung ($\mu_1 \neq \mu_2$)

## Welch's *t*

$$t = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left(\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}\right)}$$

Round down *df* to nearest integer

## Owls and dung beetles

$$t = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{4.8 - 0.51}{\sqrt{\frac{3.26^2}{10} + \frac{0.89^2}{10}}} = 4.01$$

## *Degrees of freedom*

$$df = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\left(\frac{\left(s_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(s_2^2/n_2\right)^2}{n_2 - 1}\right)} = \frac{\left(\frac{3.26^2}{10} + \frac{0.89^2}{10}\right)^2}{\left(\frac{\left(3.26^2/10\right)^2}{10 - 1} + \frac{\left(0.89^2/10\right)^2}{10 - 1}\right)} = 10.33$$

Which we round down to *df*= 10

## Reaching a conclusion

$t_{0.05(2),\ 10} = 2.23$

$t = 4.01 > 2.23$

So we can reject the null hypothesis with $P<0.05$.

Extra dung near burrowing owl nests increases the number of dung beetles eaten.

## Comparing the variance of two groups

$$H_0 : \sigma_1^2 = \sigma_2^2$$

$$H_A : \sigma_1^2 \neq \sigma_2^2$$

One possible method: the *F* test

## The test statistic *F*

$$F = \frac{s_1^2}{s_2^2}$$

Put the larger $s^2$ on top in the numerator.

## *F*...

- *F* has two different degrees of freedom, one for the numerator and one for the denominator. (Both are $df = n_i - 1$.) The numerator *df* is listed first, then the denominator *df*.
- The *F* test is very sensitive to its assumption that both distributions are normal.

## Example: Variation in insect genitalia

| | Polygamous species | Monogamous species |
|---|---|---|
| Mean | -19.3 | 10.25 |
| Sample variance | 243.9 | 2.27 |
| Sample size | 7 | 9 |

## Example: Variation in insect genitalia

$$s_1^2 = 243.9 \qquad s_2^2 = 2.27$$

$$F = \frac{243.9}{2.27} = 107.4$$

## Degrees of freedom

$$df_1 = 7 - 1 = 6$$
$$df_2 = 9 - 1 = 8$$

$$F_{0.025,6,8} = 4.7$$

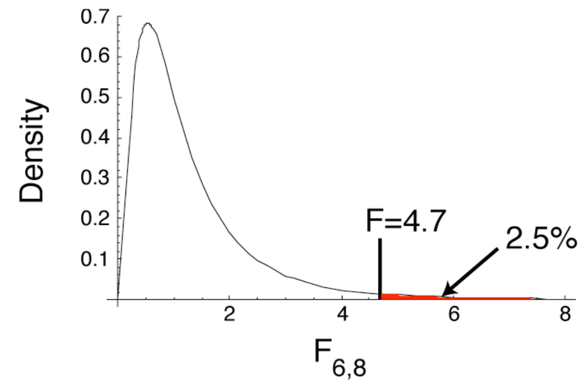
For a 2-tailed test, we compare to $F_{\alpha/2,df1,df2}$ from Table D.

# Why $\alpha/2$ for the critical value?

By putting the larger $s^2$ in the numerator, we are forcing $F$ to be greater than 1.

By the null hypothesis there is a 50:50 chance of either $s^2$ being greater, so we want the higher tail to include just $\alpha/2$.

# Critical value for $F$




# Conclusion

The $F$= 107.4 from the data is greater than $F_{(0.025), 6,8}$ =4.7, so we can reject the null hypothesis that the variances of the two groups are equal.

The variance in insect genitalia is much greater for polygamous species than monogamous species.

# The $F$ test is very sensitive to its assumption that both distributions are normal.

A more robust test to compare variances (between 2 or more groups) is:

*Levene's test*

You **should know** that Levene's test exists and why you would use it, but you do not need to know how to do it in this class. You would use a computer to do it, as the calculations are cumbersome.