Unit 8
SUPPLEMENT –
Normal, T, Chi Square, F, and Sums of Normals

1. Normal Distribution

For additional detail, see course notes 7. The Normal Distribution

When to Use:

1) tests and confidence intervals for means of continuous variables distributed normal when the population variance is *KNOWN*; and

2) tests and confidence intervals for binomial event probabilities when the available sample size is moderate to large

a. Definition

The patterns of occurrence of many continuous phenomena in nature happen to be described well using a normal distribution model. Even when the phenomena in a sample distribution are not described well by the normal distribution, the sampling distribution of sample averages obtained by repeated sampling from the parent distribution is often described well by the normal distribution (*Central limit theory*).

Normal Distribution (μ, σ^2)

A random variable X that is distributed normal with mean=μ and variance=σ^2 has probability density function

$$f_X(X=x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] \text{ where}$$

x = Value of X

Range of possible values of X: $-\infty$ to $+\infty$

Exp = e = Euler's constant = 2.71828 … *note:* $e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots$

π = mathematical constant = 3.14 *note: π = (circumference/diameter) for any circle*

μ = Expected value of X ("the long run average")

σ^2 = Variance of X. *Recall – this is the expected value of* $[X - \mu]^2$

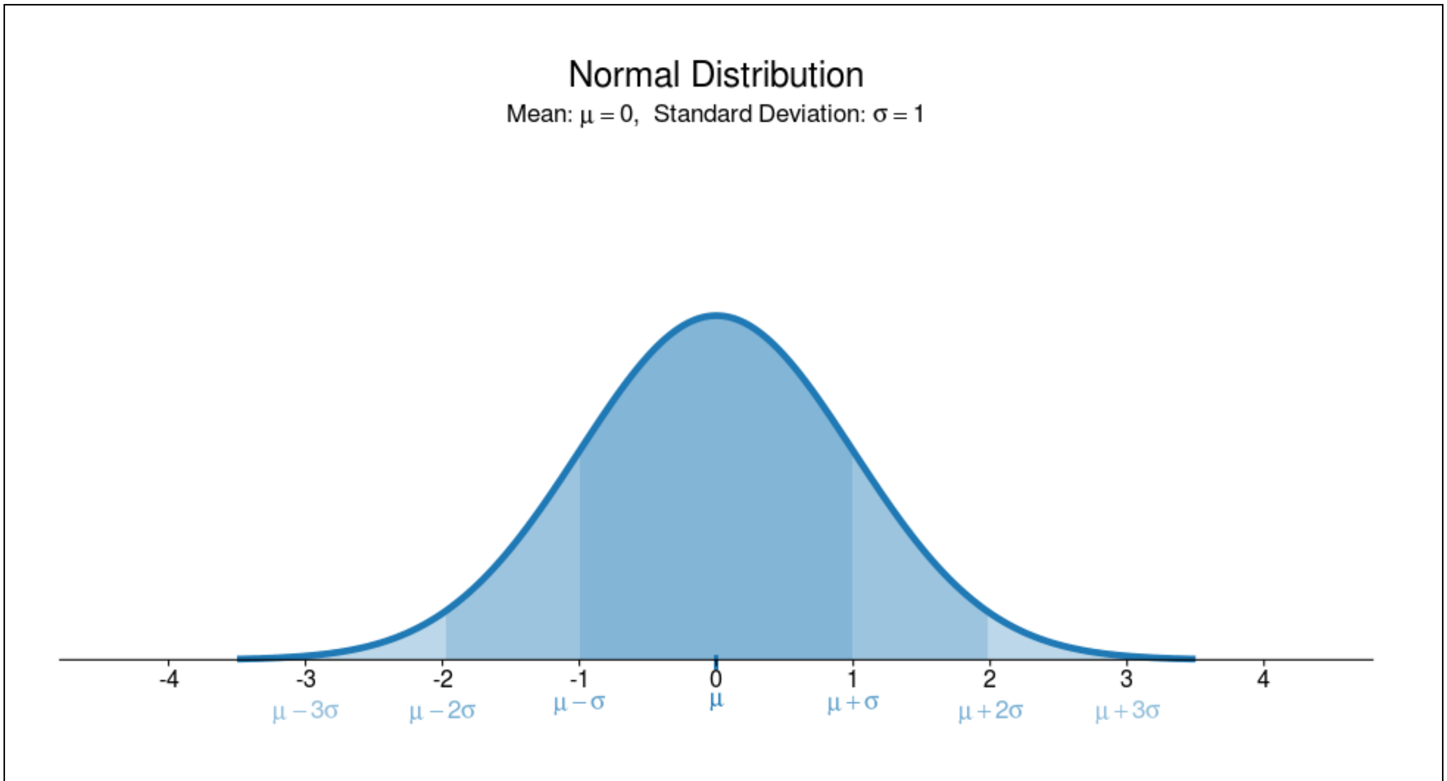
Standard Normal Distribution (μ=0, σ^2=1)

Z-scores are distributed Normal(0,1)

A random variable Z that is distributed standard normal has probability density function

$$f_Z\,(Z=z) \;=\; \frac{1}{\sqrt{2\pi}}\exp\left[\frac{-z^2}{2}\right]$$

A Feel for the Normal Distribution



https://istats.shinyapps.io/NormalDist/

(1) a smooth curve defined everywhere on the real axis that is

(2) bell shaped,

(3) symmetric about the mean; thus mean = median = mode

(4) tapered.

b. Calculations Using Online Apps

Art of Stat

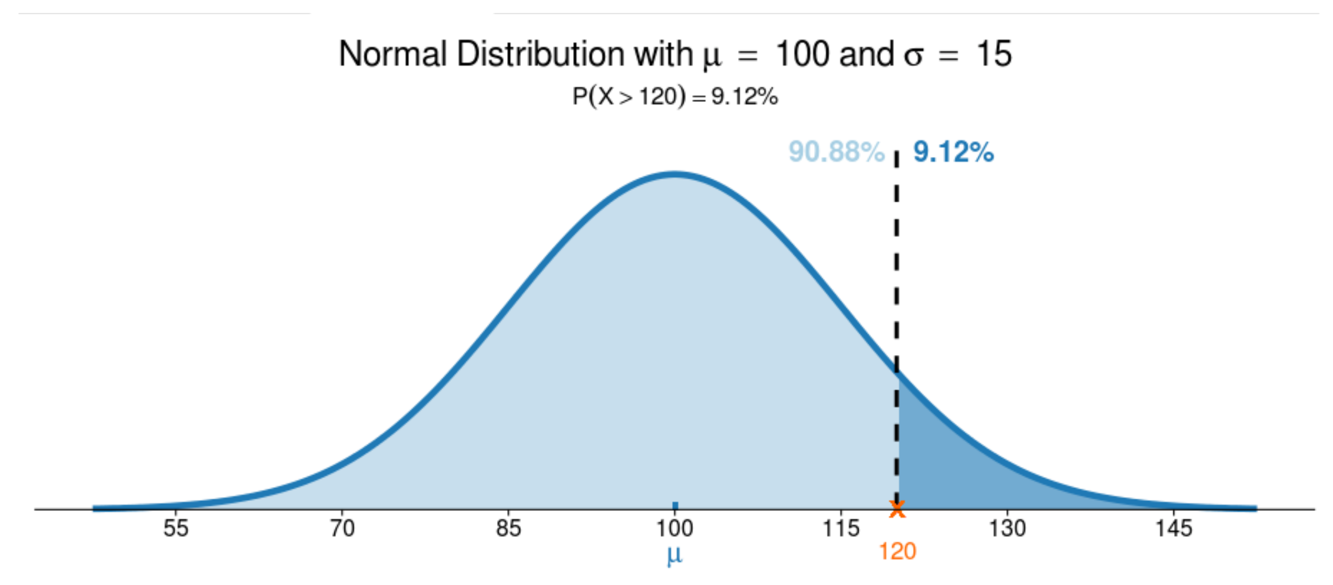
http://www.artofstat.com > Online Web Apps > Normal Distribution OR
https://istats.shinyapps.io/NormalDist/

Solve for a Probability *(Used for: P-value calculation)*

Example:
If X is distributed $\text{Normal}(\mu=100, \sigma^2=15^2)$, calculate Pr [X > 120]

Solution:
__1. Launch https://istats.shinyapps.io/NormalDist/
__2. Click tab, **Find Probability**
__3. At left, set **mean = 100** and **standard deviation = 15**
__4. Set **Type of Probability? = Upper Tail: Pr [X > x]**
__5. Set **Provide X = 120**

The calculator then returns the value **9.12%** which says **Pr [Normal(100, 15^2) > 120] = .0912**



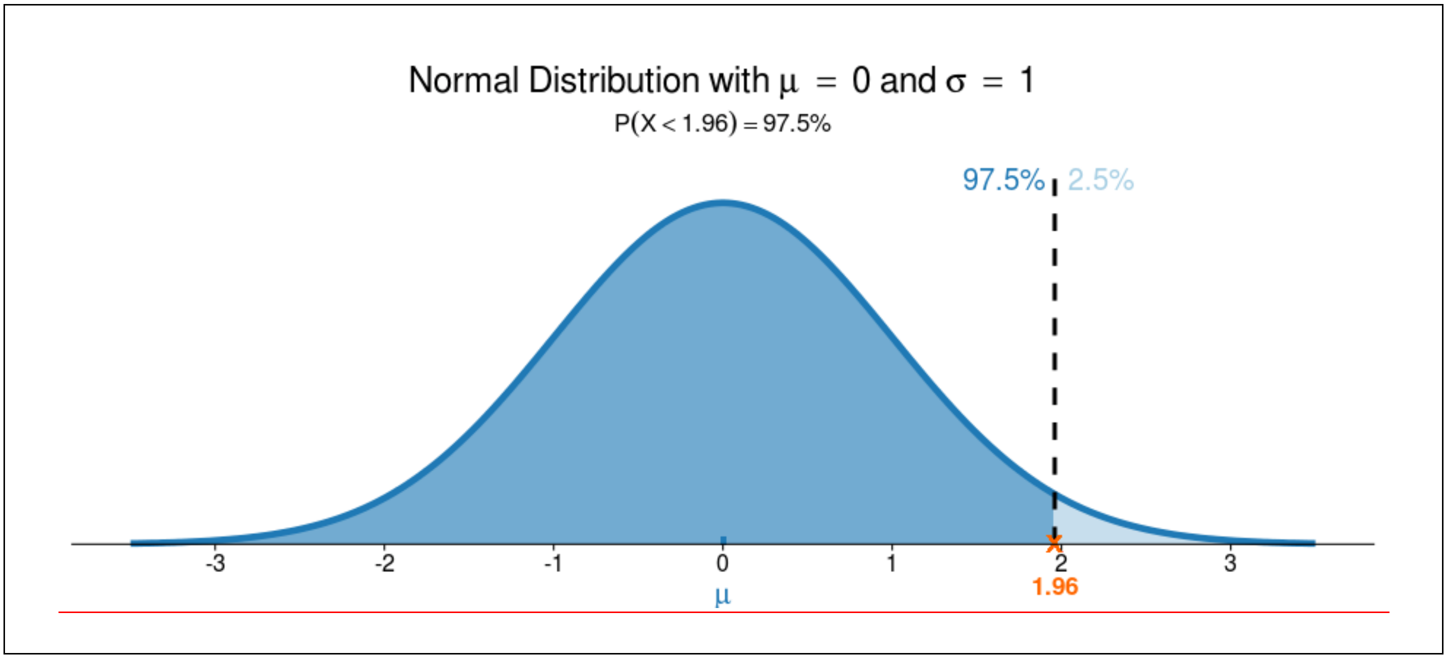
https://istats.shinyapps.io/NormalDist/

Solve for a Percentile ***(Used for: Confidence Interval Solution)***

Example:
If X is distributed $\text{Normal}(\mu{=}0, \sigma^2{=}1)$, obtain the 97.5th percentile

Solution:
__1. Launch https://istats.shinyapps.io/NormalDist/
__2. Click tab, **Find Percentile**
__3. At left, set **mean = 0** and **standard deviation = 1**
__4. Set **Type of Percentile? = Lower Tail: Pr [X < x]**
__5. Set **Probability in Lower Tail (%) = 97.5**

The calculator then returns the value **1.96** which says **Pr [Normal(0, 1) < 1.96] = .975**



https://istats.shinyapps.io/NormalDist/

c. Calculations Using R

```
Command for left-tail probability: pnnorm(x,mean=FILLIN,sd=FILLIN)
Command for right-tail probability: pnorm(x,mean=FILLIN,sd=FILLIN, lower.tail=FALSE)
Command for percentile of standard normal: qnorm(LEFTTAILPROBABILITY)

# Calculate Pr[Normal(100,15) > 120]
paste("Probability [Normal(100,15) > 120] =  ", pnorm(120,mean=100,sd=15,lower.tail=FALSE))

## [1] "Probability [Normal(100,15) > 120] =   0.0912112197258678"

# Calculate 97.5th Percentile of Standard Normal
paste("97.5th Percentile of Standard Normal  =   ", qnorm(.975))

## [1] "97.5th Percentile of Standard Normal  =    1.95996398454005"
```

d. Calculations Using Stata

```
. * Command for left-tail probability:  normal(VALUE OF ZSCORE)
. * Command for right-tail probability: 1 - normal(VALUEOFZSCORE)
. * Command for percentile of standard normal: invnormal(LEFTTAILPROBABILITY)

. display "pr [normal(100,15) > 120  =  " 1 - normal( (120-100)/15)
pr [normal(100,15) > 120  =  .09121122


. display "97th Percentile of Standard Normal  =  "  invnormal(.975)
97th Percentile of Standard Normal  =  1.959964
```

2. Student t-Distribution

When to Use:

Use the Student t-distribution in tests and confidence intervals for means of continuous variables distributed normal when the population variance is *NOT* known.

a. Definition

There are a variety of definitions of a student t random variable. A particularly useful one for us here is the following. It appeals to our understanding of the z-score.

Consider a simple random sample $X_1 \ldots X_n$ from a Normal (μ, σ^2) distribution. Calculate $\bar{X}$ and S^2 in the usual way:

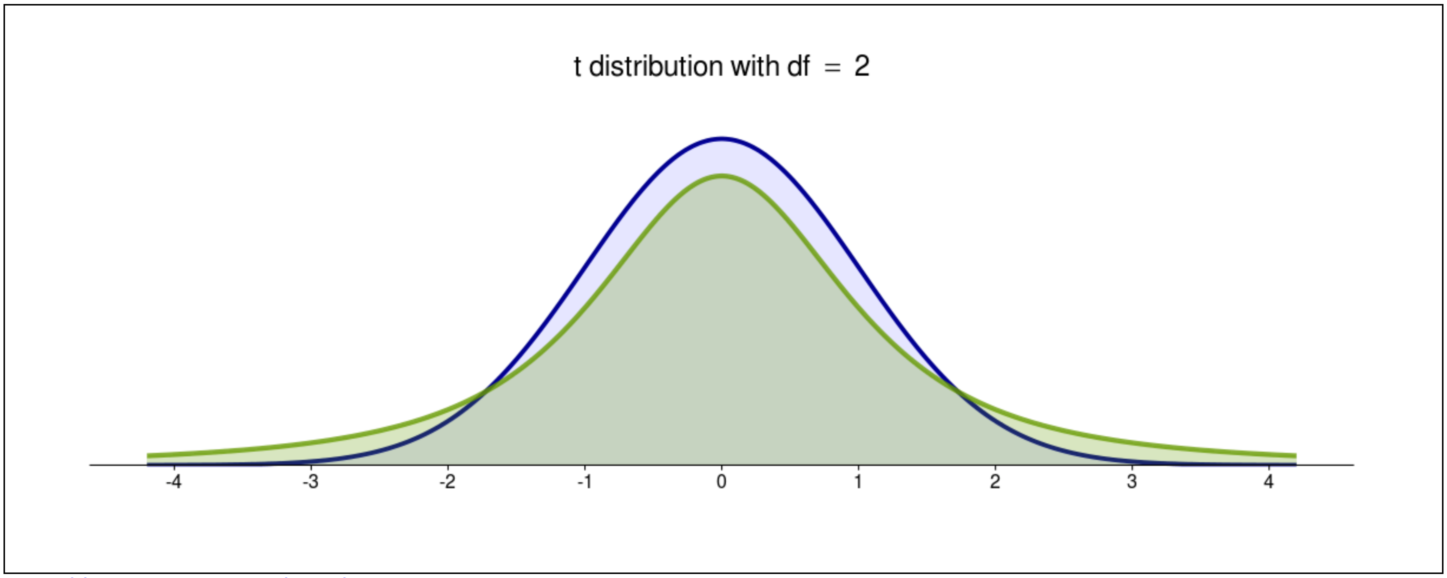
$$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} \qquad \text{and} \qquad S^2 = \frac{\sum_{i=1}^{n} \left(X_i - \bar{X}\right)^2}{n-1}$$

A student's t distributed random variable results if we construct a t-score instead of a z-score.

$$\text{t - score} = t_{DF=n-1} = \frac{\bar{X} - \mu}{s/\sqrt{n}}$$ is distributed **Student's t with degrees of freedom = (n-1)**

Note – The abbreviation "df" is often used to refer to "degrees of freedom"

A Feel for the Student-t Distribution (shaded green = Student-t, shaded blue = Normal(0,1))



https://istats.shinyapps.io/tdist/

(1) a smooth curve defined everywhere on the real axis that is

(2) bell shaped, ***but flatter than that of the Standard Normal (overlay blue)***, which means that:

(i) The variability of a student t variable greater than that of a standard normal (0,1)

(ii) Thus, there is more area under the tails and less at center

(iii) Confidence intervals tend to be wider than those based on the Normal

(3) tapered, ***with tails a bit "heavier" than those of the Standard Normal***

b. Calculations Using Online Apps

Art of Stat

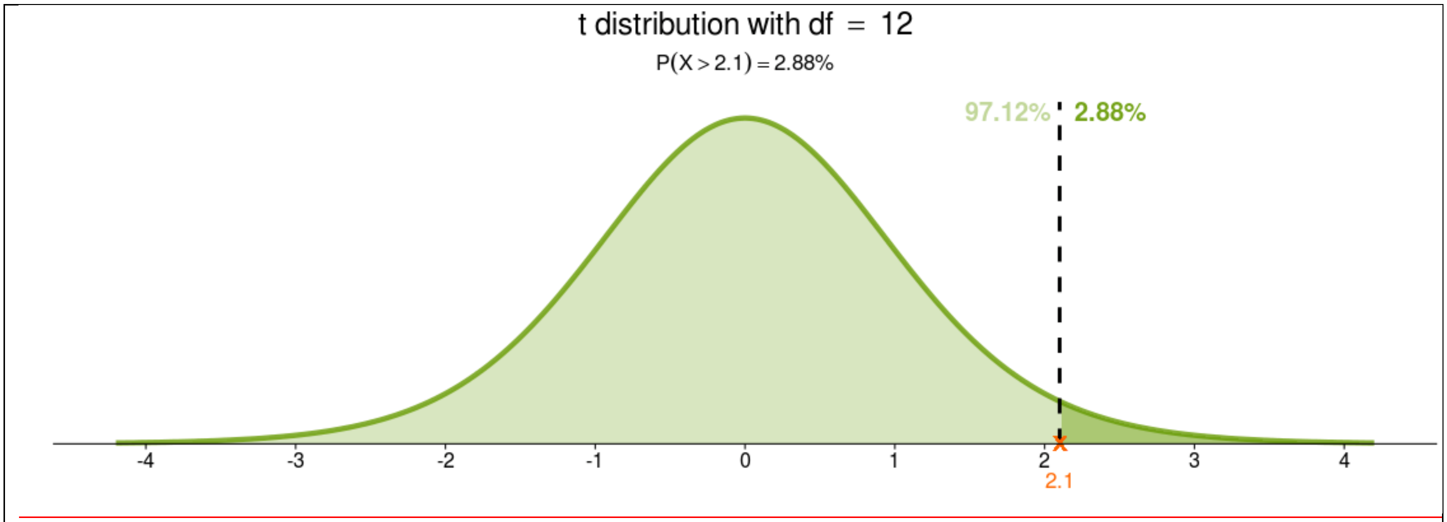
http://www.artofstat.com > Online Web Apps > t Distribution
https://istats.shinyapps.io/tdist/

Solve for a Probability *(Used for: P-value calculation)*

Example:
If X is distributed Student-t with df=12, calculate Pr [X > 2.10]

Solution:
__1. Launch **https://istats.shinyapps.io/tdist/**
__2. Click tab, **Find Probability**
__3. At left, set **Number of Degrees of Freedom=12**
__4. Set **Type of Probability? = Upper Tail: Pr [X > x]**
__5. Set **Provide X = 2.10**

The calculator then returns the value **2.88%** which says **Pr [Student-t (df=12) > 2.10] = .0288**



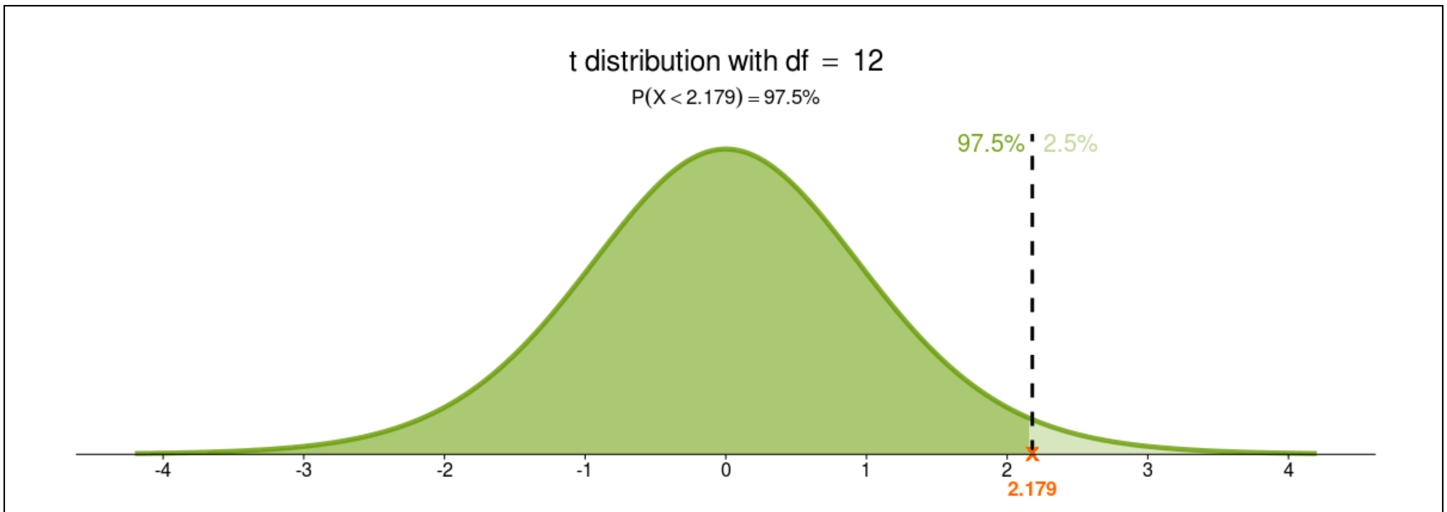
https://istats.shinyapps.io/tdist/

Solve for a Percentile ***(Used for: Confidence Interval Solution)***

Example:
If X is distributed Student-t (df=12) obtain the 97.5th percentile

Solution:
__1. Launch the following (it is from artofstat.com): **https://istats.shinyapps.io/tdist/**
__2. Click tab, **Find Percentile**
__3. At left, set **Number of Degrees of Freedom=12**
__4. Set **Type of Percentile? = Lower Tail: Pr [X < x]**
__5. Set **Probability in Lower Tail (%) = 97.5**

The calculator then returns the value **2.179** which says **Pr [Student-t (df=12) < 2.179] = .975**



https://istats.shinyapps.io/tdist/

c. Calculations Using R

```
Command for left-tail probability: pt(TVALUE,df=FILLIN)
Command for right-tail probability: pt(TVALUE,df=FILLIN, lower.tail=FALSE)
Command for percentile of student-t: qt(LEFTTAILPROBABILITY,df=FILLIN)

# Calculate Pr[Student-t (df=12) > 2.10]
paste("Probability [Student-t (df=12) > 2.10] =  ", pt(2.10, df=12,lower.tail=FALSE))

## [1] "Probability [Student-t (df=12) > 2.10] =   0.0287724693674756"

# Calculate 97.5th Percentile of Student-t with df=12
paste("97.5th Percentile of Student-t (df=12)  =   ", qt(.975,df=12))

## [1] "97.5th Percentile of Student-t (df=12)  =    2.17881282966723"
```

d. Calculations Using Stata

```
. * Command for left-tail probability:  1 – ttail(DF,VALUEOFT)
. * Command for right-tail probability: ttail(DF,VALUEOFT)
. * Command for percentile of standard normal: invttail(DF,RIGHTTTAILPROBABILITY)

. display "Pr [T (df=12) > 2.10 ]  =  " ttail(12,2.10)
Pr [T (df=12) > 2.10 ]  =  .02877247

. display "97.5th percentile of Student t (df=12)  =  " invttail(12,.025)
97.5th percentile of Student t (df=12)  =  2.1788128
```

3. Chi Square Distribution

When to Use:

(1) Use in settings of continuous outcomes that are distributed normal. Specifically, it is used in tests and confidence intervals for a single population variance or single population standard deviation.

(2) Use in the analysis of count data (stay tuned)

a. Definition

One setting (not the only one, but the one we are considering in this course) is the setting of a simple random sample from a Normal distribution. We want to test a hypothesis or construct a confidence interval estimate of the variance parameter, σ^2. To do this, we work with a new random variable Y that is defined as follows:

$$Y=\frac{(n-1)S^2}{\sigma^2},$$

In this formula, S^2 is the sample variance that you learned in Unit 1. Under simple random sampling from a Normal(μ,σ^2)

$$Y=\frac{(n-1)S^2}{\sigma^2}$$ **is distributed Chi Square with degrees of freedom = (n-1)**

The above can be stated more formally.

(1) **If** the random variable X follows a normal probability distribution with mean μ and variance σ^2,

Then the random variable V defined:

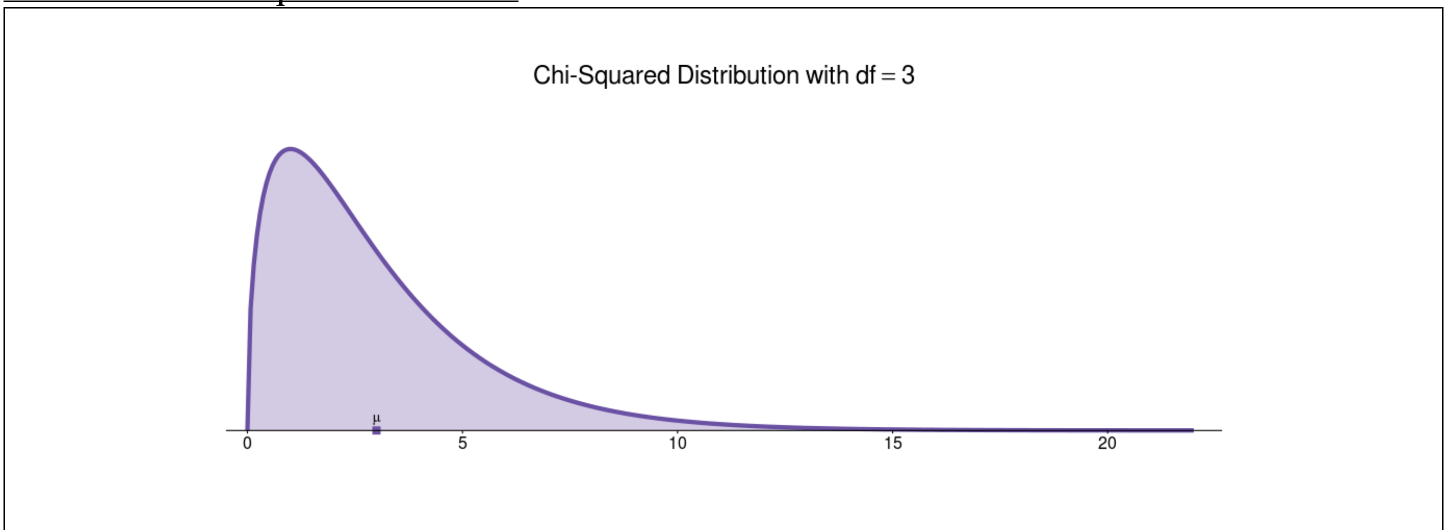
$$V=\frac{(X-\mu)^2}{\sigma^2}$$ is distributed chi square distribution with degree of freedom = 1.

(2) **If** each of the random variables V_1, ..., V_k is distributed chi square with degree of freedom = 1,
and if these are independent,
Then their sum, defined:

$\mathbf{V_1 + \ldots + V_k}$ is distributed chi square distribution with degrees of freedom = k

For the interested reader: The two definitions on the previous page are consistent because it is possible (with a little algebra) to re-write $Y=\frac{(n-1)S^2}{\sigma^2}$ as the sum of (n-1) independent chi square random variables V, each with degrees of freedom = 1. *NOTE: For this course, it is not necessary to know the probability density function for the chi square distribution.*

A Feel for the Chi Square Distribution

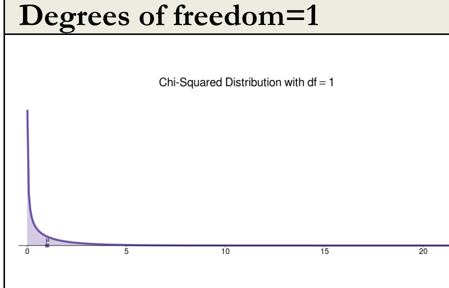
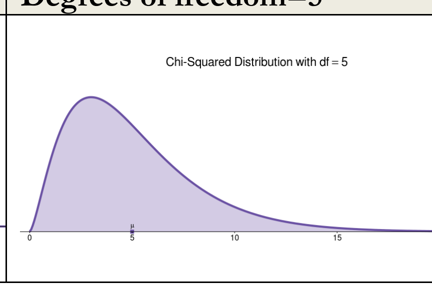
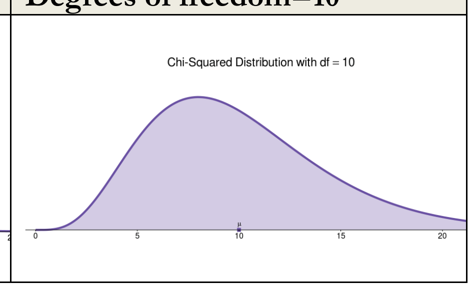


https://istats.shinyapps.io/ChisqDist/

(1) A chi square random variable can assume only non-negative values. Specifically, the probability density function has domain $[0, \infty)$ and is not defined for outcome values less than zero. Thus,

(2) **Thus, the chi square distribution is NOT symmetric. Pr[Y > y] is NOT EQUAL to Pr[Y < -y]**
Tip – This means that, in contrast to those for the Normal or Student-t, the values of chi square distribution percentiles are NOT symmetric around zero. → **For example, you have to solve for the 2.5th and 97.5th percentiles separately.**

(3) The chi square distribution is less skewed as the number of degrees of freedom increases.

Degrees of freedom=1	Degrees of freedom=5	Degrees of freedom=10
		

https://istats.shinyapps.io/ChisqDist/

b. Calculations Using Online Apps

Art of Stat

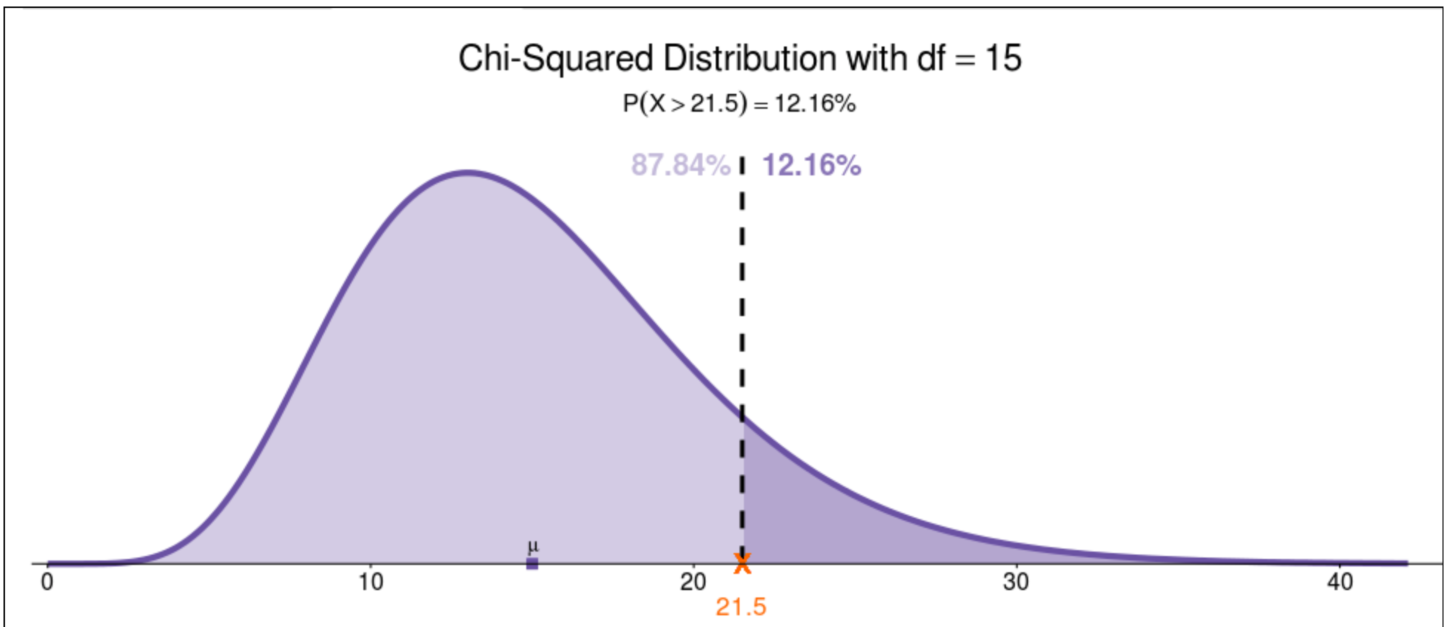
http://www.artofstat.com > Online Web Apps > Chi-Squared Distribution
https://istats.shinyapps.io/ChisqDist/

Solve for a Probability *(Used for: P-value calculation)*

Example:
If X is distributed Chi Square with df=15, calculate Pr [X > 21.5]

Solution:
__1. Launch **https://istats.shinyapps.io/ChisqDist/**
__2. Click tab, **Find Probability**
__3. At left, set **Number of Degrees of Freedom=15**
__4. Set **Type of Probability? = Upper Tail: Pr [X > x]**
__5. Set **Provide X = 21.5**

The calculator then returns the value **12.16%** which says **Pr [Chi Square (df=15) > 21.5] = .1216**



https://istats.shinyapps.io/ChisqDist/

<u>Solve for a Percentile</u> ***<u>(Used for: Confidence Interval Solution)</u>***

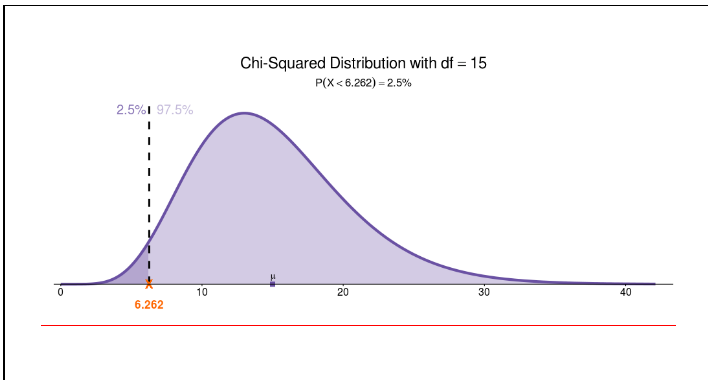
Example:

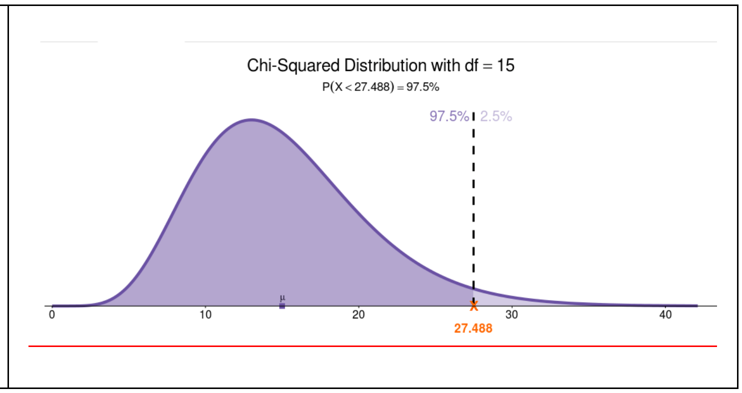
If X is distributed Chi Square (df=15) obtain the 2.5th and 97.5th percentiles. Notice that they are not symmetric about zero!

Solution:

__1. Launch **https://istats.shinyapps.io/ChisqDist/**
__2. Click tab, **Find Percentile**
__3. At left, set **Number of Degrees of Freedom=15**
__4. Set **Type of Percentile? = Lower Tail: Pr [X < x]**
__5. Set **Probability in Lower Tail (%) = 2.5**
__6. Set **Probability in Lower Tail (%) = 97.5**

The calculator then returns the values **6.262** and **27.488**, which say **Pr [Chi Square (df=15) < 6.262] = .025** and **Pr [Chi Square (df=15) < 27.488] = .975**, respectively.





https://istats.shinyapps.io/ChisqDist/

c. Calculations Using R

```
Command for left-tail probability: pchisq(VALUE,df=FILLIN)
Command for right-tail probability: pchisq(VALUE,df=FILLIN, lower.tail=FALSE)
Command for percentile of chi square: qchisq(LEFTTAILPROBABILITY,df=FILLIN)

# Calculate Pr[Chi Square (df=15) > 21.5]
paste("Probability [Chi Square (df=15) > 21.5] =  ", pchisq(21.5, df=15,lower.tail=FALSE))

## [1] "Probability [Chi Square (df=15) > 21.5] =   0.121600308000176"


# Calculate 2.5th and 97.5th Percentiles of Chi Square with df=15
paste("2.5th Percentile of Chi Square (df=15)  =   ", qchisq(.025,df=15))

## [1] "2.5th Percentile of Chi Square (df=15)  =    6.26213779504325"

paste("97.5th Percentile of Chi Square (df=15)  =   ", qchisq(.975,df=15))

## [1] "97.5th Percentile of Chi Square (df=15)  =    27.488392863443"
```

d. Calculations Using Stata

```
. * Command for left-tail probability:  1 – chi2tail(DF,VALUE)
. * Command for right-tail probability: chi2tail(DF,VALUEOFT)
. * Command for percentile of standard normal: invchi2(DF,LEFTTAILPROBABILITY)

. display "Pr [ Chi square (df=15) > 21.5 ]  =    "chi2tail(15,21.5)
Pr [ Chi square (df=15) > 21.5 ]  =    .12160031


. display "2.5th Percentile of Chi Square (df=15)  =   " invchi2(15,.025)
2.5th Percentile of Chi Square (df=15)  =   6.2621378

. display "97.5th Percentile of Chi Square (df=15)  =   " invchi2(15,.975)
97.5th Percentile of Chi Square (df=15)  =   27.488393
```

4. F Distribution

When to Use:

Use the F distribution in settings of samples from normal distributions and are used in tests and confidence intervals for the ratio of two independent variances.

a. Definition

Unlike the approach used to compare two means in the continuous variable setting (where we will look at their difference), the comparison of two variances is accomplished by looking at their ratio. Ratio values close to one are evidence of similarity. Of interest will be a confidence interval estimate of the ratio of two variances in the setting where data are comprised of two independent samples of data, each from a separate Normal distribution.

Suppose $X_1, \dots, X_{nx}$ is a simple random sample from a normal distribution with mean μ_X and variance σ_X^2. Suppose further that $Y_1, \dots, Y_{ny}$ is a simple random sample from a normal distribution with mean μ_Y and variance σ_Y^2.

If the two sample variances are calculated in the usual way

$$S_X^2 = \frac{\sum_{i=1}^{n_x}(x_i - \bar{x})^2}{n_x - 1} \quad \text{and} \quad S_Y^2 = \frac{\sum_{i=1}^{n_Y}(y_i - \bar{y})^2}{n_Y - 1}$$

Then

$$F_{n_x-1, n_y-1} = \frac{S_X^2 / \sigma_x^2}{S_Y^2 / \sigma_y^2}$$ is distributed F with two degree of freedom specifications

Numerator degrees of freedom, $df_1 = n_x - 1$
Denominator degrees of freedom, $df_2 = n_y - 1$

b. Calculations Using Online Apps

Art of Stat

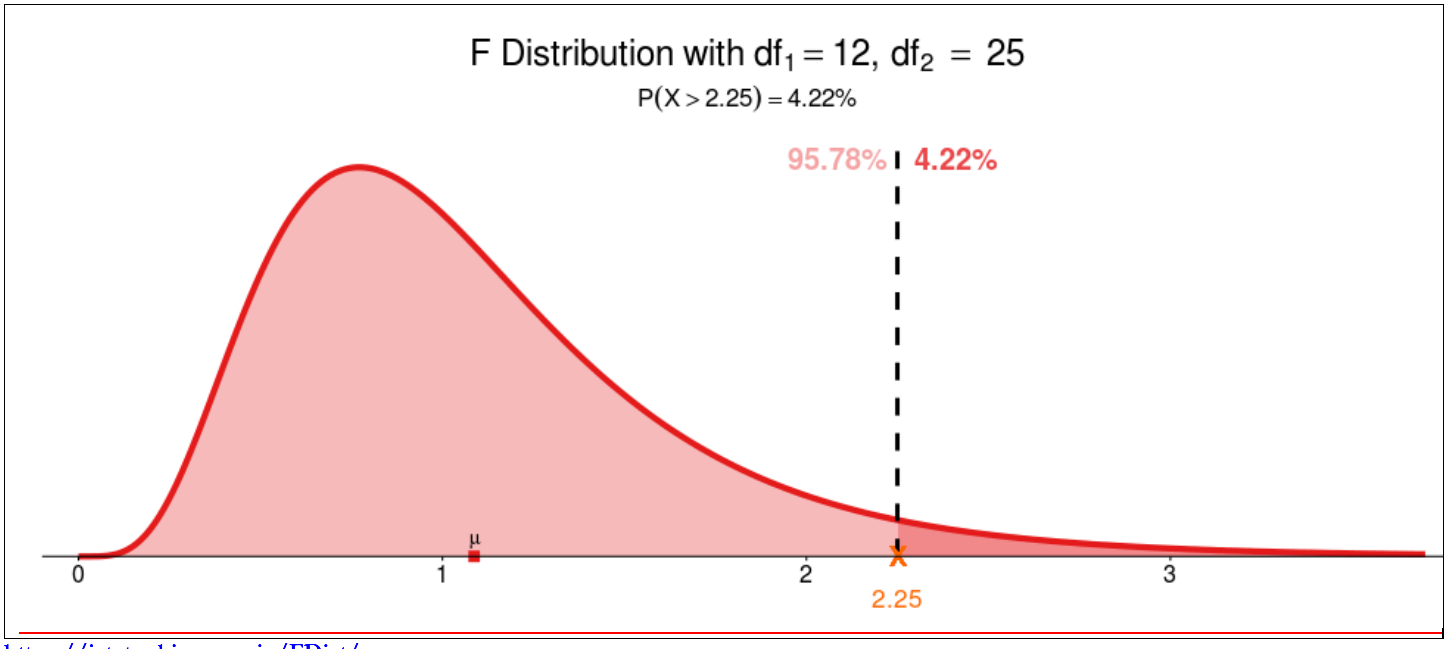
http://www.artofstat.com > Online Web Apps > F Distribution
https://istats.shinyapps.io/FDist/

Solve for a Probability *(Used for: P-value calculation)*

Example:
If X is distributed F with df_1=12 and df_2=25, calculate Pr [X > 2.25]

Solution:
__1. Launch **https://istats.shinyapps.io/FDist/**
__2. Click tab, **Find Probability**
__3. At left, set **Numerator Degrees of Freedom=12** and **Denominator Degrees of Freedom=25**
__4. Set **Type of Probability? = Upper Tail: Pr [X > x]**
__5. Set **Provide X = 2.25**

The calculator then returns the value **4.22%** which says **Pr [F (df_1=12, df_2=25) > 2.25] = .0422**



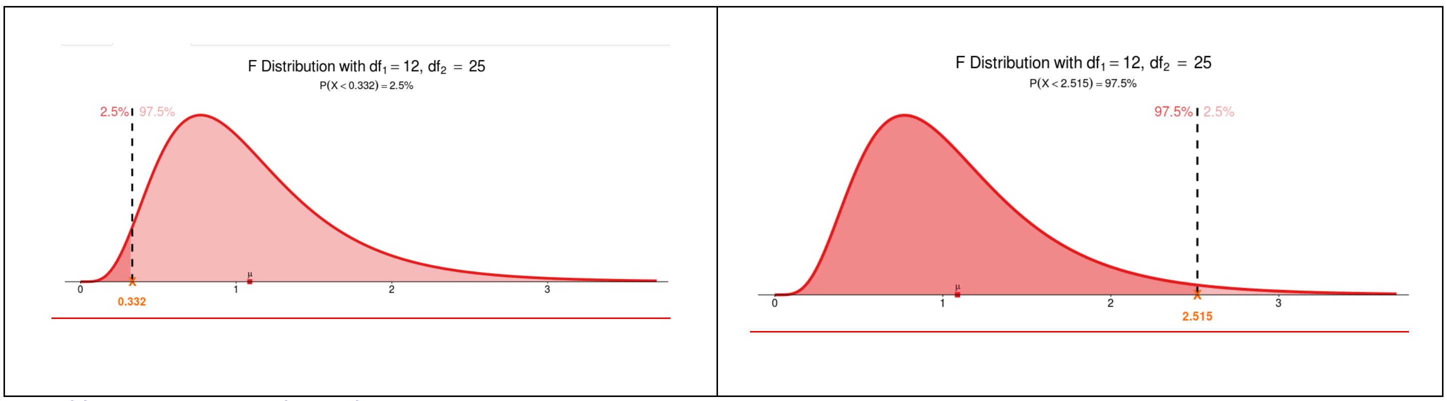
https://istats.shinyapps.io/FDist/

<u>Solve for a Percentile</u> ***<u>(Used for: Confidence Interval Solution)</u>***

Example:
If X is distributed Chi Square (df=15) obtain the 2.5th and 97.5th percentiles. Notice that they are not symmetric about zero!

Solution:
__1. Launch **https://istats.shinyapps.io/FDist/**
__2. Click tab, **Find Percentile**
__3. At left, set set **Numerator Degrees of Freedom=12** and **Denominator Degrees of Freedom=25**
__4. Set **Type of Percentile? = Lower Tail: Pr [X < x]**
__5. Set **Probability in Lower Tail (%) = 2.5**
__6. Set **Probability in Lower Tail (%) = 97.5**

The calculator then returns the values **0.332** and **2.515**, which say **Pr [F (df_1=12, df_2=25) < 0.332] = .025** and **Pr [F (df_1=12, df_2=25) < 2.515] = .975**, respectively.



https://istats.shinyapps.io/FDist/

c. Calculations Using R

```
Command for left-tail probability: pf(VALUE,df1=FILLIN,df2=FILLIN)
Command for right-tail probability: pf(VALUE,df1=FILLIN,df2=FILLIN,lower.tail=FALSE)
Command for percentile of F: qf(LEFTTAILPROBABILITY,df1=FILLIN,df2=FILLIN)

# Calculate Pr[ F (df1=12, df2=25) > 2.25]
paste("Probability [ F (df1=12, df2=25) > 2.25] =  ", pf(2.25, df1=12, df2=25,lower.tail=FALSE))

## [1] "Probability [ F (df1=12, df2=25) > 2.25] =   0.0421869738538855"

# Calculate 2.5th and 97.5th Percentile of F with df1=12, df2=25
paste("2.5th Percentile of F (df1=12, df2=25)  =   ", qf(.025,df1=12,df2=25))

## [1] "2.5th Percentile of F (df1=12, df2=25)  =    0.332475762423314"

paste("97.5th Percentile of F (df1=12, df2=25)  =   ", qf(.975,df1=12,df2=25))

## [1] "97.5th Percentile of F (df1=12, df2=25)  =    2.51489034862395"
```

d. Calculations Using Stata

```
. * Command for left-tail probability:  1 – Ftail(DF1,DF2,VALUE)
. * Command for right-tail probability: Ftail(DF1,DF2,VALUEOFT)
. * Command for percentile: invFtail(DF1,DF2,RIGHTTAILPROBABILITY)

. display "Pr [ F df=12,25 > 2.25 ]  =   " Ftail(12,25,2.25)
Pr [ F df=12,25 > 2.25 ]  =   .04218697


. display "2.5th Percentile of F (df1=12, df2=25)   =    " invFtail(12,25,.975)
2.5th Percentile of F (df1=12, df2=25)   =    .33247576

. display "97.5th Percentile of F (df1=12, df2=25)   =    " invFtail(12,25,.025)
97.5th Percentile of F (df1=12, df2=25)   =    2.5148903
```

5. Sums and Differences of Independent Normal Random Variables

Looking ahead ….

We will be doing tests, and calculating associated confidence interval estimates, of such things as the difference between two independent means (eg control versus intervention in a randomized controlled trial)

Suppose we have to independent random samples, from two independent normal distributions. eg – randomized controlled trial of placebo versus treatment groups). We suppose we want to do a hypothesis test of the difference of the two means or compute a confidence interval estimate of the difference of the means.

Point Estimator: How do we obtain a point estimate of the difference [$\mu_{Group\ 1}$ - $\mu_{Group\ 2}$] ?

- A good point estimator of the difference between population means is the difference between sample means, [$\bar{X}_{Group\ 1} - \bar{X}_{Group\ 2}$]

Standard Error of the Point Estimator: We need the standard error of [$\bar{X}_{Group\ 1} - \bar{X}_{Group\ 2}$]

Definitions
IF

- *(for group 1)*: X_{11} , X_{12} , … X_{1n1} is a simple random sample from a Normal (μ_1 , σ_1^2)
- *(for group 2)*: X_{21} , X_{22} , … X_{2n2} is a simple random sample from a Normal (μ_2 , σ_2^2)

- **This is great!** We *already* know the sampling distribution of each sample mean
  $\bar{X}_{Group\ 1}$ is distributed Normal (μ_1, σ_1^2 / n_1)
  $\bar{X}_{Group\ 2}$ is distributed Normal (μ_2, σ_2^2 / n_2)

THEN

$[\bar{X}_{Group\ 1} - \bar{X}_{Group\ 2}]$ is also distributed Normal with

$$\text{Mean} = [\mu_{Group 1} - \mu_{Group\ 2}]$$

$$\text{Variance} = \left[\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}\right]$$

Be careful!! The standard error of the difference is NOT the sum of the two separate standard errors.

Notice – You must first sum the variance and then take the square root of the sum.

$$SE\left[\bar{X}_{\text{Group 1}} - \bar{X}_{\text{Group 2}}\right] = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

A General Result
Handy!

If random variables X and Y are independent with

E [X] = μ_X and Var [X] = σ_X^2
E [Y] = μ_Y and Var [Y] = σ_Y^2

Then

$$E[aX + bY] = a\mu_X + b\mu_Y$$
$$Var[aX + bY] = a^2\sigma_X^2 + b^2\sigma_Y^2 \text{ and}$$
$$Var[aX - bY] = a^2\sigma_X^2 + b^2\sigma_Y^2$$

NOTE: **This result ALSO says that, when X and Y are independent, the variance of their difference is equal to the variance of their sum. This makes sense if it is recalled that variance is defined using squared deviations which are always positive.**