14.7 Maximum and Minimum values

1. Find the saddle points, local maximum and minimum values, of the function $f(x, y) = x^2 + y^4 + 2xy$.

$\nabla f(x,y) = \langle 2x + 2y, 4y^3 + 2x \rangle$

$= \vec{0}$ when $2x + 2y = 0 \Rightarrow x = -y$

$4y^3 + 2x = 0$

$\Downarrow$

$2y = 4y^3 \Rightarrow 0 = 2y[2y^2 - 1]$

so $y = 0$ or $y = \frac{\sqrt{2}}{2}$ or $y = -\frac{\sqrt{2}}{2}$

$\downarrow$ $\downarrow$ $\downarrow$

$x = 0$ $x = -\frac{\sqrt{2}}{2}$ $x = \frac{\sqrt{2}}{2}$

are critical points.

Now $f_{xx} = 2$, $f_{xy} = 2$ and $f_{yy} = 12y^2$ so discriminant for each critical points are:

$f_{xx} f_{yy} - f_{xy}^2$

$(0,0)$: $(2)(12[0]^2) - 2 = -2 < 0$

saddle point

$(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$: $2(12[\frac{\sqrt{2}}{2}]^2) - 2 = 10 > 0$

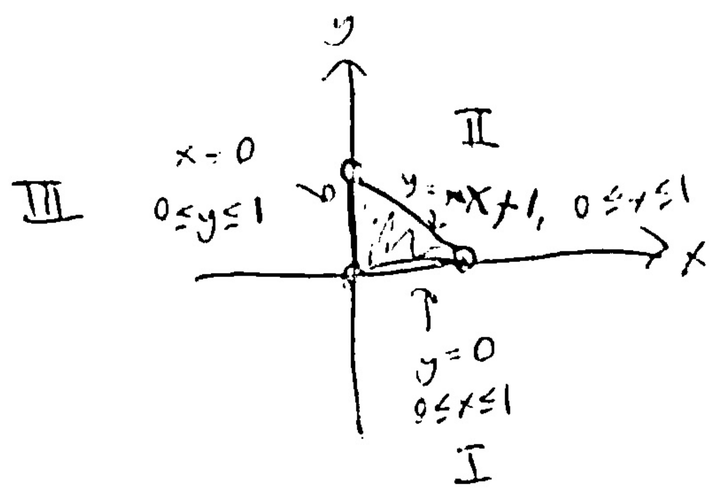
$(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$: $2(12[-\frac{\sqrt{2}}{2}]^2) - 2 = 10 > 0$

and $f_{xx} = 2 > 0$

local mins

[Ans:
$(0, 0)$: saddle point, $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$: local min, $(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$: local min]

2. Find the absolute minimum and maximum values of the function $f(x, y) = x^2+xy+y^2$ on the (closed) triangular region with vertices $(0, 0)$, $(0, 1)$, $(1, 0)$.



Check critical points inside triangle:

$\nabla f(x,y) = \langle 2x+y, x+2y \rangle$

$= \vec{0}$ when $2x+y=0 \to y=-2x$

& $x+2y=0 \to x+2(-2x)=0$

$\Downarrow$

$x=0$, $y=0$ (0,0) is only critical point.

$f(0,0)=0$.

Check each leg of triangle:

I: $f(x,0) = x^2+x(0)+(0)^2 = x^2$

crit point at $x=0$, $f(0,0)=0$.

II: $f(x,-x+1) = x^2+x(-x+1)+(-x+1)^2$

$(0 \le x \le 1)$

$= x^2 - x + x^2 - x + 1$

$= x^2 - x + 1 = g(x)$ on $[0,1]$

$g(0)=0$ and $g(1) = (1)^2-(1)+1 = 1$

$g'(x) = 2x-1$

so crit point at $x=\frac{1}{2}$:

$g(\frac{1}{2}) = (\frac{1}{2})^2 - (\frac{1}{2}) + 1 = \frac{3}{4}$

(must be a min on II)

III: $f(0,y) = (0)^2+(0)y+y^2 = y^2$

crit point at $y=0$, $f(0,0)=0$

after seeing all candidates, we conclude

[Ans. $f(0,0) = 0$ abs min, $f(1,0) = f(0,1) = 1$ abs max]

14.8 Lagrange Multipliers

3. Find the maximum and minimum values of the function $f(x, y) = xe^y$ under the constraint $x^2 + y^2 = 2$. (Hint: Here you will have to set $\lambda \neq 0$ to simplify and solve).

$\nabla f(x,y) = \langle e^y, xe^y \rangle$

$\lambda \nabla g(x,y) = \lambda \langle 2x, 2y \rangle$

so we have a system in three unknowns x, y, λ.

$e^y = \lambda 2x$

$xe^y = \lambda 2y$

$x^2 + y^2 = 2$

can't solve if $\lambda = 0$

anyways since $e^y \neq 0$ for any y.

$x(\lambda 2x) = \lambda(2y)$ (assume $\lambda \neq 0$)

$\Downarrow$

$x^2 = y$

$y + y^2 = 2$

by quad formula (or factoring)

$y = \frac{-1 \pm \sqrt{1 - 4(1)(-2)}}{2}$

$= \frac{-1 \pm 3}{2} = -2$ or 1

$\downarrow$ $x = \pm 1$

$x = \pm 1$ possibility

$x^2 \neq -2$

two input candidates found

$(-1,1)$: $f(-1,1)$

$= (-1)e^1 = \boxed{-e}$

must be min w/ constraint

$(1,1)$: $f(1,1) =$

$(1)e^{(1)} = \boxed{e}$

must be max w/ constraint

~~[Ans: At $(\frac{1}{\sqrt{2}}, 1)$ max, at $(-\frac{1}{\sqrt{2}}, 1)$ min]~~

4. Use Lagrange Multipliers to prove that the rectangle with maximum area that has a *given, fixed* perimeter p is a square.

given function: $f(x,y) = xy$ $\quad$ x - height, y - length

constraint is parameter: $g(x,y) = 2(x+y) = p$

$\nabla f(x,y) = \lambda \nabla g(x,y)$ gives system

$y = 2\lambda x$

$x = 2\lambda y$

$2x + 2y = p$

(1st step) $x = 2\lambda(2\lambda x) = 4\lambda^2 x \rightarrow (1 - 4\lambda^2)x = 0$

either $x = 0$ (boring) or $\lambda = \pm\frac{1}{2}$ (exciting)

(and) $x = 2(\frac{1}{2})y \rightarrow \boxed{x = y}$ i.e. square

so $\lambda = \frac{1}{2}$

discount negative case since it makes no geometric sense, we want $x, y > 0$.