Equations of lines and planes, distances

1. Consider the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ with equations $2x - 2y + z = 3$ and $3x + 4y = 2$ respectively.

   $\vec{n}_1 = \langle 2, -2, 1 \rangle$ $\vec{n}_2 = \langle 3, 4, 0 \rangle$

   (a) What is the angle between the two planes?

   $|\vec{n}_1 \cdot \vec{n}_2| = |\vec{n}_1||\vec{n}_2| \cos\theta_{planes} \rightarrow +2 = 3 \cdot 5 \cos\theta$

   (since plane can either use $\vec{n}$ or $-\vec{n}$ as normal, we account for sign in this)

   $\theta = \arccos\left(+\frac{2}{15}\right) \approx 82.34°$

   (b) Find a point P that belongs to both $\mathcal{P}_1$ and $\mathcal{P}_2$ and write down the vector and symmetric equations for the line of intersection L of the two planes.

   Let $x = 0$, then $2(0) - 2y + z = 3$ and $3(0) + 4y = 2 \Rightarrow y = \frac{1}{2}$ and $z = 4$

   $P = (0, \frac{1}{2}, 4)$. Line of intersection has direction $\vec{n}_1 \times \vec{n}_2 = \langle -4, 3, 14 \rangle$

   $x = -4t$, $y = 3t + \frac{1}{2}$, $z = 14t + 4$ symmetric form: $\frac{x}{-4} = \frac{y - \frac{1}{2}}{3} = \frac{z - 4}{14}$

   (c) Write down the equation of a plane P that passes through the point P and is orthogonal to both P_1 and P_2.

   $(0, \frac{1}{2}, 4)$

   orthogonal to their normals, i.e. $\vec{n}_1 \times \vec{n}_2$

   $$-4(x - 0) + 3(y - \tfrac{1}{2}) + 14(z - 4) = 0$$

   (d) Find the distance from the origin $(0, 0, 0)$ to the line L and to the plane P_1. What do you observe?

   $\vec{n}_1 = \langle 2, -2, 1 \rangle$

   using point $P = (0, \frac{1}{2}, 4)$ displacement vector $\langle 0, \frac{1}{2}, 4 \rangle$ to origin $(0,0,0)$

   a plane must be closer or equal distance since it contains the line.

   ☆ Shortest distance to line: $\dfrac{|\langle 0, \frac{1}{2}, 4 \rangle \times \langle -4, 3, 14 \rangle|}{|\langle -4, 3, 14 \rangle|} = \dfrac{|\langle -5, -16, 2 \rangle|}{\sqrt{(-4)^2 + 3^2 + 14^2}}$

   ≈ 1.1356 units

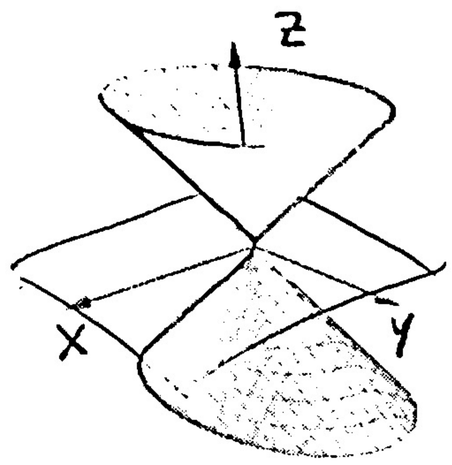
   ☆ Shortest distance to plane: $\dfrac{|\langle 0, \frac{1}{2}, 4 \rangle \cdot \langle 2, -2, 1 \rangle|}{|\langle 2, -2, 1 \rangle|} = \dfrac{3}{3} = 1$ unit

Cylinders and Quadric Surfaces

2. The following three figures all correspond to the equation $x^2 + \frac{y^2}{3} - \frac{z^2}{2} = c$ for different values of c. For each one of them, explain whether the value of c is positive, negative or zero. In all of them calculate the vertical traces $x = k$ for $k = 0$ and $k = 1$. What do you observe? (Check your answers with the table on the last page.)

A

A look at $z=0$ trace:



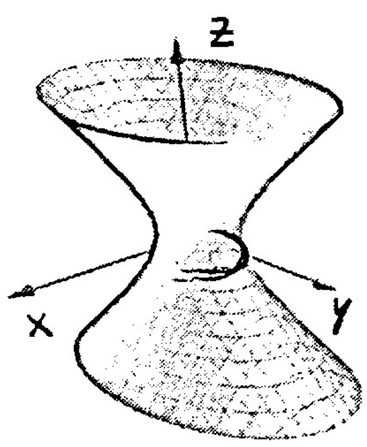
$x^2 + \frac{y^2}{3} = c$ has exactly one solution at $(0,0)$, so $0^2 + \frac{0^2}{3} = c \Rightarrow c=0$

$x=0$ trace: $\frac{y^2}{3} - \frac{z^2}{2} = 0 \to$ two lines

- $y = \pm\sqrt{\frac{3}{2}}\, z$

B

$x=1$ trace $\frac{y^2}{3} - \frac{z^2}{2} = -1 \to$ hyperbola

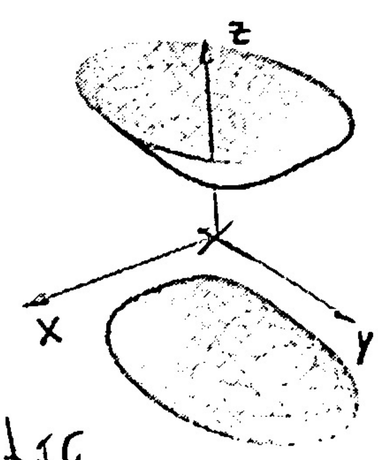


$z=0$ trace: $x^2 + \frac{y^3}{3} = c$ solutions are an ellipse, so $c>0$

$x=0$ trace: $\frac{y^2}{3} - \frac{z^2}{2} = c \to$ hyperbola

$x=1$ trace: $\frac{y^2}{3} - \frac{z^2}{2} = c-1 \to$ hyperbola

C



$z=0$ trace: NO SOLUTION, so $x^2 + \frac{y^3}{3} = c$

$\to c<0$.

$x=0$ trace: $\frac{y^2}{3} - \frac{z^2}{2} = c \to$ hyperbola

$x=1$ trace: $\frac{y^2}{3} - \frac{z^2}{2} = c-1 \to$ hyperbola

there are some subtleties regarding which axis the hyperbolas are symmetric about — think about this!

3. Identify the surfaces

(a) $3x^2 - 4y^2 + 12z^2 + 12 = 0 \longrightarrow -\frac{x^2}{4} + \frac{y^2}{3} - z^2 = 12$ hyperboloid of two sheets

[no solution at $y=0$ trace]

(b) $4x^2 - 4y + z^2 = 0$

by inspection

at $y=0$ trace we have a point

$4x^2 + z^2 = 0$

at $y=1$ trace e.g. we have an ellipse $4x^2 + z^2 = 4$

so this will be an elliptic paraboloid

4. Let $\vec{r}(t) = \left(\frac{t}{t-1}, \frac{\sin t}{t}, \ln(3-t)\right)$.

(a) Find the domain of $\vec{r}(t)$.

not defined at $t=1$

not defined at $t=0$

not defined for $t \geq 3$

So domain of $\vec{r}(t)$ is $t \in (-\infty, 0) \cup (0,1) \cup (1,3)$

(b) Find $\lim_{t\to 0} \vec{r}(t)$.

$$= \left(\lim_{t\to 0} \frac{t}{t-1}, \lim_{t\to 0} \frac{\sin t}{t}, \lim_{t\to 0} \ln(3-t)\right)$$

$$= \left(\frac{0}{0-1}, 1, \ln(3-0)\right)$$

$$= (0, 1, \ln(3))$$