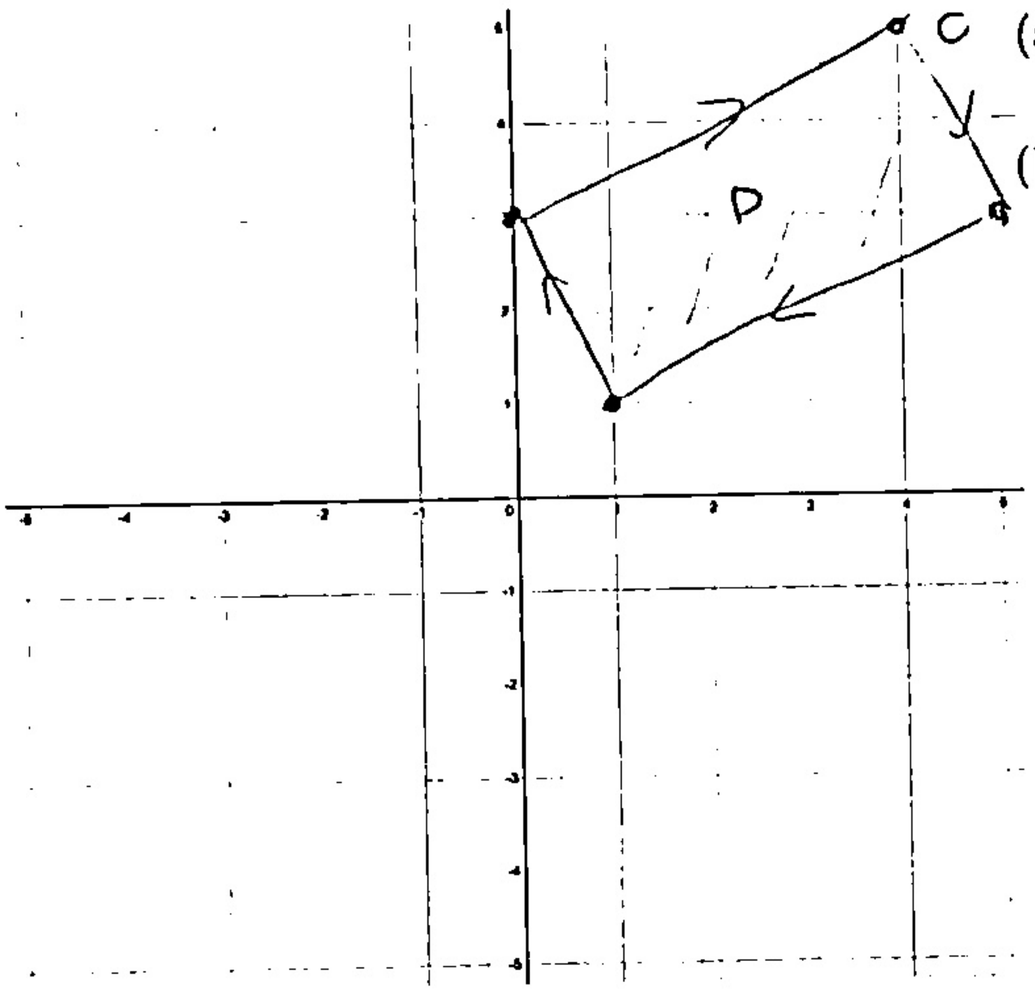
16.4 Green's Theorem

1. Let C be the boundary of the rectangular region D from vertices $(4,5)$ to $(5,3)$ to $(1,1)$ to $(0,3)$ and back to $(4,5)$.



(a) Sketch C in the coordinate plane to the left.

(b) Based on your picture, which of the following describe C? Check all that apply.

- Closed ☑ (loop)
- Simple ☑ (doesn't cross itself)
- Piecewise-smooth ☑ (union of 4 smooth curves)
- Positively oriented ☐

positive means counterclockwise

but we have a clockwise orientation!

(c) Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y) = \langle \overbrace{2y + \arcsin(\frac{x}{10})}^{P(x,y)}, \overbrace{3x - \arccos(\frac{y}{10})}^{Q(x,y)} \rangle$.

Can apply Green's Theorem for negatively oriented curve:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \Big[\overset{\frac{\partial P}{\partial y}}{(2)} - \overset{\frac{\partial Q}{\partial x}}{(3)}\Big]\, dA = -\underbrace{\iint_D dA}_{\text{area of } D} = \boxed{-10}$$

2. Let $\vec{G} = \langle P, Q\rangle$ be a continuous, conservative vector field defined on a smooth, simple, closed curve C in $\mathbb{R}^2$. Let g be a potential function for $\vec{G}$.

(a) Use the Fundamental Theorem for Line Integrals to find the value of $\oint_C \vec{G}\cdot d\vec{r}$. Explain your answer.

closed curve + conservative vector field mean $\boxed{\oint_C \vec{G}\cdot d\vec{r} = 0.}$

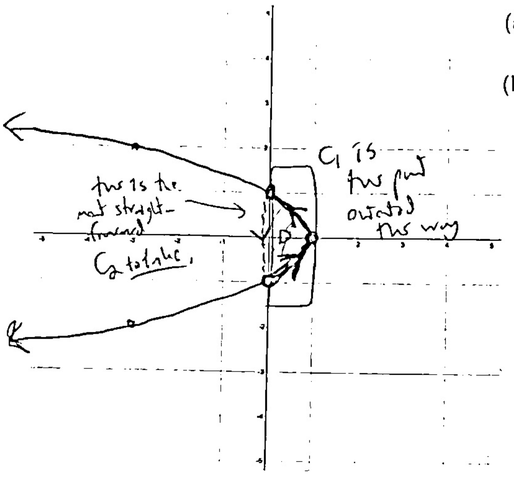
i.e. since endpoints of curve are the same point, call it a:

FT for Line Integrals: $g(a) - g(a) = 0$

(b) Use Green's Theorem to find the value of $\oint_C \vec{G}\cdot d\vec{r}$. Explain your answer.

since $\vec{G}$ conservative $\Rightarrow \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, so $\oint_C \vec{G}\cdot d\vec{r} = \boxed{\iint_D 0\, dA = 0}$ since integrand is 0.

3. Let $\vec{F}(x,y) = \langle x, y^2 \rangle$. Consider the curve C_1 being the arc of the parabola $x = 1 - y^2$ from $(0,-1)$ to $(0,1)$.



(a) Draw the given curve to show the direction is which the curve is traversed.

(b) Draw a curve/line C_2 such that $C_1 \cup C_2$ creates a simple, closed, piecewise-smooth and positively oriented curve. (Don't forget to include the arrows!)

C_2 = straight line from $(0,1)$ to $(0,-1)$

$$D = \left\{ \begin{array}{l} -1 \le y \le 1 \\ 0 \le x \le 1-y^2 \end{array} \right\}$$

(c) Using any methods you know find the value of $\int_{C_2} \vec{F} \cdot d\vec{r}$.

~~[illegible]~~ on C_2, $x = 0$ and $1 \ge y \ge -1$. So then

parametrization $r(y) = \langle 0, y \rangle$, $r'(y) = \langle 0, 1 \rangle$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_1^{-1} \langle (0), y^2 \rangle \cdot \langle 0, 1 \rangle \, dy = -\int_{-1}^{1} y^2 \, dy = -\left[\frac{1}{3}y^3\right]_{-1}^{1} = -\frac{2}{3}.$$

(c) Use Green's Theorem to find $\int_{C_1 \cup C_2} \vec{F} \cdot d\vec{r}$.

$$\int_{C_1 \cup C_2} \vec{F} \cdot d\vec{r} = \iint_D \frac{\partial}{\partial y}(x) - \frac{\partial}{\partial x}(y^2) \, dA = \iint_D 0 \, dA = 0$$

(c) Now can you figure out how to find $\int_{C_1} \vec{F} \cdot d\vec{r}$ by using the results from (c) and (d)? (This is a common method to use Green's theorem for curves that are not closed!).

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_1 \cup C_2} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} \quad \text{by linearity}$$

$$= (0) - \left(-\frac{2}{3}\right) = \frac{2}{3}.$$

[Ans. (c) $-\frac{2}{3}$, (e) $\frac{2}{3}$]

16.5 Curl and Divergence

scalar field

4. Let $f(x, y, z)$ and $\vec{F}(x, y, z)$ be a function of three variables and a vector field respectively. Pick the right answer for each of the following.

- ► div$f$ Meaningful ☐ Meaningless ☑
- ► grad$\vec{F}$ Meaningful ☐ Meaningless ☑
- ► div(grad$f$) Meaningful ☑ Meaningless ☐
- ► curl[(grad$f$) $\cdot \vec{F}$] Meaningful ☐ Meaningless ☑
- ► curl(curl(grad$f$)) Meaningful ☑ Meaningless ☐
- ► grad(div$\vec{F}$) Meaningful ☑ Meaningless ☐

analogy:
div ⟷ dot product
vector field ⟷ scalar field
curl ⟷ cross product
vector field ⟷ vector field

and could be
scalar field in → vector field out

5. Find the curl and div of the following vector fields.

(a) $\vec{F}(x, y, z) = \langle y^2x, x + y, z^3y\rangle$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(y^2x) + \frac{\partial}{\partial y}(x+y) + \frac{\partial}{\partial z}(z^3y) = \boxed{y^2 + 1 + 3z^2y}$$

$$\operatorname{curl}\vec{F} = \left\langle \frac{\partial}{\partial y}(z^3y) - \frac{\partial}{\partial z}(x+y), \frac{\partial}{\partial z}(y^2x) - \frac{\partial}{\partial x}(z^3y), \frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial y}(y^2x)\right\rangle$$

$$= \langle z^3 - 0, 0 - 0, 1 - 2yx\rangle = \boxed{\langle z^3, 0, 1-2yx\rangle}$$

(b) $\vec{F}(x, y, z) = \ln(y + 3x)\vec{i} + \ln(2y + z)\vec{j} + \ln(x + z)\vec{k}$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(\ln(y+3x)\right) + \frac{\partial}{\partial y}\left(\ln(2y+z)\right) + \frac{\partial}{\partial z}\left(\ln(x+z)\right)$$

$$= \boxed{\frac{3}{y+3x} + \frac{2}{2y+z} + \frac{1}{x+z}}$$

$$\operatorname{curl}\vec{F} = \left\langle \underbrace{\frac{\partial}{\partial y}\left(\ln(x+z)\right)}_{(0)} - \frac{\partial}{\partial z}\left(\ln(2y+z)\right), \underbrace{\frac{\partial}{\partial z}\left(\ln(y+3x)\right)}_{(0)} - \frac{\partial}{\partial x}\left(\ln(x+z)\right), \frac{\partial}{\partial x}\left(\ln(2y+z)\right) - \frac{\partial}{\partial y}\left(\ln(y+3x)\right)\right\rangle$$

$$= \boxed{\left\langle -\frac{1}{2y+z}, -\frac{1}{x+z}, -\frac{1}{y+3x}\right\rangle}$$