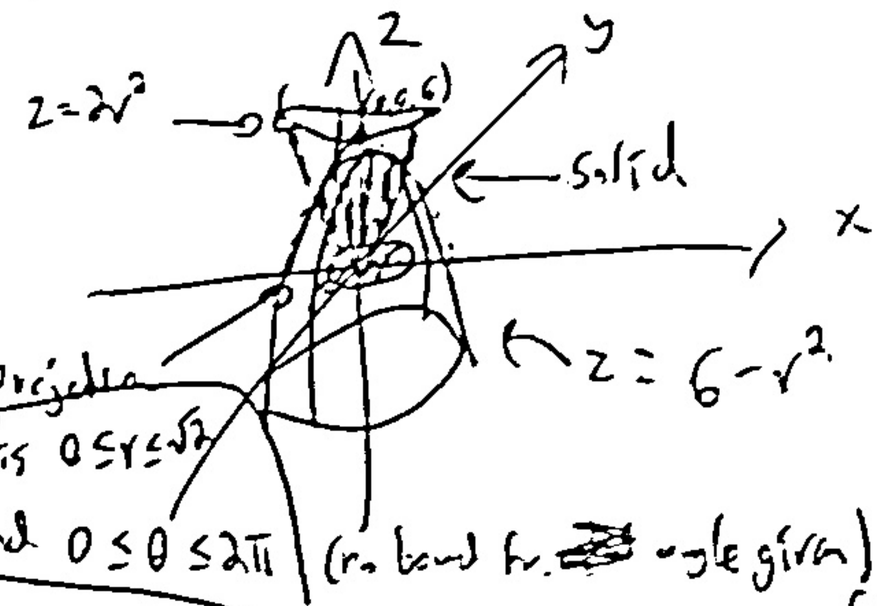
15.3 Double Integrals in Polar Coordinates

1. Use polar coordinates to find the volume of the solid bounded by the paraboloids $z = 6 - x^2 - y^2$ and $z = 2x^2 + 2y^2$. $\quad$ Intersection at $6 - r^2 = 2r^2 \Rightarrow r^2 = 2 \to r = \sqrt{2}$

   (a) Sketch the two paraboloids, the solid and its projection on the xy-plane.



this is D for (b)

   (b) Set up a double integral of the form $\iint_D f(x,y)\,dA$ that evaluates the volume of the solid and express its region in polar coordinates. $\to \iint_D (6 - x^2 - y^2) - (2x^2 + 2y^2)\,dA$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \left((6-r^2) - (2r^2)\right) r\,dr\,d\theta$$

remember this!

$$\iint_D 6 - 3x^2 - 3y^2\,dA$$

   (c) Rewrite the double integral from part (b) as an iterated integral using the polar parameters r and θ and evaluate it. (multiplying the r into the integrand)

$$3\int_0^{2\pi}\int_0^{\sqrt{2}} 2r - r^3\,dr\,d\theta$$

no dependence on θ

$$= 3 \cdot 2\pi \cdot \left[r^2 - \frac{1}{4}r^4\right]_0^{\sqrt{2}}$$

$$= 6\pi\left(2 - \frac{1}{4}\cdot 4\right) = \boxed{6\pi}$$

[Ans. 6π]

2. Use a double integral to find the area of the region *inside* the circle $(x-1)^2+y^2=1$ and *outside* the circle $x^2+y^2=1$.

Hint: 1

$(r\cos\theta - 1)^2 + (r\sin\theta)^2 = 1$

$r^2 - 2r\cos\theta + 1 = 1$

$-1 \quad -1$

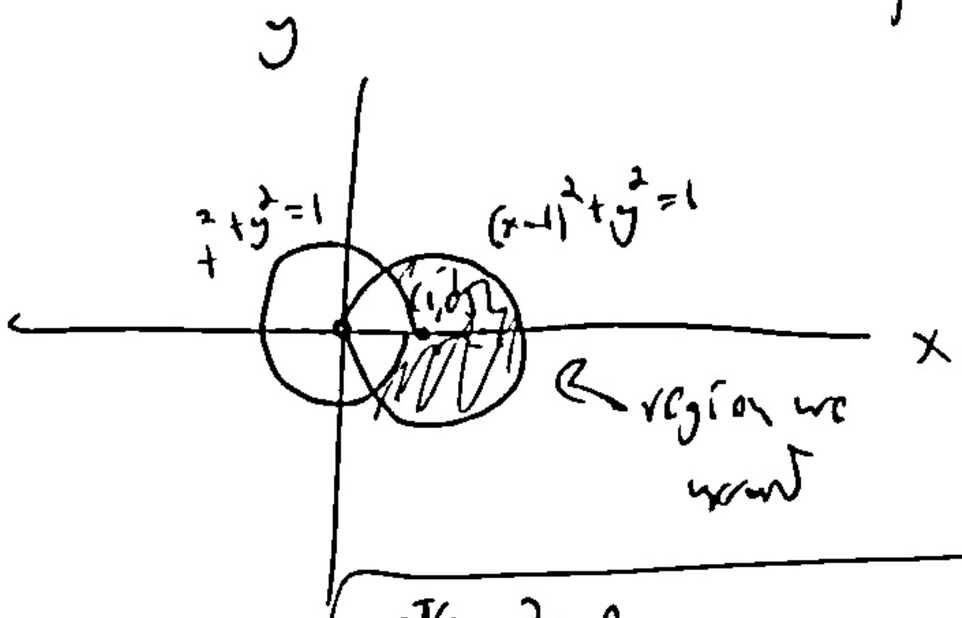
($r \neq 0$ on region so divide thru)

$r = +2\cos\theta$

is upper bound on r

(describes radius on off center circle)

and $r=1$ is lower bound



intersection
$1 = 2\cos\theta$
$\downarrow$
$\frac{1}{2} = \cos\theta$
so $\theta = -\pi/3$ lower bound
and $\theta = \pi/3$ upper bound

$$\int_{-\pi/3}^{\pi/3}\int_1^{2\cos\theta} r\,dr\,d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left[r^2\right]_1^{2\cos\theta} d\theta$$

$$= \frac{1}{2}\int_{-\pi/3}^{\pi/3} 4\cos^2\theta - 1$$

[Ans. $\frac{\pi}{3}+\frac{\sqrt{3}}{2}$]

$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} \cancel{4}^{2}\left(\frac{1+\cos(2\theta)}{\cancel{2}}\right) - 1 \; d\theta$$

$$= \frac{1}{2}\left[\frac{1}{2}\sin(2\theta) + \theta\right]_{-\pi/3}^{\pi/3}$$

$$= \frac{1}{2}\left(\left(\frac{1}{2}\frac{\sqrt{3}}{2} + \frac{\pi}{3}\right) - \left(\frac{1}{2}\cdot -\frac{\sqrt{3}}{2} - \frac{\pi}{3}\right)\right)$$

$$= \boxed{\frac{\sqrt{3}}{2} + \frac{\pi}{3}}$$

15.5 Surface Area

3. Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2+y^2=1$.

$r^2 = 1 \to r = 1$

$\cancel{z = (r\cos\theta)(r\sin\theta) = r^2\cos\theta\sin\theta}$

$\nabla z = \langle y, x\rangle$ factor

$\cancel{\nabla z(r,\theta) = \langle r\cos\theta\sin\theta, r^2[\cos^2\theta - \sin^2\theta]\rangle}$

$$\int_0^{2\pi}\int_0^1 \sqrt{1 + x^2 + y^2}\; r\,dr\,d\theta$$

needs to depend on r, θ

$$= \int_0^{2\pi}\underbrace{\int_0^1 r\sqrt{1+r^2}\,dr}_{\text{no dependence on } \theta}\,d\theta$$

$$\Downarrow$$

$$2\pi\cdot\left[\frac{\cancel{2}}{3}\cdot\frac{1}{\cancel{2}}(1+r^2)^{3/2}\right]_0^1 \to \boxed{\frac{2\pi}{3}(2\sqrt{2}-1)}$$

cancel power rule

cancel chain rule

[Ans. $\frac{2\pi}{3}(2\sqrt{2}-1)$]

not on quiz

15.6 Triple Integrals

4. Evaluate the iterated integral $\int_0^{\pi}\int_0^{1}\int_0^{\sqrt{1-z^2}} z\sin x\,dy\,dz\,dx$.

$$= \int_0^{\pi}\int_0^{1} (z\sin x)\left[y\right]_0^{\sqrt{1-z^2}}\,dz\,dx$$

$$= \int_0^{\pi}\int_0^{1} (\sin x)\, z\sqrt{1-z^2}\,dz\,dx$$

$$= \int_0^{\pi} (\sin x)\left[\frac{\cancel{2}}{3}\cdot\frac{-1}{\cancel{2}}(1-z^2)^{3/2}\right]_0^{1} dx$$

+ z=1 goes away

[Ans. $\frac{2}{3}$]

$$= \int_0^{\pi} \sin x\left(\frac{1}{3}\right) dx$$

$$= \frac{1}{3}\left[-\cos(x)\right]_0^{\pi}$$

$$= \frac{1}{3}\left(-(-1)-(-1)\right)$$

$$= \boxed{\frac{2}{3}}$$