Introduction

• Why should we care about stellar astrophysics?
  – stars are a major constituent of the visible universe
  – understanding how stars work is probably the earliest major triumph of astrophysics
  – stars are responsible for the chemical composition of the universe
  – mass and energy feedback (supernovae)
  – stellar astrophysics is the foundation for the astronomy and astrophysics in general
    * cosmic yardsticks to estimate the distances (e.g., Cepheid variables)
    * kinematic status of large scale structure (redshifts)
    * time scales and mechanisms for the formation of stellar populations and galaxies (stellar ages)

• Review the course syllabus
  A lot to cover. Emphasize on the physical principles, methodologies, and connections to what you might be interested in doing in your research.

• Approaches
  – lectures are mainly to provide a coherent thread of the course and to cover essential and/or difficult parts.
  – reading materials will be discussed and quizzed
  – homework is to consolidate and expand your understanding
  – Research and writing assignments to further expand and integrate your knowledge.

Scopes of this chapter:

• introduce some basic concepts and physical processes.
• estimate various time and physical scales of stars.
• paint a preliminary picture of the stellar interior.

To be expanded later to the modern development of stellar structure and evolution.
See next chapter for a narrative description of the development.
Assumptions: single isolated, spherical symmetric, and mechanically steady.
1 Basic Equations

What is a star? A steady shining, self-sustained, getting most energy from nuclear burning. How is the balance achieved in a star?

1.1 Mass conservation

Consider a shell with a thickness of $dr$, the mass included is

$$dM_r = 4\pi r^2 \rho dr.$$  

(1)

The integration of this equation gives the mass $M_r$ within the radius $r$. Since the radius of a star can change greatly during its lifetime, sometimes quickly, while the change in mass is relatively small and slow, it is convenient to use the $M_r$ as the coordinate; i.e., expressing various equations in the Lagrangian form.

1.2 Momentum Conservation and Hydrostatic Equilibrium

Stars are mostly steady, spending most of their lifetime converting H into He. Therefore, mostly various forces are balanced:

- gravity - only the mass within $r$ matters and acts as if it is at the center of a star (the gravitational potential is different, though)
- pressure - most importantly its differences between the inner and outer boundaries of a shell considered here.
- the equation of motion

$$\rho \ddot{r} = -\frac{dP}{dr} - \frac{GM_r \rho}{r^2},$$

(2)

- For a steady star, the balance of forces leads to the hydrostatic equilibrium:

$$\frac{dP}{dr} = \frac{GM_r \rho}{r^2},$$

(3)

which in the Lagrangian form is

$$\frac{dP}{dM_r} = \frac{GM_r}{4\pi r^3}.$$  

(4)

This form is especially useful when the stellar evolution (hence the structure change) is discussed.
1.3 Application example: constant-density model

We can now construct a simple model of a star with a constant density, which satisfies the above hydrostatic equilibrium equation.

First, we need to eliminate $r$. The constant density implies

$$ M_r = \frac{r^3}{R^3} M $$

Now, we have

$$ \frac{dP}{dM_r} = -\frac{GM^{4/3}}{4\pi R^4} M_r^{-1/3}. $$

Integrating this equation from the center to $M_r$ gives

$$ P - P_c = -\frac{3}{2} \frac{GM^{4/3}}{4\pi R^4} M_r^{2/3}. $$

Using the boundary condition: $P = 0$ at $M_r = M$, we have

$$ P_c = \frac{3}{2} \frac{GM^2}{4\pi R^4} $$

and

$$ P = P_c [1 - \left(\frac{M_r}{M}\right)^{2/3}] $$

This central pressure is a lower limit if density always decreases outward.

By using the equation of state (EoS) for a monatomic ideal gas, $P = n k T$, one can get the temperature structure.

But what is an ideal gas? It is composed of a set of randomly-moving, non-interacting point particles. We also need to know how to calculate $n$.

1.4 Molecular Weights

The thermodynamic relations between $P$, $\rho$, and $T$, as well as the calculation of stellar opacity requires knowledge of the systems mean molecular weight (defined as the mass per unit mole of material, or, alternatively, the mean mass of a particle in Atomic Mass Units). Recall that a mole of any substance contains $N_A = 6.0225210^{23}$ atoms. Thus, the number density of ions is related to the mass density, $\rho$, by

$$ n \equiv \frac{\rho}{\mu m_A} = n_i + n_e $$

where $m_A$ is the mass that is equivalent to 1 A.M.U.

$$ \frac{1}{\mu} \equiv \frac{1}{\mu_i} + \frac{1}{\mu_e} $$
If the mass fraction of species $i$ is $x_i$, then we have

$$\frac{1}{\mu_i} = \sum_i \frac{x_i}{A_i}$$  \hspace{1cm} (12)$$

$$\frac{1}{\mu_e} = \sum_i \frac{Z_i x_i f_i}{A_i}$$  \hspace{1cm} (13)$$

where $A_i$ is the atomic weight of the species, $Z_i$ be the atomic number of species $i$, and $f_i$ be the species ionization fraction, i.e., the fraction of electrons of $i$ that are free. Note that in the case of total ionization ($f_i = 1$), this equation simplifies greatly. Since $Z_i/A_i = 1$ for hydrogen, and $\sim 1/2$ for everything else,

$$\mu_e = \left( X + \frac{1}{2}(Y + Z) \right)^{-1} = \frac{2}{1 + X}$$  \hspace{1cm} (14)$$

**Comment:** $1/\mu_i \approx (X + Y/4)$.

For a “zero-age main sequence” star (ZAMS), $X \approx 0.7$, $Y = 0.3$, and $Z \approx 0.03$, $\mu_i = 1.3$, $\mu_e = 1.2$ and $\mu = 0.6$.

Using this value in the EoS, we can now get the temperature structure. In particular, the central temperature for the Sun can be estimated as

$$T_c = \frac{1}{2} \frac{GM\mu_m A}{kr} = 1.15 \times 10^7 \mu K$$  \hspace{1cm} (15)$$

To balance the gravity, a star must have a high pressure, which is realized with both high temperature and density. How to maintain such a high temperature? We need to consider both the energy generation and transportation.

### 1.5 Energy Generation

Neglecting terms due to the energy input or loss due to neutrinos and to gravitational expansion or contraction, we can express the energy equation as

$$\frac{dL_r}{dM_r} = \epsilon.$$  \hspace{1cm} (16)$$

Over some sufficiently restricted range of $T$, $\rho$, and composition, one may approximate $\epsilon$ in a power law form

$$\epsilon = \epsilon_0 \rho^\lambda T^\mu,$$  \hspace{1cm} (17)$$

where $\lambda = 1, 1, 2$ and $\mu \approx 4, 15,$ and $40$ for pp-chain, CNO-cycles, and triple-$\alpha$ modes, respectively.
For the conversion of hydrogen to Helium, about 0.7% of the rest mass energy is released, which is $6 \times 10^{18}$ ergs for energy gram of hydrogen consumed. With this energy efficiency, how long will sun last? (assuming that about 10% of the Sun’s mass will be fused and 70% of it is hydrogen, $t \sim 10^{10}$ yrs).

### 1.6 Energy Transportation

Now, what determines the luminosity of a star? The energy leak makes a star shine. But this leakage must be slow to maintain a star steady. A feedback mechanism is needed, like a thermometer. The energy generation also needs to be balanced by energy removal, but not too fast. Then we call the material is in “thermal balance”.

Three major modes of energy transportation: radiation (photon) transfer, convection of hotter and cooler materials, and heat conduction (only important under degenerate condition, i.e., in white dwarfs).

For the time being we consider the radiation transfer or diffusion.

Consider a system of particles diffusing across a boundary in the $z$ direction. If the motions of the particles are isotropic, on average, about 1/3 of the particles will have their motion primarily in the $z$ direction and about 1/2 of them moving toward the boundary, as opposed to away from it. The net number flux of particles diffuse across the boundary is

$$F \approx \frac{1}{6} \bar{v} [n_{z-l} - n_{z+l}] = -\frac{1}{3} vl \frac{dn}{dz}, \quad (18)$$

where $l \equiv 1/n\sigma$ is the particle mean free path, while $\sigma$ is the cross section of the collision. We have the Fick’s law of diffusion:

$$F = -D \frac{dn}{dz}, \quad (19)$$

where $D = \frac{\bar{v}}{3n\sigma}$ is the diffusion coefficient.

Similarly, we can compute the energy flux of radiative energy across a boundary. Treat the photons as particles and recall the energy density $U = aT^4$ and $l_{ph} = 1/\kappa\rho$, where $\kappa$ is the absorption coefficient. Thus, the energy transportation equation is

$$F = -\frac{cl_{ph}}{3} \frac{dT}{dz} = -\frac{4caT^3}{3\kappa\rho}dTdz, \quad (20)$$

or

$$L_r = -\frac{(4\pi r^2)^2 4caT^3}{3\kappa} \frac{dT}{dM_r}, \quad (21)$$

where $a = 4\sigma_s/c$ and $\sigma_s = 5.7 \times 10^{-5}$ cgs is the Stefan-Boltzmann’s constant.

The calculation of the opacity is a whole industry. Here we use a generic opacity form
\[ \kappa = \kappa_0 \rho n T^{-s} \text{ cm}^2 \text{ g}^{-1}. \] (22)

For the Thompson scattering in an ionized medium, \( n = 0 \) and \( s = 0 \), where \( n = 1 \) and \( s = 3.5 \) for Kramers’ opacity, characteristic of radiative processes involving atoms.

In summary, under the steady and spherical assumptions, we have describe the basic equations (mass, momentum, energy, and heat transfer). These equations, together with an equation of the state as well as the energy generation and opacity forms, allow for the construction of a complete stellar model (e.g., solving for dependent variables: \( r, \rho, T, \) and \( L \) as function of \( M_r \)).

While the above outlines the general approach to construct stellar models, we will have in-depth look of the physics of the EoS, the heat transfer, and energy generation, before showing how the above equations can actually be solved. But we may also get some ideas about various scales and dependencies among various stellar properties, based on some simple but very useful techniques. To demonstrate this, let us get familiar with what we need to understand.

## 2 The HR diagram

Please draw a HR diagram, both theorist’s and observer’s versions, and explain the axes, as well as directions of stellar radius and mass).

An important way to characterize the properties of stars: power output vs. temperature or equivalent. The exact units of the axes depend on the context and who present them. For historic reason, the temperature axis has the highest value on the left.

As the radiation from a star photosphere is close to a black-body, one can define \( T_{\text{eff}} \) as

\[ T_{\text{eff}} \equiv \left[ \frac{L}{4 \pi \sigma_s R^2} \right]^{1/4}, \] (23)

where \( R \) is the radius of the photosphere.

The \( L - T_{\text{eff}} \) diagram itself gives no further information than \( L, T_{\text{eff}}, \) and \( R \) and says nothing about mass, composition, or state of evolution. But these may be inferred from the distribution of stars in the diagram, plus the modeling.

Show an example of the diagram and point out the main sequence, giants, white dwarf etc. Density of stars depends on the lifetime of individual processes that govern the evolution at different stages.
3 Virial Theorem

It is easy (though tedious) to derive the so-called Virial Theorem based on Newton’s 2nd law:
\[
\frac{1}{2} \frac{d^2I}{dt^2} = 2K + \Omega,
\] (24)

where the momentum of inertia \( I = \sum_i m_i r_i^2 \), the kinetic energy \( K = \frac{1}{2} \sum_i m_i v_i^2 \), and the potential energy \( \Omega = -\sum \frac{Gm_im_j}{r_{ij}} \). For a gaseous system (e.g., a star), \( K = \int_M \frac{3P}{2\rho}dM_r \) and \( \Omega = -\int_M \frac{GM_r}{r}dM_r \). The system is called to be virialized, if the r.h.s of Eq. 24, averaged over time, is equal to zero (i.e., statistically does not change with time).

Comment: How to get \( R, M, \) and age of a star? In the HR diagram?

Note that the above expression of the theorem applies to the entire system (e.g., the entire star). For a portion of a star, one needs to account for external force (e.g., the pressure at the boundary).

Comment: This section may be given right before § 1.5 at the beginning of 2nd class to allow for dimension analysis of \( T_c \) and various times scales such as KH time scale and nuclear burning time scale, etc. The \( K \) and \( \Omega \) gas forms can also be derived: from \( d\Omega = -\int F(dM_r)dr \) and \( \frac{1}{2} < m_i v_i^2 > = 3/2kT \) and \( \Sigma_i = \int dM_r/(\mu m_i) \)

4 Various scale estimates

contraction and nuclear and their time scales. What is the dynamic scale? We can use simple dimensional analysis and central temperature estimate.

5 Stellar Dimensional Analysis

This is a very useful technique, not only for the study of stars, but for other astrophysical problems. Without getting all the solutions, such analysis can provide insights into how physical quantities depend on each other and scaled.

Consider the four equations that we have discussed to model the stellar structure of stars, excluding all the multiplicative constants for simplicity:
\[
\frac{dr}{dM_r} \propto \frac{1}{r^2 \rho},
\] (25)
\[
\frac{dP}{dM_r} \propto -\frac{M_r}{r^4},
\] (26)


\[
\frac{dL_r}{dM_r} \propto \rho^\lambda T^\mu, \tag{27}
\]

\[
L_r \propto -\frac{r^4 T^3}{\rho^n T^{-s}} \frac{dT}{dM_r}, \tag{28}
\]

assuming the radiation transfer dominates. We have also implicitly assumed that the chemical composition is uniform in each star and same for stars of different masses.

These four equations, together with an equation of state, assumed to be \( P \propto \rho^\lambda T^\mu \), and appropriate boundary conditions, allow one to solve for \( r, \rho, L_r \), and \( T \) as functions of \( M_r \). When all the constants and exponents are assumed to be the same, solutions from one star to another with different masses are scalable. This family of stars is said to be in a homologous sequence.

To show this, we do the following variable transformation in the above equations:

\[
r = r_0 M^{\alpha_r}, \tag{29}
\]

\[
\rho = \rho_0 M^{\alpha_r}, \tag{30}
\]

\[
L_r = L_{r_0} M^{\alpha_L}, \tag{31}
\]

\[
T = T_0 M^{\alpha_T}, \tag{32}
\]

and

\[
M_r = M_{r_0} M, \tag{33}
\]

where \( M \) is any constant.

We then obtain, for example,

\[
M^{\alpha_r-1} \frac{d r_0}{dM_{r_0}} \propto M^{-2\alpha_r-\alpha_\rho} \frac{1}{r_0^2 \rho_0^{\alpha_\rho}}, \tag{34}
\]

If we have \( \alpha_r - 1 = -2\alpha_r - \alpha_\rho \) or \( 3\alpha_r + \alpha_\rho = 1 \), the above two mass equations are the exactly the same. Similarly, we have

\[
4\alpha_r + \chi_\rho \alpha_\rho + \chi_T \alpha_T = 2,
\]

\[
\lambda \alpha_\rho - \alpha_L + \nu \alpha_T = -1,
\]

\[
4\alpha_r - n \alpha_\rho - \alpha_L + (4 + s) \alpha_T = 1
\]

This set of linear equations can be solved (if the determinant is not zero) to get the \( \alpha_\ast \) values.

For example, consider upper MS stars as a homologous sequence. For these stars, the opacity is mostly due to electron scattering (i.e., \( n = s = 0 \)) and the nuclear reaction is due to CNO cycles (i.e., \( \lambda = 1 \) and \( \mu = 15 \)). The pressure is still dominated by the ideal gas law (i.e., \( \chi_\rho = \chi_T = 1 \)). Although the inner region of these stars are convective, the radiative transport dominates in the outer regions. The inferred \( \alpha_r = 0.78 \) and \( \alpha_L = 3.0 \) may be compared with the empirical fitted values of 0.75 and 3.5, respectively. In addition, we get \( \alpha_T = 0.22 \) and
\( \alpha_\rho = -1.3 \) so that \( T \) \textbf{should increase with the mass whereas} \( \rho \) \textbf{decreases}. This is indeed what happens.

For lower MS stars, the physics are different (different \( \lambda, \mu, \chi_\rho, \) and \( \chi_T \)). The convection is also more important. The homology does not work as well.


6 Review

You should be able to derive the mass conservation, hydrostatic equilibrium, energy, and heat transfer equations (sort of from the 1st principle).

How is the hydrostatic balance achieved in a star?

What is an ideal gas?

What is the molecular weight (which you should be able to derive, depending on the chemical composition of the gas)?

Now what is the lifetime of the Sun with the nuclear power?

What is the virial theorem?

Can you have a quick estimate of the central temperature of the Sun, based on a simple dimensional analysis? How may the temperature depend on the average density and mass of a star (assuming ideal gas)?

Similarly, can you estimate the lifetime of the Sun if it were powered by the gravitational energy alone?

What is the dynamic scale of the Sun?

Can you derive an expression of the gravitational potential for a star (assuming spherical symmetry)?

What is the H-R diagram?

Point out the directions of increasing temperature, luminosity, mass, and radius.

Mark the locations of various types of stars you know.

Trace evolutionary tracks of stars.

How do the central density and temperature of a main-sequence star depend on its mass, qualitatively?
7 Reading assignment

HKT Chapters 1 and 2