Stellar Modeling

1 The Equations of Stellar Structure

We consider the modeling of stars in hydrostatic and thermal equilibrium (i.e., time-dependent processes are ignored for now). From the modeling, we hope to know how such global properties as the luminosity and radius of a star depend on its mass and initial chemical composition. To do so, we need to solve a set of four differential equations as we discussed in Chapter 1, which are equivalent to a fourth-order differential equation:

The mass conservation

$$\frac{dr}{dM_r} = \frac{1}{4\pi r^2 \rho}, \tag{1}$$

hydrostatic equation

$$\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}, \tag{2}$$

energy equation

$$\frac{dL_r}{dM_r} = \epsilon, \tag{3}$$

and the heat transfer method (radiative, conduction, and/or convection), or the form of the model

$$\nabla \equiv \frac{d\ln T}{d\ln P}. \tag{4}$$

To implement the specific heat transfer, compute

$$\nabla_{rad} = \frac{3}{16\pi ac} \frac{P\kappa}{T^4} \frac{L_r}{GM_r}, \tag{5}$$

which assumes that the transfer is all due to the radiation (the conduction is neglected here). Then we may set

$$\nabla = \nabla_{rad} \quad \text{if} \quad \nabla_{rad} \leq \nabla_{ad} \tag{6}$$

for pure diffusive radiative transfer or conduction, or

$$\nabla = \nabla_{ad} \quad \text{if} \quad \nabla_{rad} > \nabla_{ad} \tag{7}$$

when adiabatic convection is present locally — as in a mixing length theory.

Four boundary conditions are required to close the system. For simplicity, we choose "zero" conditions which are $r = L_r = 0$ at the center ($M_r = 0$), and $\rho = T = 0$ at the surface ($M_r = M$). Here M was specified beforehand.

Of course, we assume that we already know the microscopic constituent physics, as we have discussed; i.e., the quantities P, E, κ, and ϵ as functions of ρ, T, and X, where X is shorthand for composition. These quantities should be available on demand either in analytic and numerical forms.

It should be noted that the availability of the quantities does not guarantee that a solution to the equations always exists or unique!

Before going into how the above stellar structure equations are solved in practice, we first consider a simplified case, which is both useful practically and illumining.

2 Polytropic Equations of State and Polytropes

We define a polytropic stellar model (or polytrope) to be one in which the pressure is given by

$$P(r) = K\rho^{1+1/n}(r) \tag{8}$$

where both the *polytropic index* n and K are constant. This model allows us to avoid dealing with both the heat transfer and thermal balance.

We have encountered such power law before; e.g., the EoS for a zero temperature, completely degenerate electron gas ($P_e = 1.0 \times 10^{13}(\frac{\rho}{\mu_e})^{5/3}$ dyn cm^{-2} if non-relativistic). The polytrope is also a good approximation for certain types of adiabatic convection zones. For a region with efficient convection, i.e., $\nabla = \nabla_{ad} = \left(\frac{\partial \ln T}{\partial \ln P}\right)_{ad} = 1 - 1/\Gamma_2$. If Γ_2 is assumed constant, then

$$P(r) \propto T^{\Gamma_2/(\Gamma_2-1)}(r). \tag{9}$$

If in addition, the gas is idea, then $P(r) \propto \rho^{\Gamma_2}(r)$.

For a polytrope, we can derive from the hydrostatic and mass conservation equations (in Euclidean coordinates) the following

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -\frac{G}{r^2}\frac{dM_r}{dr} = -4\pi G\rho \tag{10}$$

Now perform the transformations to make the the equation dimensionless: $\rho(r) = \rho_c\theta^n(r)$ and $r = r_n\xi$, we then have the *Lane-Emden equation*:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta_n}{d\xi}\right) = -\theta_n^n \tag{11}$$

where P_c is defined (from the EoS) as $P_c = K\rho_c^{1+1/n}$ and $r_n^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2}$. The solutions are called "Lane-Emden solution" and denoted by $\theta_n(\xi)$.

Note that if the EoS for the model material is an ideal gas with constant μ, θ_n measures temperature $T(t) = T_c\theta_n(r)$, where $T_c = K\rho_c^{1/n}(N_A k/\mu)^{-1}$.

Since the *Lane-Emden equation* is a second-order differential equation, we need two real boundary conditions: First for ρ_c to really be the central density, we require that $\theta_n(\xi = 0) = 1$; Second, the spherical symmetry at the center (dP/dr vanishing at $r = 0$) requires that $\theta_n'(\xi = 0) = 0$, with the resulting regular solutions called "E-solutions" (the abandoned divergent solutions at center may be used in part of a star, however).

The surface of a model star is where the first zero of θ_n occurs, $\theta_n(\xi_1) = 0$, where ξ_1 is the location of the surface. This interpretation of the solution is not a boundary condition.

Analytical E-solutions for θ_n are obtainable for $n = 0, 1,$ and 5, as given in the textbook. For example, for $n = 0$, $\rho(r) = \rho_c$. This constant-density sphere has the solution (when the boundary conditions at $\xi = 0$ is used)

$$\theta_0(\xi) = 1 - \frac{\xi^2}{6}. \tag{12}$$

Clearly, from the outer boundary condition, $\xi_1 = 6^{1/2}$.

Numerical methods must be used for general n. So given n and K, we can in principle find the dependence of P and ρ on ξ. However, to get the absolute physical numbers, we need $R = r_n\xi_1$, which depends on ρ_c, as shown above. These two parameters are linked by the stellar mass, which we wish to specify via

$$\begin{aligned} M &= \int_0^R 4\pi r^2 \rho(r) dr \\ &= 4\pi r_n^3 \rho_c \int_0^{\xi_1} \xi^2 \theta_n^n d\xi \\ &= 4\pi r_n^3 \rho_c (-\xi^2 \theta_n')_{\xi_1} \end{aligned} \tag{13}$$

For a given n, $(-\xi^2\theta_n')_{\xi_1}$ is known, the above equation then gives ρ_c, and hence P_c, in terms of M.

A useful quantity that depends only on n is the ratio

$$\frac{\rho_c}{< \rho >} = \frac{1}{3}\left(\frac{\xi}{-\theta_n'}\right)_{\xi_1} \tag{14}$$

For the pressure of a completely degenerate, but non-relativistic electron gas ($\propto \rho^{5/3}$), $n = 1.5$. For the completely degenerate and fully relativistic case, $n = 3$. For an ideal gas convection zone ($\propto \rho^{5/3}$), $n = 1.5$. Unfortunately, neither of these values have analytic E-functions.

3 Numerical calculation of the Lane-Emden equation

A convenient way to numerically solve such an equation is to cast the second-order problem in the form of two first-order equations by introducing the new variables $x = \xi$, $y = \theta$, and $z = (d\theta_n/d\xi) = (dy/dx)$:

$$\begin{aligned} y' &= \frac{dy}{dx} = z \\ z' &= \frac{dz}{dx} = -y^n - \frac{2}{x} z \end{aligned} \tag{15}$$

Here we use a simple "shooting method", whereby one "shoots" from a starting point and hopes that the shot will end up at the right place; e.g., using a "Runge-Kutta" integrator. The solution is "leap-frogged" from x to $x+h$, where h is called the "step size". Suppose we know the values of y and z at some point x_i and call these values y_i and z_i. If h is some carefully chosen, then we can use the above equations to find y_{i+1} and z_{i+1} at $x_{i+1} = x_i + h$.

Care needs to be taken at the origin, where z' is indeterminate because both x and z are equal to zero. The resolution to this problem is to expand $\theta_n(\xi)$ in the Lane-Emden equation in a series about the origin. Inserting $\theta_n(\xi) = a_0 + a_1\xi + a_2\xi^2...$ into the equation, compare the coefficients of individual ξ terms, and apply the boundary condition to establish the constants in the expansion, we get

$$\theta_n(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4... \tag{16}$$

For $\xi \to 0$, find that $z' \to -1/3$, which may be used to start the integration.

This way, the calculation may march from the origin to the surface, when $y = \theta_n$ cross the zero.

4 Newton-Raphson and Henyey Methods

A more powerful technique to solve the stellar structure equations is the "integration" over the model, instead of shooting from one point to another.

Consider a second-order system as an example

$$\begin{aligned} \frac{dy}{dx} &= f(x, y, z) \\ \frac{dz}{dx} &= g(x, y, z) \end{aligned} \tag{17}$$

with boundary conditions on y and z specified at the endpoints of the interval $x_1 \leq x \leq x_N$ and may be generally expressed as

$$\begin{aligned} b_1(x_1, y_1, z_1) &= 0 \\ b_N(x_N, y_N, z_N) &= 0 \end{aligned} \tag{18}$$

where y_i and z_i are $y(x_i)$ and $z(x_i)$.

Assuming that f, g, b_1, and b_2 are well behaved, the differential equations can be cast in a "finite difference form" over a predetermined "mesh" in x; i.e., $x_1, x_2, ..., x_N$ at which y and z are to be evaluated. For simplicity, consider that the mesh interval is constant; i.e., $x_{i+1} - x_i = \Delta x$ for all i. The equations can hen be expressed as

$$\begin{aligned}\frac{y_{i+1} - y_i}{\Delta x} &= \frac{1}{2}(f_{i+1} + f_i)\\ \frac{z_{i+1} - z_i}{\Delta x} &= \frac{1}{2}(g_{i+1} + g_i)\end{aligned} \tag{19}$$

where f_i, for example, is shorthand for the function $f(x_i, y_i, z_i)$. The above expressions then represent $2N - 2$ equations, which together with the two boundary conditions can in principle be used to solve $2N$ variables y_i and z_i. Thus, this is *not* an initial value problem. However, the difficulty is that the variables are all mixed up among the equations, generally in a nonlinear fashion.

4.1 Newton-Raphson Method

One way to get out of this difficulty is to use the Newton-Raphson method: find the solution by linearized the equations and the boundary conditions.

Suppose that we have a "guessed" solution (e.g., from the shooting method) that gives y_i and z_i for all i, which generally do not satisfy the equations. we may make corrections

$$\begin{aligned}y_i &\to y_i + \Delta y_i\\ z_i &\to z_i + \Delta z_i\end{aligned} \tag{20}$$

so that the new y_i and z_i might satisfy both the equations and the boundary conditions. We now *estimate* the values of Δy_i and Δz_i for all i by letting the new y_i and z_i satisfy the linearized equations and boundary conditions.

For example, the first equation becomes

$$\begin{aligned} y_{i+1} + \Delta y_{i+1} - y_i - \Delta y_i = \\ \frac{\Delta x}{2}\left[f_{i+1} + \left(\frac{\partial f}{\partial y}\right)_{i+1} \Delta y_{i+1} + \left(\frac{\partial f}{\partial z}\right)_{i+1} \Delta z_{i+1}\right] \\ + \frac{\Delta x}{2}\left[f_i + \left(\frac{\partial f}{\partial y}\right)_i \Delta y_i + \left(\frac{\partial f}{\partial z}\right)_i \Delta z_i\right] \end{aligned} \tag{21}$$

Some manipulation leads to

$$\begin{aligned} y_{i+1} - y_i - \frac{\Delta x}{2}(f_{i+1} + f_i) = \\ \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_i + 1\right] \Delta y_i + \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_{i+1} - 1\right] \Delta y_{i+1} \\ + \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_i\right] \Delta z_i + \left[\frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_{i+1}\right] \Delta z_{i+1} \end{aligned} \tag{22}$$

Note that the left-hand side of these equations are zero when the difference equations are satisfied; that is when Δy_i and Δz_i go to zero. Similarly, the boundary conditions can be linearized into

$$b_{(1 \text{ or } N)} + \left(\frac{\partial b}{\partial y}\right)_{(1 \text{ or } N)} \Delta y_{(1 \text{ or } N)} + \left(\frac{\partial b}{\partial z}\right)_{(1 \text{ or } N)} \Delta z_{(1 \text{ or } N)} = 0. \tag{23}$$

We can arrange all these equations in a matrix form

$$\mathbf{M} \cdot \mathbf{U} = \mathbf{R}. \tag{24}$$

in which,

$$\mathbf{U} \equiv (\Delta y_1, \Delta z_1, \Delta y_2, \Delta z_2, \cdots, \Delta y_N, \Delta z_N)^{\mathbf{T}} \tag{25}$$

where the superscript "T" indicates transpose;

$$\mathbf{R} = (-b_1, Y_{3/2}, Z_{3/2}, \cdots, Y_{N-1/2}, Z_{N-1/2}, -b_N)^{\mathbf{T}} \tag{26}$$

where

$$\begin{aligned} Y_{i+1/2} &\equiv y_{i+1} - y_i - \frac{\Delta x}{2}(f_{i+1} + f_i) \\ Z_{i+1/2} &\equiv z_{i+1} - z_i - \frac{\Delta x}{2}(g_{i+1} + g_i); \end{aligned} \tag{27}$$

and finally

$$\left|\begin{array}{cccccccc}
\left(\frac{\partial b}{\partial y}\right)_1 & \left(\frac{\partial b}{\partial z}\right)_1 & 0 & 0 & 0 & 0 & \cdots & \cdots \\
A_1+1 & B_1 & A_2-1 & B_2 & 0 & 0 & \cdots & \cdots \\
C_1 & D_1+1 & C_2 & D_2-1 & 0 & 0 & \cdots & \cdots \\
0 & 0 & A_2+1 & B_2 & A_3-1 & B_3 & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & \cdots & \cdots & \left(\frac{\partial b}{\partial y}\right)_N & \left(\frac{\partial b}{\partial z}\right)_N
\end{array}\right| \tag{28}$$

where

$$\begin{aligned}
A_i &\equiv \frac{\Delta x}{2}\left(\frac{\partial f}{\partial y}\right)_i, \quad C_i \equiv \frac{\Delta x}{2}\left(\frac{\partial g}{\partial y}\right)_i \\
B_i &\equiv \frac{\Delta x}{2}\left(\frac{\partial f}{\partial z}\right)_i, \quad D_i \equiv \frac{\Delta x}{2}\left(\frac{\partial g}{\partial z}\right)_i .
\end{aligned} \tag{29}$$

Once the solution set $\mathbf{U}$ is found, then new values of y_i and z_i are obtained by adding Δy_i and Δz_i to the corresponding old guesses. If all goes well, then the corrections decrease as the square of their absolute values. We iterate the above procedure until Δy_i and Δz_i become sufficiently small.

4.2 Eigenvalue Problems and the Henyey Method

However the above scheme needs to be slightly modified if the location of the boundary is not known before-hand. In the polytropic stellar structure we want to get, the radius ξ_1 needs to be found as part of the solution. To do so, we do a simple conversion,

$$x \to x = \xi/\lambda \tag{30}$$

where $\lambda = \xi_1$. After this conversion, x is within the closed interval $[0, 1]$ and can be divided into a grid, upon which the difference equations and boundary conditions can be applied. The converted f and g are generally also depend on λ; e.g., the Lane-Emden equation becomes

$$\begin{aligned} y' &= \frac{dy}{dx} = \lambda z \\ z' &= \frac{dz}{dx} = -\lambda y^n - \frac{2}{x} z. \end{aligned} \tag{31}$$

Now we have one more parameter λ, or an *eigenvalue*, to determine. This can be done since we have one more boundary condition, $y = 0$ at the new $x = 1$, in addition to the two at the center $(y = 1, z = 0)$.

To get the solution, we just need to add the correction $\lambda \to \lambda + \Delta\lambda$ in the lionization of f, g, and the three boundary conditions; e.g.,

$$f \to f + \left(\frac{\partial f}{\partial y}\right)_{(z_i,\lambda)} \Delta y_i + \left(\frac{\partial f}{\partial z}\right)_{(y_i,\lambda)} \Delta z_i + \left(\frac{\partial f}{\partial \lambda}\right)_{(y_i,z_i)} \Delta\lambda. \tag{32}$$

Each difference equation thus contributes terms in $\Delta\lambda$ to the matrix algebra problem; i.e., there is now an extra row in the main matrix corresponding to the additional boundary condition and an extra column for the extra unknown, $\Delta\lambda$.

5 The Eddington Standard Model

This gives a simple example of the use of polytropes in making a stellar pseudo-model, which approximately incorporates the energy and radiative transfer equations.

Recall that in case of no convection the radiative transfer equation can be expressed as

$$\nabla = \frac{3}{16\pi ac} \frac{P\kappa}{T^4} \frac{L_r}{GM_r}, \tag{33}$$

where

$$\nabla \equiv \frac{d\ln T}{d\ln P} = \frac{1}{4} \frac{P}{P_{rad}} \frac{dP_{rad}}{dP}. \tag{34}$$

Here we have introduced the radiative pressure $P_{rad} = aT^4/3$ so that T can be replaced to get

$$\frac{dP_{rad}}{dP} = \frac{\kappa L_r}{4\pi c G M_r}. \tag{35}$$

We define

$$< \varepsilon(r) > \equiv \frac{L_r}{M_r} = \frac{\int_0^r \varepsilon dM_r}{\int_0^r dM_r} \tag{36}$$

where the energy equation $dL_r/dM_r = \varepsilon$ is used. We further define

$$< \eta(r) > \equiv \frac{< \varepsilon(r) >}{< \varepsilon(R) >} = \frac{L_r/M_r}{L/M}. \tag{37}$$

The transfer equation then becomes

$$\frac{dP_{rad}}{dP} = \frac{L}{4\pi cGM}\kappa(r)\eta(r). \tag{38}$$

Assuming that the surface pressure is equal to zero, the integration of the above equation gives

$$P_{rad}(r) = \frac{L}{4\pi cGM} < \kappa(r)\eta(r) > P(r) \tag{39}$$

where the average expression is

$$< \kappa(r)\eta(r) > = \frac{1}{P(r)} \int_0^{P(r)} \kappa\eta dP. \tag{40}$$

If we can assume that $< \kappa(r)\eta(r) >$ varies weakly with position in a star, or close to a constant, as Eddington did, then the ratio of $1 - \beta \equiv P_{rad}/P$ is a constant and so is β. This constancy may be translated into a T vs. ρ relation as follows. If the pressure is contributed by ideal gas plus radiation only, then

$$P_{rad} = P - P_{gas} = (1/\beta - 1)P_{gas} = \frac{1-\beta}{\beta}\frac{N_A k}{\mu}\rho T = aT^4/3 \tag{41}$$

$$T(r) = \left(\frac{1-\beta}{\beta}\frac{3}{a}\frac{N_A k}{\mu}\right)^{1/3} \rho^{1/3}(r). \tag{42}$$

$$P = \frac{P_{gas}}{\beta} = \frac{N_A k}{\mu}\frac{\rho T}{\beta} = K\rho^{4/3}(r), \tag{43}$$

where

$$K = \left[\frac{1-\beta}{\beta^4} \frac{3}{a} \left(\frac{N_A k}{\mu} \right)^4 \right]^{1/3} . \tag{44}$$

So we have a polytrope with $n = 3$, which can be readily solved numerically.

6 Review

Key concepts: Polytropes, Lane-Emden equation, Newton-Raphson and Henyey Methods, the Eddington Standard Model

What are the basic equations and the boundary conditions that are needed to construct a normal stellar interior model? What microscopic physics should be implemented in such a modeling?

Please give two examples of the situations in which a polytropic equation of state may be used?

What is the basic approach of the Newton-Raphson or Henyey Method in solving the stellar equations? How is it different from a simply "shooting method"?

What is the key assumption made in the Eddington Standard Model? Why may this assumption be reasonable?