Russell's Logicism

Kevin C. Klement, University of Massachusetts Amherst

klement@philos.umass.edu

Draft of March 20, 2016; do not cite.

1 Introduction

Logicism is typically defined as the thesis that mathematics reduces to, or is an extension of, logic. Exactly what "reduces" means here is not always made entirely clear. (More definite articulations of logicism are explored in section 5 below.) While something like this thesis had been articulated by others (e.g., Dedekind 1888 and arguably Leibniz 1666), logicism only became a widespread subject of intellectual study when serious attempts began to be made to provide complete deductions of the most important principles of mathematics from purely logical foundations. This became possible only with the development of modern quantifier logic, which went hand in hand with these attempts. Gottlob Frege announced such a project in his 1884 *Grundlagen der Arithmetik* (translated as Frege 1950), and attempted to carry it out in his 1893–1902 *Grundgesetze der Arithmetik* (translated as Frege 2013). Frege limited his logicism to arithmetic, however, and it turned out that his logical foundation was inconsistent.

Working at first in ignorance of Frege, Bertrand Russell's interests in the fundamental principles of mathematics date back to the late 1890s. He published *An Essay on the Foundations of Geometry* in 1897 and soon thereafter began work on

the nature and basis of arithmetic.[1] He found his work hampered somewhat by a mismatch between his earlier generally Kantian views on mathematics and the staunch realist metaphysics he had adopted at the end of the 1890s largely under the influence of G. E. Moore.[2] This tension ended when Russell attended the International Congress of Philosophy in July 1900. There Russell became acquainted with Giuseppe Peano and his work, which Russell describes as one of the most significant events of his philosophical career (*MPD*: 11). After being exposed to and quickly mastering Peano's style of symbolic logic, Russell became convinced that it was possible to analyze all the concepts of pure mathematics in logical terms, and provide deductions of them from logical axioms.[3]

The result was Russell's first major work of lasting significance, *The Principles of Mathematics* (*POM*), published in 1903. In it, Russell laid out his views on philosophical logic and argued informally for logicism. He described the project as offering in part:

> … the proof that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts, and that all its propositions are deducible from a very small number of fundamental logical principles … (*POM*: v; cf. §434)

His initial plan was to carry out the actual deductions symbolically in a second volume, and recruited his Cambridge colleague and former teacher, A. N. Whitehead, as a co-author. By the time their technical work was ready for publication, it had grown so large, and their views had changed so significantly in response to certain paradoxes (see section 3 below), that they decided to rename their work *Principia*

[1] Early drafts of his work on this subject can be found in *Papers* 2 and *Papers* 3.

[2] See Hylton (1990) and Griffin (1991) for discussion of Russell's philosophical conversion to realism.

[3] See Proops (2006) for an account of the philosophical motivations behind Russell's initial acceptance of logicism.

Mathematica (*PM*). It was published in three large volumes from 1910 to 1913. (A proposed fourth volume, which was to be Whitehead's responsibility primarily, was never produced.) Russell later describes the process as having been exhausting, and seems to have been ready to retire from technical work in mathematical logic after its publication. Nonetheless, in 1919, he published *Introduction to Mathematical Philosophy*, which provides an informal introduction to the theories and results of *PM*, and in the mid-1920s, Russell oversaw the publication of a second edition of *PM*, wherein he added a new introduction and various appendices exploring certain new theories and made note of what he saw as important contributions made by others in the intermediate decade. While Russell did not write any additional lengthy pieces on the philosophy or foundations of mathematics after the second edition of *PM*, comments on various issues concerning the nature of mathematics are scattered among his other writings.

Russell's attempt to establish logicism in *PM*, and the logical system developed therein for the purposes of the project, have had a remarkable impact on the history of logic and philosophy. Many of the most important figures in the history of analytic philosophy (from Wittgenstein and Carnap to Quine and Putnam, etc.) describe Russell's logical work as a significant influence. The logic of *PM* is perhaps the single greatest inspiration for the kind of predicate logic nearly all philosophy students are now required to learn. While it is not widely accepted today that *PM* in fact succeeded in establishing logicism, I think a case that it did can still be made.

2 The Regressive Method and Its Prior Successes

According to Russell, progress in mathematics can proceed in either of two directions (*IMP*: 1–2). Most research proceeds in the "forward" or "constructive" direction, in which new results are proven from previously known findings. In this way,

one attains the results of newer, more complex, and "higher" mathematics. Russell saw his own work, however, as proceeding in a different direction, employing an "regressive" or "analytic" method. Here, one begins with a body of knowledge already accepted as at least mostly true. The goal is to work backwards from these known results to a deductive basis for them, a set of principles more general, less complex, and employing a smaller undefined vocabulary. If it can be shown that the original body of knowledge—or at least all of it deemed worth preserving—can be deductively recovered from this more austere basis, we gain insight into the real nature of the truths. Logical connections between the terms employed are revealed. The new basis might also remove puzzling or unwanted aspects of the original theory. Finally, the process organizes the new theory as a deductive system, which invites and facilitates the discovery of new results.[4]

Russell portrays the truth of logicism as something emerging from already-established results in the "regressive" or backwards journey from accepted mathematical principles to their deductive origins. Russell situates his work as the next in a series of recent successes in the that direction (see *Papers* 5: 574–78, *IMP*: chaps. 1–2).

According to Russell's story, a number of late 19th century mathematicians, including Dedekind, Weierstrass, and the members of Giuseppe Peano's Italian school, had succeeded in "arithmetizing" most of pure arithmetic, and even many areas of pure geometry. That is, they showed how it is that these areas of mathematics could be deductively recovered starting only with the basic principles governing natural numbers. Moreover, Peano and his associates had made further "regressive" progress in number theory by reducing its basis to five principles, now known as the Peano-Dedekind axioms:

1. Zero is a natural number.

[4]It is worth comparing Russell's description of philosophical progress in the study of mathematics, with his overall description of "analysis" as a method employed in other areas of philosophy; see, e.g., Hager (2003) for discussion.

2. Every natural number has a successor, which is also a natural number.
3. Zero is not the successor of any natural number.
4. No two natural numbers have the same successor.
5. Whatever holds of zero, and always holds of the successor of any natural number of which it holds, holds of all natural numbers. (The principle of mathematical induction.)

At the time, it was widely believed that all of pure arithmetic could be derived from these principles together with the usual assumptions of logic and set or class-theory (and even today, it is generally agreed that no axiomatic system for number theory represents an improvement on this).

However, according to Russell, the work of Peano's school had not completed the backwards journey to the basic principles of mathematics, because an even more minimal basis is possible. Russell portrays Frege as having taken the next step in this direction, by showing how what amount to Peano's "axioms" for number theory could be derived from logical axioms, removing the need for any basic assumptions beyond the purely logical. Achieving this next step required first a way of defining the non-logical terminology used in Peano's postulates: "zero", "successor" and "natural number". There are in fact different ways of defining these, but let us first focus on natural numbers considered as cardinal numbers. Taking Frege's talk of "extensions of concepts" in his *Grundgesetze* as interchangeable with talk of "classes", Frege defined the cardinal number of class α as the class of all classes whose members could be put in 1–1 correspondence with the members of α. *Zero* could then be defined as the cardinal number of the null class. The relation of *successor* could be defined as holding between n and m when there is a class α and member of that class x such that n is the number of α and m is the number of the class consisting of all members of α except x. In that case, the members of n are all those containing one more member than contained in members of m. The *natural numbers* could

be defined as all those (cardinal) numbers following zero in the series created by the successor relation: zero, zero's successor (1), zero's successor's successor (2), and so on (Frege 2013, §§41–46; cf. Frege 1950, §§73–83). In particular, n is a natural number when it has every property possessed by 0 and always passed from something having it to that something's successor. Frege also provided an axiomatic system for logic, including principles governing classes, in which Peano's "axioms", interpreted using these definitions, could be derived as theorems.

Frege's proposed basis for arithmetic, although fully logicist (in not employing any specifically non-logical vocabulary or non-logical principles) and clearly more "regressive" or "analyzed" (in Russell's sense of these words) than Peano's basis, was flawed in that its assumptions about classes (or "extensions", as Frege called them) led to contradiction. In particular, Russell discovered that Frege's Basic Law V led to the antinomy now known as "Russell's paradox" (discussed in sec. 3 below). Russell communicated this problem to Frege in June 1902 (Frege 1980, 130–33), and Frege himself admitted the flaw in his system (Frege 2013, vol. 2, afterword). Russell sees his work as an attempt to improve upon Frege's advance by taking yet another "regressive" step towards fundamentality. Russell sought to recover what was worthwhile in Frege's proposed basis but at the same time purge it of inconsistency by making use of a yet even smaller logical basis, discarding Frege's problematic assumptions about classes. Here we have an example of how the regressive method, as Russell sees it, does not necessarily preserve *every* aspect of the original body of knowledge being analyzed. It can reasonably be hoped that sometimes the analytic process might remove problems with the original theory or show dubious assumptions to be unnecessary. Changes to the original body of knowledge could be acceptable if what was theoretically important was preserved.

Considered as anything like a historical record of progress in research in the foundations of mathematics, this story Russell tells must be taken with several large grains

of salt. Frege did not see himself as making further progress on a basis established by Peano; Frege's core theses on the nature of arithmetic had been formed, and at least partially defended, before Peano's major works were published. For his own part, Russell had formed the logicist thesis and had worked out logical definitions of the key concepts of number theory independently from Frege, and only discovered Frege's work later (*POM*: xviii). While Russell himself discovered Peano before Frege, and no doubt found Frege to be an improvement over Peano once he had found him, this does not describe very accurately how anyone else would have surveyed the developments. Moreover, it has been suggested that Russell's later depiction of his work on higher mathematics as merely exploiting the "arithmetization" already achieved in the late 19th century does not do adequate justice to the strides towards generality achieved even by Russell himself (and his co-author Whitehead) in their technical work.[5] Nonetheless, this story, as misleading as it is, tells us something about how he understood the achievement of his work and what he counted as "success" when it came to philosophical research in mathematics. It also underscores what for him would have been the most important part of mathematics to be shown to have a logical foundation: the basic principles of number theory.

Russell's regressive method also has implications for how he understood the epistemological dimensions of his project.[6] Unlike Frege (at least as usually read), Russell was not attempting to argue that simple workaday mathematical truths such as that $2 + 3 = 5$ gained their epistemological justification from the more general logical axioms that could be used to deduce them. In fact, Russell saw the issue the other way around: the obviousness and certainty of such elementary mathematical truths provide "inductive support" for a more minimalist theory or list of basic assumptions from which they can all be derived (*Papers* 5: 572–77).

[5] On this point, see especially Gandon (2012, chaps. 5–6) and note 13 below.

[6] For further discussion on this point, see Irvine (1989).

3 The Paradoxes and the Development of Russell's Logic

Russell describes the period just after his discovery of Peano's work in 1900 as a kind of intellectual honeymoon. Peano's logic allowed him as he saw it to give precise definitions for all of the fundamental notions of mathematics and he "discovered what appeared to be definitive answers" (*Auto*: 148) to the philosophical doubts he previously had entertained. By the end of this year, he reports (*MPD*: 73; *Auto*: 148), he had finished an initial draft of what was to become *POM*. What ended the honeymoon was the discovery in 1901 that the assumptions he had been making about the existence of classes, and other entities such as predicates and propositions, were impossible because they led to contradictions (and that the assumptions made by Peano and Frege were similarly flawed).

The best known difficulty is now known as "Russell's paradox of classes". If we consider it intelligible to ask whether or not a given class x is a member of itself, $x \epsilon x$, and consider every intelligible condition (every open sentence) to define a class, then we can consider the class of all such x which are not members of themselves—which Russell writes as "$\hat{x}(x \sim\epsilon x)$"—and ask whether it is or is not a member of itself. By the naïve assumptions he made at the time, it would be a member of itself just in case it met its defining condition, i.e., just in case it was not a member of itself. This is a contradiction. A similar problem threatens if one assumes that there is a quality or monadic universal, or "predicate" as Russell would say at this time, corresponding to every condition defined by an open sentence. It would seem then that there is a quality q which a predicate a has just case a does not have quality a. But then we get a contradiction if we consider whether or not q has quality q (*POM*: §101). This version has come to be known as "Russell's paradox of predication".

Russell discovered these contradictions by considering Cantor's result that every class has more subclasses than members (*POM*: §100, *MPD*: 75). Cantor's reasoning

for this result proceeded by showing that a contradiction results from the assumption that any class as many members as subclasses as follows. Suppose there is a mapping from subclasses to members that yields a distinct member for every subclass. One can then define a "diagonal" subclass s of all those members which are not in the subclass mapped to them. Since s is a subclass, it itself should be mapped to some member, a. But then we can ask whether or not a is in s: since s is defined as the class of all those members not in the subclass mapped to them, a would be a member of s just in case it isn't. Russell arrived at his contradiction more or less by applying Cantor's argument to such classes as the universal class or the class of all classes, for which it would appear impossible that they would have more subclasses than members, since their subclasses *are* members.[7]

Russell also discovered more complicated versions of similar kinds of problems, such as a paradox of relations (*Papers* 5: 588), and another involving propositions (*POM*: §500). At the time of *POM*, Russell considered a proposition to be a mind-independent complex object somewhat like a state of affairs. If propositions are genuine objects, by Cantor's result there ought to be more subclasses of the class of all propositions than members, i.e., there ought to be more classes of propositions than propositions. However, it seems possible, for each class of propositions, m, to generate a distinct proposition, such as the proposition that every member of the class is true, i.e., $(p)(p \in m \supset p)$. Cantor's reasoning would invite us to consider the class w of all propositions of the form $(p)(p \in m \supset p)$ not in the class m they are "about". Now we consider the proposition $(p)(p \in w \supset p)$ and ask whether or not it is in w, and a contradiction arises from either answer.

Russell mentioned these problems in *POM*, and made certain suggestions in the direction of solving some of them, but even by his own admission, he needed to do more work to provide a complete solution before the formal project of *PM* could

[7] For further historical details of the context of Russell's discovery, see Griffin (2004).

proceed. Russell found this task more difficult than he anticipated, and attempting to solve such paradoxes was his chief intellectual preoccupation from 1903 to 1907. In his *Autobiography* (154), he describes himself as having spent the period largely staring at a blank page, failing to make progress. In fact, Russell considered a large variety of different solutions during this period, but usually found that his attempts were only partly successful, solving some but not all versions of the paradoxes. For example, the rather elegant theory found in his 1906 withdrawn-from-publication paper "On the Substitutional Theory of Classes and Relations" (*Papers* 5: 236–61) solved the paradoxes involving classes and predicates, but fell prey to complicated versions of the kind of paradox of propositions mentioned above.[8] Most of the manuscripts in which Russell explored various options are now published in *Papers* 4 and *Papers* 5. Many of his attempts are fascinating, and still worthy of further study, but they cannot be discussed in detail here.

Central to the solution Russell settled on for *PM* was the suggestion that many apparent "abstract" entities such as classes and propositions are not genuine entities at all, but only "logical constructions" or "logical fictions". Apparent terms for them are not genuine terms, but "incomplete symbols", meaningful not by naming entities but rather contributing to the meaning of statements in which they appear in a more complicated way. *PM* *20.01 offered the following contextual definition of a class abstract "$\hat{z}\ldots z\ldots$" for a class of individuals:

$$f\{\hat{z}(\psi z)\} =_{\mathrm{df}} (\exists\phi)[(x)(\phi!x \equiv \psi x)\ \&\, f(\phi!\hat{z})]$$

(A similar contextual definition *20.08 was given for class abstracts for classes of classes.) Roughly, to say something f about the class $\hat{z}(\psi z)$ is really to say that there is a (predicative–see below) property ϕ coextensive with ψ of which f holds. It

[8] For discussion, see Landini (1998, chap. 8).

follows from this definition that claims made about classes of individuals must be reinterpretable as claims about *properties of* individuals rather than claims directly about individuals. A formula of the form $x \in \alpha$ would only be meaningful if x and α were of different *logical types*, and therefore the entire question of whether or not some thing, x, is a member of itself, $x \in x$, is syntactically illegitimate and unintelligible. The class of all classes not members of themselves cannot be spoken of under this theory, and thus leads to no paradox. Russell dubbed this method of eliminating commitment to classes in favor of higher-order quantification, somewhat oxymoronically, as the "no classes theory of classes."

Russell's higher-order logic made use of a hierarchy of variables of different logical types. Unfortunately, the exposition of the syntax of the logic of *PM* does not live up to contemporary standards of rigor, and as a result, there is a fair amount of disagreement about the precise nature of its type theory.[9] At the bottom of the hierarchy, there are variables $(x, y, z, \ldots)$ for individuals. Russell speaks of the values of higher-order variables as "propositional functions", or sometimes, less formally, as "properties" or "relations". Specific propositional functions could be represented by open sentences, or formulas with one or more free variables, "x is green", "x is left of y", or "$x = x$". Part of the controversy, however, is whether Russell regarded propositional functions to be extra-linguistic entities in some sense *denoted* by such expressions, or merely spoke of "propositional functions" as a loose way of speaking about such open sentences themselves and their functioning in the logic in the "material mode", to borrow a phrase from Carnap (1937, §64). At any rate, if the variable ϕ is used to quantify over (predicative) propositional functions of individuals, a formula such as "$(\phi)\phi!a$" is taken to entail the truth of every one of an entire class of propositions formed by replacing "ϕ" with an open sentence and letting a be taken as the value of its variable: "a is green", "a is left of b", and so on. There is

[9] For a variety of positions, see Church (1976), Hatcher (1981, chap. 4), Landini (1998, chap. 10), Linsky (1999, chap. 4) and Klement (2013).

then a new class of open sentences containing such variables as ϕ occupying the next highest type, and then variables for quantifying over such propositional functions, and so on. Moreover, variables for propositional functions with two arguments, or two variables, would be considered as a different logical type from those with one, and so on. Combined with the "no classes" theory, this meant that one could introduce eliminable variables for classes of individuals, and variables for classes of classes of individuals, and so on, forming a hierarchy of types of class variables and class abstracts, but it would be impossible to have a single logical type for *all* classes.

The foregoing is compatible with what is now known as "simple type theory". However, in addition to the sorts of divisions mentioned above, Russell also regarded it as unintelligible for a higher-order quantifier to quantify over properties involving the same kind of quantification as itself or anything of a higher type or order. Consider our previous example of "$(\phi)\phi!a$". Loosely, this can be interpreted as saying that "a has every property", where properties are represented by open sentences such as "x is green", "x is left of b", and so on. But what of the open sentence, "$(\phi)\phi!x$" or "x has every property": is this included among the properties a must have if "$(\phi)\phi!a$" were to be true? Arguably, the suggestion that it is so included leads to a vicious circle in the truth conditions of "$(\phi)\phi!a$", and Russell took such vicious circles to be what generates such "semantic" paradoxes as the Liar paradox and Richard's Paradox (*PM*: 60–65). Hence Russell distinguished the "order-type" of propositional functions not involving quantification over their own or higher types—so called "predicative" propositional functions—from those which do quantify over predicative functions of the same type, and those which quantify over those, and so on. This is known as Russell's "ramified" hierarchy. There is some disagreement over how and to what extent this hierarchy is fully enshrined in the logic of *PM* itself.[10] However, Russell found it necessary to add an assumption,

[10]In particular, Landini (1998, chap. 10, 2011b, chap. 3) has argued that *PM* should be interpreted to make use of no *object-language* variables except predicate variables, making the syntactic rules

the "axiom of reducibility" (*12.1), according to which every monadic propositional function ϕ is equivalent to a predicative one f, marking predicative ones with an exclamation point ! (as well as similar principles for higher types and two-argument functions):

$$(\exists f)(x)(\phi x \equiv f!x)$$

Notice that the quantified ϕ in Russell's contextual definition of classes, above, is restricted to predicative values. If the function ψ were not equivalent to a predicative ϕ, then according to this definition, all statements $f\{\hat{z}(\psi z)\}$ about the class $\hat{z}(\psi z)$ would come out as false. Thus, Russell took assuming the axiom of reducibility to be a necessity for recapturing an adequate class theory and sometimes also called it "the axiom of classes" (*Papers* 5: 540, 607).

4 Logicism in *Principia Mathematica*

The no classes theory of classes, and *PM*'s similar treatment of "relations in extension" (*21), greatly simplified its syntactic primitives and reduced its basic assumptions. In fact, the only signs taken as primitive in *PM* are two statement operators, "$\sim$" (negation), "$\vee$" (disjunction), the universal and existential quantifiers, and variables of the various logical types. Other truth-functional statement operators are defined from negation and disjunction in the usual ways. For instance, material implication $p \supset q$ is defined as $\sim p \vee q$. The axioms of the system, apart from the various forms of the axiom of reducibility mentioned above, include only those for standard classical propositional and (higher-order) quantifier logic.[11] From this basis alone,

much simpler than what is suggested by the traditional interpretation found in e.g., Church (1976), where variables of infinitely many orders within the same type are possible. Landini reads the non-predicative variable ϕ occurring in the axiom of reducibility as a metalinguistic schematic letter.

[11] Strictly speaking, *PM* offers us two different treatments of the logic of quantification, one in *9 and one in *10, with the more austere *9 disallowing quantifiers subordinate to truth-functional operators except as abbreviation. For the sake of simplicity, I pass over such complications here.

Whitehead and Russell sought to define all of the notions of pure mathematics and derive the laws governing them. (There are two other principles sometimes misleadingly *called* "axioms" in the discussions of Russell's logicism, viz., the "axiom of infinity" and the "multiplicative axiom"; more on these below.)

Russell's treatment of cardinal numbers is very similar to Frege's, with the substitution of the no classes theory of classes for Frege's naïve theory of extensions the main point of departure. Cardinal numbers are treated as classes of classes alike in size, i.e., which can be put in 1–1 correspondence with each other. The number 0 is then the class of all empty classes, of which there is of course only one (the null class). The number 1 is the class of all single-membered classes; the number 2 the class of all couples, and so on. One caveat, introduced by Russell's type distinctions and its consequences for the no classes theory, is that numbers are duplicated across various types. For example, the null class of individuals must be treated differently in the no classes theory from the null class of classes of individuals, and consequently, the "zero" which contains only the former is in a different type from the "zero" containing only the latter. In the summary of the formal treatment below, variables such as x, y and z are used for individuals, Greek letters such as α, β, etc., as special variables for classes (eliminable in favor of propositional function variables on the no classes theory), variables R, S, etc., for relations in extension (also eliminable); however definitions for higher-type analogues of these concepts would be quite similar.[12]

$$\begin{array}{rrcl} \text{identity} & x = y & =_{\text{df}} & (\phi)(\phi!x \supset \phi!y) \\ \text{non-membership} & x \sim\epsilon\ \alpha & =_{\text{df}} & \sim(x \epsilon \alpha) \\ \text{universal class} & \mathrm{V} & =_{\text{df}} & \hat{x}(x = x) \end{array}$$

[12] I here try to make use of Whitehead and Russell's own notation as much as possible, although note that they used a dot, or concatenation, rather than "&" for conjunction, and also used dots in place of parentheses for grouping; see *PM*: 9–10. I also here omit certain intermediately defined signs employed in *PM* itself, and simply write the unabbreviated forms where these are used later.

complement	$-\alpha$	$=_{\mathrm{df}}\ \hat{x}(x \sim\epsilon\ \alpha)$
null class	Λ	$=_{\mathrm{df}}\ -\mathrm{V}$
union	$\alpha \cup \beta$	$=_{\mathrm{df}}\ \hat{x}(x \epsilon \alpha \vee x \epsilon \beta)$
intersection	$\alpha \cap \beta$	$=_{\mathrm{df}}\ \hat{x}(x \epsilon \alpha\ \&\ x \epsilon \beta)$
subset	$\alpha \subset \beta$	$=_{\mathrm{df}}\ (x)(x \epsilon \alpha \supset x \epsilon \beta)$
singleton y	$\iota`y$	$=_{\mathrm{df}}\ \hat{x}(x = y)$
domain	$\mathrm{D}`R$	$=_{\mathrm{df}}\ \hat{x}((\exists y)\, xRy)$
converse domain	$\mathrm{Ɔ}`R$	$=_{\mathrm{df}}\ \hat{x}((\exists y)\, yRx)$
class of many-one relations	$\mathrm{Nc} \rightarrow 1$	$=_{\mathrm{df}}\ \hat{R}((x)(y)(z)(xRy\ \&\ xRz \supset y = z))$
class of one-many relations	$1 \rightarrow \mathrm{Nc}$	$=_{\mathrm{df}}\ \hat{R}((x)(y)(z)(yRx\ \&\ zRx \supset y = z))$
class of one-one relations	$1 \rightarrow 1$	$=_{\mathrm{df}}\ \mathrm{Nc} \rightarrow 1 \cap 1 \rightarrow \mathrm{Nc}$
cardinal similarity	$\alpha\ \mathrm{sm}\ \beta$	$=_{\mathrm{df}}\ (\exists R)(R \epsilon 1 \rightarrow 1\ \&\ \alpha = D`R\ \&\ \beta = \mathrm{Ɔ}`R)$
cardinal number of α	$\mathrm{Nc}`\alpha$	$=_{\mathrm{df}}\ \hat{\beta}(\beta\ \mathrm{sm}\ \alpha)$
class of cardinal numbers	NC	$=_{\mathrm{df}}\ \hat{\beta}((\exists\alpha)\beta = \mathrm{Nc}`\alpha)$
zero	0	$=_{\mathrm{df}}\ \mathrm{Nc}`\Lambda$
successor of α	$\mathrm{Succ}`\alpha$	$=_{\mathrm{df}}\ \hat{\beta}((\exists x)(x \epsilon \beta\ \&\ \beta \cap -\iota`x \epsilon \alpha))$
class of natural numbers	$\mathbb{N}$	$=_{\mathrm{df}}\ \hat{\alpha}((\beta)((0 \epsilon \beta\ \&\ (\gamma)(\gamma \epsilon \beta \supset \mathrm{Succ}`\gamma \epsilon \beta)) \supset \alpha \epsilon \beta))$

Notice here that the successor of a number α is the class of classes which are such that if a member is removed one gets a member of α: the class of all classes with one more member than the members of α. The class of natural numbers are those contained in *all classes* containing 0 and the successor of anything they contain, and hence would be the class containing exactly 0, 1, 2, 3, ... Using these definitions, formal versions of the Peano-Dedekind axioms could be stated as follows:

1. $0 \epsilon \mathbb{N}$
2. $(\alpha)(\alpha \epsilon \mathbb{N} \supset \mathrm{Succ}`\alpha \epsilon \mathbb{N})$

3. $\sim(\exists\alpha)(0 = \text{Succ}`\alpha)$
4. $(\alpha)(\beta)((\alpha \in \mathbb{N} \;\&\; \beta \in \mathbb{N} \;\&\; \text{Succ}`\alpha = \text{Succ}`\beta) \supset \alpha = \beta)$
5. $(\alpha)((0 \in \alpha \;\&\; (\gamma)(\gamma \in \alpha \supset \text{Succ}`\gamma \in \alpha)) \supset \mathbb{N} \subset \alpha)$

Using nothing beyond the usual assumptions of higher-order quantifier logic (which also provide a proxy for class-logic via the no classes theory), four of these five can be derived as theorems relatively straightforwardly. (An interested reader may consult *PM*, parts I–II, for details.)

The problematic one of the bunch is the fourth, which states that no two natural numbers have the same successor. In order for this to hold, no natural number can be empty: i.e., for every natural number there must be at least one class with that many things. Notice further that in order for this to be true for every natural number, the number of things in total must be infinite. Suppose instead that there were only finitely many individuals; for convenience, let us assume there are only three: a, b and c. Then the class consisting of a, b and c would be the only three membered class, and the only member of the number 3. The successor of 3 (viz., 4) would be the class of all those classes which are such that, if you removed a member from them, you'd get a member of 3. But since a, b and c are the only individuals there are, no class could meet this condition. Hence, 4 would be empty. The successor of 4 (viz., 5) would also be empty, since it is clearly impossible to get a member of the null class by removing an element from a class, as the null class has no members. Hence, 3 and 4 would both be natural numbers, and both would have the same successor (the null class), but they would not be identical to each other, and so the fourth Peano "axiom" would be false.

During earlier periods of his career, when he took propositions and classes to be more than just "logical fictions", Russell had regarded it as possible to deduce the existence of infinitely many individuals purely logically. By the time of *PM*, however, Russell rejected such deductions. Such proofs only *appear* to succeed because

of a failure to take type distinctions into account. Consider what is now called the Zermelo sequence: $\Lambda, \iota`\Lambda, \iota`\Lambda, \iota`\iota`\Lambda, \ldots$ (the null class, its singleton, its singleton's singleton, and so on). It might appear that this sequence is unending, and thus infinite, and that there must therefore be infinitely many things. However, notice that each member of this sequence is of a higher type than the previous member, and so it cannot be used to establish an infinity of *individuals*, or an infinity of things in any one type at all. A similar difficulty makes it impossible for Russell to make use of Frege's argument (Frege 1950, §82) alleging that all the numbers proceeding a given natural number shows that this number must apply to at least one collection (i.e., the class containing only 0 has one member, the class of 0 and 1 has two members, the class of 0, 1 and 2 has three, and so on). Russell also rejects an argument (adapted from Dedekind) that alleges that the series: Socrates, the idea of Socrates, the idea of the idea of Socrates, the idea of the idea of the idea of Socrates, ..., is infinite, on the grounds that the relevant sort of ideas can only be assumed to exist when actually thought by some mind, and hence we have no reason to think that this list goes on beyond some finite number of steps (*IMP*: 139).

Russell therefore concludes that one cannot derive something like the fourth Peano axiom stated above without assuming that every natural number is nonempty, or what he sometimes calls *the axiom of infinity*:

$$(\alpha)(\alpha \in \mathbb{N} \supset (\exists\beta)(\beta \in \alpha)) \tag{Inf}$$

Despite his use of the word "axiom" here, in *PM* itself this is not taken as an axiom in the usual sense, and is simply left as an undischarged antecedent on many results, including the fourth Peano axiom and all results depending on it. The question as to whether or not this compromises the success of Russell's logicism is revisited in section 6 below.

For reasons mentioned in section 2 above, natural number theory is perhaps the most important test case for Russell's logicism, but of course, *PM* covers far more than simply the theory of natural numbers. In addition to the treatment of cardinal numbers just sketched, *PM* (parts IV and V; mostly in volume 2) also provides a treatment of ordinal numbers, which are taken as a subclass of a broader notion of "relation numbers", defined as classes of relations which are structurally isomorphic to each other. Finite ordinal numbers provide yet another interpretation of "natural numbers" for the purposes of the Peano-Dedekind axioms, though a similar problem with infinity arises there. For both cardinals and ordinals, they discuss infinite numbers (if there are any collections or relations exemplifying such). To capture multiplication for infinite numbers and achieve the expected results (i.e., those suggested by Cantor's work), Whitehead and Russell discuss an assumption they call the *multiplicative axiom*, which asserts that for every class of disjoint (non-overlapping) classes, there is a class which has among its members exactly one member from each member of the class of disjoint classes (*PM*: *88). This assumption is equivalent with Zermelo's axiom of choice, which is notoriously independent of the other basic assumptions of set theory. Like the so-called "axiom" of infinity, the multiplicative axiom is not formally taken as an axiom or basic assumption in *PM*, but is simply abbreviated as "Mult ax" and left as an antecedent on the various results which require it, such as the principle that all classes can be well-ordered.

In Parts V and VI (mostly in volume III) of *PM*, Whitehead and Russell provide treatments of integers, ratios, complex numbers and finally real numbers in terms of what is in common between certain systems of relations, of which relationships between segments of ratios provide only a single instance.[13] Along the way, they give formal logical definitions of such notions as *limit* and *continuity*. Part VI goes

[13] In *IMP* (chap. 7), Russell defines real numbers simply *as* segments of ratios or rational numbers according to the "Dedekind cut" method, and does little to explain the more general application of *PM*'s theory of quantity; this oversimplification of the more general treatment provided in *PM* has been bemoaned by Gandon (2012, 9) among others.

further yet and generalizes on these notions to provide a generic theory of quantity and measurement. The planned fourth volume of *PM* was slated to provide logicist accounts of many aspects of pure geometry as well, though one can glean certain aspects of how this might have proceeded from Russell's chapters on projective, metric and descriptive geometry making up the later parts of his earlier *POM*.

5 Gödel's Results and the Scope of Logicism

We now turn to evaluation of Russell's logicism. Perhaps the most common problem one hears cited against it, or logicism in general, especially in casual conversation, appeals to Gödel's famous incompleteness results. However, I think the relevance of these results for assessing logicism, at least on the most reasonable understandings of the logicist thesis, is greatly exaggerated.

Using *PM* as an example,[14] Gödel showed that any deductive system for mathematics having certain features contains undecidable sentences, i.e., sentences such that neither they nor their negations can be derived as theorems. Especially when combined with well-known corollaries shown by Rosser, Tarski, and others,[15] the features can for all intents and purposes be reduced to consistency, having the requisite strength to capture basic elements of number theory, and having a recursive axiomatization. Since recursive functions have been shown equivalent with Turing-computable functions and λ-definable functions, if we accept the Church-Turing thesis that these and only these functions are effectively computable, having a recursive axiomatization amounts more or less to there existing an effective decision procedure for determining what does or does not count as a deduction in this sys-

[14]It is likely that Gödel's exposition of *PM* was not entirely faithful to the original; in particular, Gödel did not fully take into account that numerals, as signs for classes, in *PM* were incomplete symbols and not genuine terms. This issue, and the complications arising from it, cannot be explored in depth without providing a full reconstruction of *PM*, which cannot be attempted here.

[15]Gödel's original results were published in Gödel (1931), and a summary of them and the most important corollaries can be found in most textbooks on mathematical logic, e.g., Mendelson (2010).

tem. The feasibility or desirability of a non-recursively axiomatized system is basically nil: what good could a deductive system be if there was no effective way to tell what counts as a deduction in it and what doesn't? Gödel's "first incompleteness theorem" roughly establishes that systems with the desirable features must contain an undecidable sentence abbreviated $\mathcal{G}$, which, roughly asserts of itself that it is not a theorem of the system (or really, asserts something analogous of a number which corresponds to it in an arithmetization of the system's syntax). Clearly, it would be undesirable for $\mathcal{G}$ to be a theorem, as then one would have a false theorem in the system. But if it is not a theorem, it is *true*, and hence we have what appears to be a number-theoretic truth not derivable from the axioms. Gödel's "second incompleteness theorem" establishes that in such systems one can form a sentence which roughly asserts that the system is consistent, i.e., that no contradiction is derivable within, but that this sentence is also undecidable, at least if the system is consistent.

How relevant such results are for evaluating Russell's (or anyone else's) form of logicism depends largely on how strong we take the logicist thesis to be. Suppose we define a "purely logical deductive system" as one whose axioms are all logical truths and whose inference rules are sound on logical grounds alone, and so cannot lead from a logical truth or truths to something that isn't also a logical truth. A very strong interpretation of the logicist thesis (LT) would be the following:

> There is a practical purely logical deductive system, S, such that for every mathematical truth, p, p is a theorem of S. (LT-a)

It is likely that Russell believed that (LT-a) was true early on, and that *PM* was such a deductive system. However, even if we interpret the "practical" in (LT-a) to restrict us to recursively axiomatized systems, additional assumptions would be required to argue that the Gödel results undermine (LT-a). Russell was aware, even in 1910, that there were undecidable sentences in *PM*. The (so-called) axiom of infinity, for ex-

ample is neither derivable nor refutable therein. Russell would not have taken this to undermine his main thesis, or even (LT-a), since he would regard neither the axiom of infinity nor its negation as a "mathematical truth"; if it is true, it is contingently true (*IMP*: 141). Unless it could be shown that some of the undecidable sentences proven to exist by Gödel, or their negations, are *mathematically* true, there is no problem. Russell even suggested a response along these lines in 1963 when asked about the importance of Gödels results in a letter.[16] Unfortunately, as such, the response seems a bit hasty. Gödel's $\mathcal{G}$ and the consistency of *PM* itself, unlike the axiom of infinity (arguably), we do seem to know to be true, and on *a priori* grounds. If they are not mathematical truths, it is very hard to see what *other* kind of *a priori* truths they could be. And if they are mathematical truths, then this would seem to undermine *PM*'s claim to justify (LT-a), as they cannot be proven in *PM*.

However, it is natural to wonder whether or not the heart of Russell's logicism requires anything quite as strong as (LT-a). Russell saw himself as arguing against the Kantian thesis that mathematical truths were importantly different from logical truths, with the former understood as *synthetic a priori* and the latter as *analytic*.[17] From this perspective, it seems that Russell's principal thesis was that mathematical truths don't have a special nature or essence distinct from that possessed by logical truths. Whatever it is that makes logical truths special or sets them apart from other kinds of truths is also possessed by mathematical truths: *mathematical truths are a species of logical truths*. It seems possible to hold that thesis without insisting on anything as strong as (LT-a). Unfortunately, the issue cannot be formally spelled out

[16] See Russell's letter to G. Simons, 1 October 1963; document RA2.710.111928 at the Bertrand Russell Archives, McMaster University, and compare *Papers* 11: 65.

[17] Interestingly, Russell's exact attitude of where Kant had erred seems to have changed over the years; in *POM* (§434), Russell suggests that Kant was wrong not about mathematics but about logic, alleging that even logical truths can be understood as synthetic a priori in Kant's sense. Later on, Russell claims instead that logicism shows that mathematics is analytic and not synthetic (e.g., *HWP*: 740). It is likely, however, that the different attitudes are as much the result of employing different definitions of "analytic" and "synthetic" as they are a reflection of a change of mindset; see Landini (2011b, 223–25) and Korhonen (2013, chap. 1).

without saying exactly what it is that makes logical truth special, an issue Russell himself struggled with.[18] Early on, he seems to have thought that any truth which could be stated using only logical constants counts as a "logical truth" (*PM*: §10); later on, however, he suggests that they must have a special "tautological" form, but neglects to provide an exact specification of what that amounts to (*IMP*: 203).

In contemporary research in metalogic, it is often the case that two distinct criteria are given for what makes formulas logically necessary: a deductive or proof-theoretic one (derivability from the axioms), and a semantic one (usually amounting to something like truth in all acceptable models or interpretations). If a deductive system is both sound and complete, the two characterizations are equivalent: something is a logical truth in one sense if and only if it is in the other. However, some deductive systems are incomplete. For example, it is a well-known corollary of Gödel's results that second- and higher-order logical calculi (including *PM*), when interpreted according to "standard semantics",[19] cannot be complete when axiomatized recursively. There are formulas that are logically true according to the semantic criterion that cannot be derived as theorems. Obviously, this result should not be interpreted as showing that some logical truths are not logical truths, and similarly, it is unclear why the fact that certain apparently "mathematical" truths cannot be derived in a certain deductive calculus should be taken to establish that those truths are not logical truths. In other words, if the logicist thesis were interpreted along the lines of (LT-b) below,[20] then it is not clear how or why Gödel's results should be

[18] For further discussion, see Klement (2015).

[19] Standard semantics is often contrasted with "general" or "Henkin" semantics; the former, unlike the latter, requires that every subset of the domain of the first-order quantifiers be the extension of some value of the domain of quantification for second-order variables, and similarly for other types.

[20] Rayo (2007, 240) provides a list of different interpretations of logicism, though curiously he does not bother to make distinctions similar to my (LT-a) and (LT-c) for forms focused on deductive consequence or derivability.

taken to pose a problem for it:

> Every mathematical truth has the semantic feature which sets apart logical truths semantically from others. (LT-b)

Of course, *PM* does not explicitly provide a semantic characterization of logical truth nor a direct argument for anything like (LT-b). Nonetheless, the deductions of various mathematical truths provided in *PM* could be relevant for semantic characterizations of logicism, provided it could be established that the axioms of *PM* are logical truths in this sense and that the inference rules preserve this feature.

Moreover, even for those primarily interested in a deduction-centered account of logical truth, it is not obvious that something as strong as (LT-a) needs to be maintained by a logicist. Consider the following similar, but slightly weaker, thesis:

> For every mathematical truth, p, there is a practical purely logical deductive system, S, such that p is a theorem of S. (LT-c)

This is much like (LT-a) except that it does not require there to be a single logical system in which every mathematical truth is captured: it requires only that each such truth be captured in some such system *or other*. To my knowledge, there is nothing in Gödel's results to suggest that (LT-c) must be false. Consider again the situation that we seem to know that Gödel's $\mathcal{G}$ is true, despite its undecidability in *PM*. So long as our knowledge of this relies only on logical principles, perhaps there is some purely logical extension of *PM* in which $\mathcal{G}$ is derivable. (And if our knowledge of $\mathcal{G}$ requires something beyond logic, for this to pose a problem for Russell, it would need to be shown that it *doesn't* require anything beyond mathematics, or his earlier response is sufficient.) If this new system is recursive, it too will have undecidable (true) sentences, but perhaps they are derivable in another logical system of the appropriate type. This is more or less in keeping with Russell's "official" response to

Gödel, which suggests that there is a hierarchy of logical languages or logical systems, and what cannot be captured in one is captured higher in the hierarchy (*Papers* 11: 159; *MPD*: 114).

Indeed, when the issue is posed in terms of the *special nature* of mathematics and its relation to logic as a whole—the issue of what makes mathematical truths different from others, and whether or not this is the same as the special nature of logical truths—what *PM* does succeed in establishing seems far more important than any worries stemming from the few rather recondite undecidable sentences shown to exist by Gödel. Consider every mathematical truth taught in primary or even secondary school. Consider all the mathematics of quantity and measurement used in engineering. Consider Kant's famous example of the allegedly synthetic $5 + 7 = 12$. It appears that all of these can be captured in any higher-order system interpreting the number theory provided by Peano arithmetic. If *PM* or similar work succeeds in showing that this portion of mathematics consists in nothing other than logical truths, it becomes almost comical to suggest that the logicist thesis as a whole is mistaken. If nearly all of the mathematics anyone uses or knows about, the truths about numbers which most readily come to mind when considering arithmetic, reduce to logic, can it seriously be maintained that the *essence* of mathematics is extra-logical, merely because there are certain highly complex, and application-free truths in certain formal systems which cannot be shown to be logical in precisely the same way within a (single) recursive axiom system? If mathematics has a special essence, a special nature, surely that nature expresses itself already in Peano arithmetic, so here again it seems right to maintain that capturing Peano arithmetic is the best test case for Russell's logicism as a whole.

Of course, doubts may still be raised as to whether or not *PM* adequately captures Peano arithmetic, but they are distinct from any stemming from Gödel's results. Such doubts are considered in the next section.

6 The Controversial "Axioms" and If-Then-ism

Perhaps the next most commonly heard objection to Russell's logicism—and the most common from specialists in the area—points to its employment of certain basic assumptions which do not seem, or at least do not obviously seem, like logical truths.[21] There are three controversial (so-called) axioms here: reducibility, infinity and the multiplicative axiom. Of course, it must be remembered that the latter two were not treated as axioms in *PM*, but instead simply left as undischarged antecedents on results depending on them. This point, however, simply invites stating the criticism in a different form. The charge, called *if-then-ism* by the critics that offer it,[22] is that Whitehead and Russell do not prove the basic principles of mathematics from logic outright, but merely derive conditionals with certain mathematical assumptions as antecedents and their logical consequences as consequents. For example, rather than proving that no two natural numbers have the same successor (the fourth Peano axiom), they only prove that this is true *if* there are infinitely many individuals. In the eyes of these critics, this greatly compromises the success of *PM*'s logicism. After all, for any theory which can be organized as a deductive system, even an obviously empirical and non-logical one such as a theory in physics, the conditionals from the axioms to their logical consequences can take the form of logical truths. This tells us nothing about the essence of its subject matter.

For what it is worth, Russell himself admitted that the truth of the axiom of reducibility was less than fully obvious. Indeed, even when first stating it he admitted that the grounds for accepting it were "inductive", in keeping with his general "regressive" method. It could be used as a premise from which many desirable results

[21] See, e.g., Kneale and Kneale (1962, 669), Roselló (2012, 50), Tennant (2013, §1.3.1) and Soames (2014, 488).

[22] See, e.g., Putnam (1967), Musgrave (1977), Coffa (1981) and Boolos (1998, chap. 16). For a variety of responses, see Griffin (1982), Landini (2011b), Gandon (2012), Klement (2012), Galaugher (2013), and Kraal (2014).

could be proven and he knew of no more plausible or no more general principle from which as many important results followed (*PM*: 59). Russell also dedicated much of the new material added to the second edition of *PM* to exploring a less than fully successful[23] attempt to do away with the axiom by adopting instead a doctrine of extensionality he took to be motivated by the philosophical treaties of Wittgenstein's *Tractatus* (see *PM* 2: xiv and appendices). However, it should be remembered that Russell's main motivation for adopting the axiom was to recover the ability to speak of classes, which for him amounted simply to things considered in groups, or plurally,[24] without a stronger assumption of "real" classes taken as genuine things. It is now widely acknowledged that something like quantification over collections or pluralities, or in general, what one finds in a "standard" higher-order logic, is necessary for capturing some logical forms used in every-day reasoning. Perhaps the most famous example is the Kaplan-Geach sentence, "some critics only admire one another", which cannot be interpreted in first-order logic. The history of logic does have its first-order purists, like Quine, who considered higher-order logic to be "set-theory in sheep's clothing" (Quine 1986, 66), and there are even hold-outs among contemporary philosophers (e.g., Burgess 2005, 201–14). Still, I think it safe to say that these represent the minority. Something like the ability to consider objects in groups seems to be a necessary central element in reasoning about nearly every topic, and allowing for this is not to take on a special extra-logical or specifically mathematical assumption. Russell's axiom of reducibility is, if anything, no stronger and arguably weaker than adopting the usual comprehension principles allowed in simple type theory or most plural logics (cf. Yi 2013).[25] The ideal formulation of a

[23]The difficulties with the attempt were partly appreciated by Russell himself, partly not. For further discussion, see Landini (1996), Hazen and Davoren (2000), Landini (2007), and Linsky (2011).

[24]For more on Russell's attitude about the relationship between discourse about classes and about plurals, see Oliver and Smiley (2005), Bostock (2008) and Klement (2014).

[25]Indeed, if Landini is right that the object-language variables of *PM* are only predicative variables (see note 10 above), then system *PM* is formally speaking the same as a simple type theory, and the axiom of reducibility just *is* its comprehension principle; see Landini (1998, 264–67).

logical apparatus that allows us to speak of classes, or collections, or pluralities, remains an active area of research, discussion and controversy, but when compared to the alternatives, Russell's adoption of something like the axiom of reducibility seems all-things-considered to be among the most restrained.

The failure to prove the multiplicative axiom or obtain its results outright in *PM* also does not seem to pose a serious problem for the success of Russell's logicism. While the very similar axiom of choice is perhaps widely accepted among mathematicians, and it is also known to be equivalent to a number of other principles with a similar status (the well-ordering principle, Zorn's lemma, the multiplicative axiom itself, etc.), no one has been able to provide a proof of it from more obvious or more fundamental principles, and indeed, it is well-known to be independent of the other axioms of ZF set theory. The multiplicative axiom is not needed for results in finite arithmetic, but only for certain results in less-firmly settled infinite arithmetic. The axiom of choice and the multiplicative axiom hardly count as uncontroversial examples of definite mathematical truths. The only results which are considered definite are their mutual equivalence and their relationships with these other principles, which are captured in *PM*. It does not seem reasonable to argue that the logicism of *PM* is a failure merely because the multiplicative axiom, or its logical consequences, are not derived in non-conditional form.

Similar concerns surrounding the axiom of infinity seem to pose a far greater challenge to a positive evaluation of Russell's logicism. Without it, as we have seen, Russell cannot recapture Peano arithmetic in its usual form, as the axiom of infinity would be needed to obtain the fourth axiom. A defender of Russellian logicism seems to be forced into one of the following positions: (a) argue that the axiom of infinity can reasonably be taken as a basic logical truth, (b) provide a demonstration of the axiom of infinity from other plausible purely logical assumptions (not already included in *PM*), (c) deny that the fourth Peano axiom is, without qualification or

antecedent, a mathematical truth, or (d) admit that certain very important mathematical truths cannot be derived from a logical starting place. Response (d) seems tantamount to abandoning logicism. Responses (a) and (b) are perhaps worthy of further investigation, but (c) seems to have been Russell's own considered view. Can anything be said in its favor?

Proponents of the if-then-ist criticism of Russell sometimes point to early passages such as in *POM* (§1), where Russell claims that *all* mathematical propositions can be seen as taking the form of conditionals:

> Pure mathematics is the class of all propositions of the form "p implies q," where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants.

However, there are a number of reasons for being cautious about reading this in favor of if-then-ism. For one, in the logic of *POM*, all other connectives (negation, disjunction, conjunction, etc.) are defined in terms of material or formal implication (§§18–19), and so any logically complex proposition would be of the requisite form, and obviously a logical deduction of a statement relies on its having the kind of complexity that might make its derivation from basic principles possible. Russell's example in this context (§6) in which he rewords "$1 + 1 = 2$" as "If x is one and y is one, and x differs from y, then x and y are two" helps explain how uncovering the true logical complexity of a proposition is necessary in order to see its real logical status. As stressed by Griffin (1982, 89), Russell (even later on) thought many mathematical truths took the form conditionals not because he thought of them as representing the relationships between the axioms of a theory and the consequences, but rather because he took them as *general* truths with definite application conditions where those conditions were stated in the antecedents applied to variables. Applying mathematics involves finding values of the variables satisfying the antecedents,

and therefore accepting the relevant instances of the consequents for those values.

There is precedent in Russell's earlier work for recognizing situations where required antecedents have been left off of stated theorems by practicing mathematicians because they took for granted the most likely practical application conditions for their work. Past geometers who assumed that actual (physical) space was Euclidian took for granted that they could presume features of Euclidian space when stating their results. As Russell argues, however, the real "pure" matheamtical truth should be taken as the conditional from their assumptions about the structure of space to the consequences thereof (*POM*: §§353, 412), rather than the unqualified statements which only are true in certain application conditions. Hence, it appears Russell would argue that the *true form* of a given mathematical truth may reasonably be taken to be conditional in form if the antecedent involves special and restricted conditions for its application, and the consequent itself, even if true for a given instance, is not true for that instance on mathematical grounds alone, but only because the antecedent is satisfied. In the geometrical context, if the antecedent is not satisfied, we have a different kind of space, where a different kind of geometry may be applicable; this does not make it in any way a logically or mathematically defective space.

Is the case of the relationship between the axiom of infinity and the fourth Peano axiom at all similar? The suggestion on the table then is that the fourth Peano axiom (and its consequences) should not be considered mathematical truths, full stop, just like geometrical statements only true if space is Euclidian. Hence, it is not required for logicism that we derive them, full stop, from a logical basis. It is enough if we can demonstrate that they hold given the right conditions. Russell himself does not explicitly argue this way to my knowledge, and at first blush it seems a fairly radical suggestion. Should we really conclude that 4 is distinct from 5 only *given cer-*

tain application conditions? Yet, I think the proposal is not wholly without merit.[26] Without presupposing infinity, our usual assumptions about the behavior of numbers will only start going "awry" for those greater than the number of individuals there in fact are. If we are applying the mathematics of *PM* to a domain of individuals with 3 members, then all the usual results for numbers up to including 3 would be what we expect, and at higher types, where there are more and more classes formed of our three things or formed of class of them or classes of them, etc., the "well-behaving" numbers will reach higher and higher (*IMP*: 134). In short, nothing will go wrong in our actual conditions of application. Perhaps "non-infinite" arithmetic, where the series of numbers runs out after a certain stage, or goes in circles, much like non-Euclidian geometry, should be seen as a new area of study. Indeed, one could make the case that modular ("clock") arithmetic already exemplifies this. Some philosophers take a hard conservative line and demur from any philosophical proposals which could be seen as suggesting "reforms" to mathematics (e.g., Lewis 1986, 109). Russell himself, as we have seen, thought it an advantage of the process of analysis that it did not necessarily preserve the analyzed body of knowledge as originally conceived.

Whatever may be said for the above proposal, Russell himself seems to have been disappointed by his failure to establish in any definitive or a priori way the existence of the complete series of numbers. In fact, while the logic of *PM* presupposes there is at least one individual—something Russell eventually came to regard as "a defect of logical purity" (*IMP*: 203n)—there is no way in it to show that there is any more than one, and therefore, no logical way to arrive at any numbers beyond one, at least at the lowest type. Ivor Grattan-Guinness provides us with a report, perhaps apocryphal, from an acquaintance of Russell who allegedly heard him bemoan this fact after having finished *PM*:

[26] See Landini (2011a) for a sympathetic treatment of such an approach.

> Bertrand Russell called me aside as a mathematician I suppose and likely to appreciate the gravity of his statement—'I have just realised that I have failed—it is easy to establish the unit one but I have omitted to establish a second like unit'—(I won't guarantee the precise wording but it's not far off). He went on to say 'I have finished'. (Quoted in Grattan-Guinness 2000, 401)

Did, however, Russell fail? When we consider the effects of Russell's pursuit of logicism on the history of both logic and philosophy, there can be no question that it has had a lasting and positive impact. It shaped not only Russell's philosophy but has made modern symbolic logic an irreplaceable part of almost every analytic philosopher's toolkit. Even when it comes to the question of the logicist thesis for mathematics in particular, there is a lot more reason to think that Russell's work was a success than is generally realized.

References

Boolos, George, 1998. *Logic, Logic and Logic*. Cambridge, Mass.: Harvard University Press.

Bostock, David, 2008. "Russell on 'the' in the Plural." In *Russell vs. Meinong: The Legacy of "On Denoting"*, edited by Nicholas Griffin and Dale Jacquette, pp. 113–143. New York: Routledge.

Burgess, John, 2005. *Fixing Frege*. Princeton: Princeton University Press.

Carnap, Rudolf, 1937. *The Logical Syntax of Language*, trans. by A. Smeaton. London: Routledge & Kegal Paul.

Church, Alonzo, 1976. "Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski." *Journal of Symbolic Logic* 41: 747–760.

Coffa, J. Alberto, 1981. “Russell and Kant.” *Synthese* 46: 247–263.

Dedekind, Richard, 1888. *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg.

Frege, Gottlob, 1950. *The Foundations of Arithmetic*, trans. by J. L. Austin. Evanston: Northwestern University Press.

——, 1980. *Philosophical and Mathematical Correspondence*, trans. by H. Kaal, edited by B. McGuinness. Chicago: University of Chicago Press.

——, 2013. *Basic Laws of Arithmetic*, trans. by P. Ebert and M. Rossberg. Oxford: Oxford University Press.

Galaugher, Jolen, 2013. *Russell’s Philosophy of Logical Analysis 1897–1905*. Basingstoke: Palgrave Macmillan.

Gandon, Sébastien, 2012. *Russell’s Unknown Logicism*. Basingstoke: Palgrave Macmillan.

Gödel, Kurt, 1931. “Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandter Systeme I.” *Monatshefte für Mathematik und Physick* 38: 173–98.

Grattan-Guinness, Ivor, 2000. *The Search for Mathematical Roots 1870–1940*. Princeton: Princeton University Press.

Griffin, Nicholas, 1982. “New Work on Russell’s Early Philosophy.” *Russell* 2: 69–83.

——, 1991. *Russell’s Idealist Apprenticeship*. Oxford: Clarendon Press.

——, 2004. “The Prehistory of Russell’s Paradox.” In *One Hundred Years of Russell’s Paradox*, edited by Godehard Link, pp. 349–371. Berlin: Walter de Gruyter.

Griffin, Nicholas and Bernard Linsky, eds., 2013. *The Palgrave Centenary Companion to Principia Mathematica*. Basingstoke: Palgrave Macmillan.

Hager, Paul, 2003. "Russell's Method of Analysis." In *The Cambridge Companion to Bertrand Russell*, edited by Nicholas Griffin, pp. 310–31. Cambridge: Cambidge University Press.

Hatcher, William, 1981. *The Logical Foundations of Mathematics*. Oxford: Pergammon Press.

Hazen, Allen P. and J. M. Davoren, 2000. "Russell's 1925 Logic." *Australasian Journal of Philosophy* 78: 534–556.

Hylton, Peter, 1990. *Russell, Idealism and the Emergence of Analytic Philosophy*. Oxford: Oxford University Press.

Irvine, A. D., 1989. "Epistemic Logicism and Russell's Regressive Method." *Philosophical Studies* 55: 303–27.

Klement, Kevin C., 2012. "Neo-logicism and Russell's Logicism." *Russell* n.s. 32: 127–59.

——, 2013. "*PM*'s Circumflex, Syntax and Philosophy of Types." In Griffin and Linsky (2013), pp. 218–47.

——, 2014. "Early Russell on Types and Plurals." *Journal for the History of Analytical Philosophy* 2: 1–21.

——, 2015. "The Constituents of the Propositions of Logic." In *Acquaintance, Knowledge and Logic: New Essays on Bertrand Russell's* The Problems of Philosophy, edited by Donovan Wishon and Bernard Linsky, pp. 189–229. Stanford: CSLI Publications.

Kneale, W. and M. Kneale, 1962. *The Development of Logic*. Oxford: Clarendon Press.

Korhonen, Anssi, 2013. *Logic as Universal Science: Russell's Early Logicism and Its Philosophical Context*. Basingstoke: Palgrave Macmillan.

Kraal, Anders, 2014. "The Aim of Russell's Logicism." *Synthese* 191: 1493–1510.

Landini, Gregory, 1996. "The Definability of the Set of Natural Numbers in the 1925 *Principia Mathematica*." *The Journal of Philosophical Logic* 25: 597–615.

——, 1998. *Russell's Hidden Substitutional Theory*. Oxford: Oxford University Press.

——, 2007. *Wittgenstein's Apprenticeship with Russell*. Cambridge: Cambridge University Press.

——, 2011a. "Logicism and the Problem of Infinity: The Number of Numbers." *Philosophia Mathematica* 19: 167–212.

——, 2011b. *Russell*. New York: Routledge.

Leibniz, G. W., 1666. *Dissertatio de Arte Combinatoria*. Leipzig: J. S. Fick and J. P. Seubold.

Lewis, David, 1986. *On the Plurality of Worlds*. Oxford: Blacwell.

Linsky, Bernard, 1999. *Russell's Metaphysical Logic*. Stanford: CSLI Publications.

——, 2011. *The Evolution of* Principia Mathematica*: Bertrand Russell's Manuscripts and Notes for the Second Edition*. Cambridge: Cambridge University Press.

Mendelson, Elliot, 2010. *Introduction to Mathematical Logic*, 5th ed. Boca Raton: CRC Press.

Musgrave, Alan, 1977. "Logicism Revisited." *British Journal for the Philosophy of Science* 28: 99–127.

Oliver, Alex and Timothy Smiley, 2005. "Plural Descriptions and Many-valued Functions." *Mind* 114: 1039–68.

Proops, Ian, 2006. "Russell's Reasons for Logicism." *Journal of the History of Philosophy* 44: 267–92.

Putnam, Hilary, 1967. "The Thesis that Mathematics Is Logic." In *Bertrand Russell: Philosopher of the Century*, edited by Ralph Schoenman, pp. 273–303. London: Allen & Unwin.

Quine, W. V., 1986. *Philosophy of Logic*, 2nd ed. Cambridge, MA: Harvard University Press.

Rayo, Agustín, 2007. "Logicism Reconsidered." In *The Oxford Handbook of Philosophy of Mathematics and Logic*, edited by Stewart Shapiro, pp. 203–35. Oxford: Oxford University Press.

Roselló, Joan, 2012. *From Foundations to Philosophy of Mathematics*. Newcastle upon Tyne: Cambridge Scholars Publishing.

Russell, Bertrand, 1897. *An Essay on the Foundations of Geometry*. Cambridge: Cambridge University Press.

——, 1919. *Introduction to Mathematical Philosophy*. London: Allen & Unwin.

——, 1946. *History of Western Philosophy*. London: Allen & Unwin.

——, 1958. *My Philosophical Development*. London: Allen & Unwin.

——, 1990. *The Collected Papers of Bertrand Russell*, vol. 2, *Philosophical Papers 1896–99*, edited by N. Griffin and A. C. Lewis. London: Unwin Hyman.

——, 1993. *The Collected Papers of Bertrand Russell*, vol. 3, *Toward the "Principles of Mathematics" 1900–02*, edited by G. H. Moore. London: Routledge.

——, 1994. *The Collected Papers of Bertrand Russell*, vol. 4, *Foundations of Logic 1903–05*, edited by A. Urquhart. London: Routledge.

——, 1997. *The Collected Papers of Bertrand Russell*, vol. 11, *Last Philosophical Testament 1943–68*, edited by J. G. Slater. London: Routledge.

——, 1998. *Autobiography*. London: Routledge.

——, 2014. *The Collected Papers of Bertrand Russell*, vol. 5, *Towards "Principia Mathematica" 1905–08*, edited by G. H. Moore. London: Routledge.

Soames, Scott, 2014. *The Analytic Tradition in Philosophy*, vol. 1, *The Founding Giants*. Princeton: Princeton University Press.

Tennant, Neil, 2013. "Logicism and Neologicism." *The Stanford Encyclopedia of Philosophy*, `http://plato.stanford.edu/entries/logicism/`, accessed 19 March 2016.

Whitehead, A. N. and Bertrand Russell, 1925–1927. *Principia Mathematica*, 2nd ed., 3 vols. Cambridge: Cambridge University Press.

Yi, Byeong-uk, 2013. "The Logic of Classes of the No-Class Theory." In Griffin and Linsky (2013), pp. 96–129.