

## Modeling turbulent dissipation at low and moderate Reynolds numbers

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The dissipation of kinetic energy is one of the key features of turbulent flows that must be modeled accurately in order to obtain useful engineering predictions. At high Reynolds numbers the assumption of scale separation can be invoked in the modeling of the dissipation process. This paper focuses on the more difficult issue of modeling the dissipation process at moderate and low Reynolds numbers. The low and moderate Reynolds number range is very important for tuning turbulence models and for many practical engineering problems. To approach this problem, an alternative formulation to the classic dissipation scale equation is proposed. The interesting feature of this formulation, an inverse lengthscale equation, is that it captures both the high Reynolds number and low Reynolds number decay limits. A careful assessment of existing data then allows us to clearly identify the region of transition between high and low Re and propose a very simple equation system which can accurately model dissipation at any Reynolds number. The equivalent  $K/\varepsilon$  model is derived and the proposed model is compared with a number of other low Re dissipation modifications for the  $K/\varepsilon$  equation system. To complete the discussion, the issue of near-wall dissipation modeling is carefully examined and shown to be fundamentally different from the low Reynolds number limit. This is shown to be an important distinction of practical modeling importance.

*Keywords:* Dissipation; Low Reynolds number; Isotropic decay; Turbulence modeling.

### 1. Introduction

The turbulent energy cascade, and associated dissipation of energy, is one of the key nonlinear processes that govern the behavior of turbulence. Significant theoretical work has been performed concerning how this process behaves at high Reynolds numbers. Considerably less theoretical analysis is available at moderate Reynolds numbers. However, the modeling of dissipation at low and moderate Reynolds numbers is of direct engineering interest. Very low turbulent Reynolds numbers are often present in the free-stream of external flows. Moderate Reynolds numbers are fairly common in many applications (I.C. engines being one example). Furthermore, the very useful technique of tuning and developing turbulence models based on direct numerical simulation (DNS) and experimental data requires that the moderate Reynolds number regime be well understood (since all DNS simulations, and many experiments, occur at moderate Reynolds numbers).

While this work does not provide any additional mathematical analysis of the moderate and low Reynolds number dissipation regime, it does provide the very interesting and useful

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observation that certain (nonstandard) two-equation model systems can capture the high and low Reynolds number dissipation limits without using any model constants. In addition, we show that this property is fundamentally not achievable by the classic  $k/\varepsilon$  two-equation model system. In itself, accurate prediction of the dissipation alone does not fix the entire turbulence modeling problem. Nevertheless, this work is a step in the direction of developing turbulence models that predict real physical effects.

The canonical flow situation that focuses on the dissipation process is isotropic homogeneous decaying turbulence. A number of very different analyses [1–5] all suggest that decaying turbulence should have a power law behavior in time,

$$K = K_0 \left( 1 + \frac{\varepsilon_0 t}{n K_0} \right)^{-n}. \quad (1)$$

In this expression,  $K_0$  and  $\varepsilon_0$  are the initial turbulent kinetic energy and dissipation rate at the time  $t = 0$  and  $n$  is a universal exponent. While the power law is found to hold, the exponent is not universal. Values of the decay exponent of between 1.1 and 1.3 [22] are often observed in high Reynolds number wind tunnel experiments. Compte–Bellot and Corrsin [18] report a value of 1.26. Additional experimental and simulation values are shown in table 1. Batchelor and Townsend [6] presented the first analysis and experiments for very low Reynolds number turbulent decay and suggested that the value of the decay exponent should be  $5/2$  in that regime. This particular low Reynolds number limit is widely known and frequently used in turbulence models, though perhaps incorrectly.

It is now understood that the decay exponent is closely related to the low wavenumber portion of the 3D energy spectrum [7]. For isotropic turbulence in which the low wavenumber portion of the spectrum goes as  $k^2$  (where  $k$  is the wavenumber) the low Reynolds number exponent was shown to be  $3/2$  (not the frequently used  $5/2$ ) and the high Reynolds number limit to be  $6/5$  [7] (in the range of what is commonly observed in experiments). However, if the low wavenumber portion of the spectrum goes as  $k^4$ , the exponent is indeed  $5/2$  in the low Reynolds number limit and a slightly higher  $10/7$  in the high Reynolds number limit [2]. The analysis of Kolmogorov for the high Re  $k^4$  limit is only approximate because it assumes the Loitzianskii integral is invariant. Simulation data [12] and EDQNM results [28] suggest this integral varies quite slowly in time and the actual value is not  $10/7$  but 3% (Lesieur) to 6% (Chasnov) lower (1.38–1.34 instead of 1.428). At this time, it is not clear which (if either) of the two different spectral behaviors  $k^2$  or  $k^4$  is physically more correct. It is possible that both are viable alternatives, though most experiments seem to the authors to be closer to the  $k^2$  values. The book by Lesieur [28] gives a detailed description of how both the low and high Re asymptotic limits are determined.

Two-equation models can (and in some sense, must) capture the low wavenumber portion of the energy spectrum into the model formulation. The model’s assumed low wavenumber behavior is always implicit in the choice of certain model constants. This paper does not actually concern itself with the debate of  $k^2$  versus  $k^4$  and so, unlike most dissipation models, we leave the low wavenumber behavior as an explicit free parameter (for the user’s preference)

Table 1. Summary of theoretical values for the decay exponent  $n$  and the  $k/\varepsilon$  constant  $C_{\varepsilon 2}$  for various low wavenumber exponents and Re numbers.

	$n$ -High Re	$n$ -Low Re	$C_{\varepsilon 2}$ -High Re	$C_{\varepsilon 2}$ -Low Re
$k^2$	$6/5$	$3/2$	$11/6$	$5/3$
$k^4$	$10/7$	$5/2$	$17/10$	$7/5$
$k^p$	$2(p+1)/(p+3)$	$(p+1)/2$	$(3p+5)/(2p+2)$	$(p+3)/(p+1)$

in the model. The authors currently use  $k^2$  in applied problems but the following analysis is never bound to that choice and the  $k^4$  alternative is given equal consideration in this paper.

The  $K/\varepsilon$  model is a useful and familiar point to begin the discussion of two-equation dissipation modeling. Note that a capital  $K$  is used for turbulent kinetic energy and a lower case  $k$  for wavenumber. For isotropic decaying turbulence, this model is very simple and is given by

$$\frac{dK}{dt} = -\varepsilon \quad (2)$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2} \frac{\varepsilon^2}{K}. \quad (3)$$

The first equation is exact and the right-hand side of the second equation represents the model. Substituting the power law expression (equation 1) into equation (2) shows that the dissipation also has a power law behavior,  $\varepsilon = \varepsilon_0(1 + \frac{\varepsilon_0 t}{nK_0})^{-n-1}$ . Substituting this power law expression into equation (3) and again using equation (1) reveals the relation between the model constant and the decay exponent,  $C_{\varepsilon 2} = \frac{(n+1)}{n}$ . Since the decay exponent is not a universal constant, this analysis makes it clear that  $C_{\varepsilon 2}$  is not actually a model constant either, but is a model parameter that really should be a function of the turbulent Reynolds number and the low wavenumber spectrum. For convenience, the theoretical behavior of the exponent and model constant are summarized below. The general expression for  $k^p$  at high Re is exact for  $p < 4$ . It is only a good approximation when  $p = 4$ .

There are a large number of proposals concerning the functional form for  $C_{\varepsilon 2}$ . Almost all of these proposals are a function of the turbulent Reynolds number,  $Re_T = \frac{K^2}{\nu\varepsilon}$ . A representative sample of commonly used expressions is presented below.

$$C_{\varepsilon 2} = 1.92(1 - 0.3e^{-Re_T^2}) \quad [8] \quad (4a)$$

$$C_{\varepsilon 2} = C_{\varepsilon 2}^{\infty} \min\left(1, \frac{Re_T^{1/2}}{6}\right) \quad [9] \quad (4b)$$

$$C_{\varepsilon 2} = C_{\varepsilon 2}^{\infty} + (C_{\varepsilon 2}^{\infty} - 1.4)e^{-(Re_T/6)^2} \quad [10] \quad (4c)$$

$$C_{\varepsilon 2} = 1.4 + 0.38(1 - e^{-0.5Re_T^{1/2}} + 0.62e^{-40Re_T^{-1/2}}) \quad [11] \quad (4d)$$

The last two expressions have correct low Reynolds number behavior (consistent with the  $k^4$  spectrum) but typically use values more consistent with the  $k^2$  spectrum for the high Re limit. The second model goes to zero in the low Reynolds number limit. This is a common feature of models that are tuned/developed to work near-walls, since this type of behavior fixes some near-wall modeling problems. Unfortunately a zero low Re limit causes other problems (described in section 6) away from walls (in the free-stream). The first expression was developed before the exact asymptotic limits were well known, but gets reasonably close based purely on data fitting. The functional choices behind these models were motivated largely by simplicity and the fact that each has the correct low and high Reynolds number asymptotic limits desired by the developers.

In this paper, a model for  $C_{\varepsilon 2}$  is derived from very basic modeling assumptions and a functional form for  $C_{\varepsilon 2}$  is arrived at rather than hypothesized. Section 2 of the paper describes a  $K/\lambda$  model (where  $\lambda$  is an inverse turbulent lengthscale) where both the high and low Reynolds number limits and low wavenumber behavior are represented well and independently of any model constants. This interesting formulation (which is not possible with  $K/\varepsilon$ ) forms the basis for our dissipation model. The one constant that does exist in this model sets the region of transition from low to high Reynolds number. This constant is very carefully determined

in section 3 using DNS and experimental data. In addition, a comparison with the commonly used low Reynolds number dissipation models described above, by equation (4), is performed. Section 4 is a short aside on how the decay exponent is determined within this paper, while section 5 derives the equivalent expression for  $C_{\varepsilon 2}$  that corresponds to the proposed  $K/\lambda$  model. The problem of near-wall dissipation modeling, which is also a low Reynolds number situation, is extensively discussed in section 6 and the importance of distinguishing between near-wall and low Re effects is discussed. A summary of the paper's conclusions and a brief discussion is presented in section 7.

## 2. The $K/\lambda$ model

We begin with the following general two-equation model for the decay of isotropic turbulence:

$$\frac{dK}{dt} = - \left\{ \alpha_L \nu \lambda^2 + \alpha_H K^{\frac{1}{2}} \lambda \right\} K \quad (5a)$$

$$\frac{d\lambda}{dt} = - \left\{ \beta_L \nu \lambda^2 + \beta_H K^{\frac{1}{2}} \lambda \right\} \lambda, \quad (5b)$$

where  $\lambda$  is the inverse lengthscale of the turbulence. The inverse lengthscale goes to zero in a laminar flow and is therefore an easier quantity to model than the lengthscale itself. Equation (5b) essentially replaces the dissipation (or omega) equation as the second ‘scale’ equation in the two-equation model system. The problems with starting directly with the dissipation equation are analyzed in detail in section 5. We note that these equations can easily be generalized to anisotropic turbulence.

Equation (5a) was motivated largely by how the energy equation looks in wavenumber space and how the exact two-point correlation equation looks. In those cases, the viscous and nonlinear (cascade) terms in the energy equation are separate and additive. In wavenumber space the viscous term does not require a model and has a form almost identical to the first term in equation (5a). It is linear in the energy and quadratic in the wavenumber (which also has units of inverse length). The high Re number nonlinear cascade term uses the simplest model possible that is dimensionally correct and excludes the viscosity. Equation (5b) was largely modeled after (5a). The first term will make  $\lambda \sim t^{-1/2}$  which is classical behavior for a lengthscale under the influence of viscosity. The high Reynolds number (second) term in (5b) is the simplest dimensionally correct term possible. Like the energy equation, this equation also assumes that the viscous and nonlinear effects are separate and additive. All modeling requires some assumptions, but we consider these assumptions to be very reasonable and very weak.

Like the standard  $K/\varepsilon$  equation system, this system has power law solutions. They are of the form  $K = K_0(1 + \frac{t}{t_0})^{-n}$  and  $\lambda = \lambda_0(1 + \frac{t}{t_0})^{-m}$ . Substituting these solutions into equations (5a) and (5b) gives the relations

$$\frac{K_0}{t_0} n \left(1 + \frac{t}{t_0}\right)^{-n-1} = \alpha_L \nu \left(1 + \frac{t}{t_0}\right)^{-2m-n} K_0 \lambda_0^2 + \alpha_H \left(1 + \frac{t}{t_0}\right)^{-3n/2-m} K_0^{3/2} \lambda_0 \quad (6a)$$

$$\frac{\lambda_0}{t_0} m \left(1 + \frac{t}{t_0}\right)^{-m-1} = \beta_L \nu \left(1 + \frac{t}{t_0}\right)^{-3m} \lambda_0^3 + \beta_H \left(1 + \frac{t}{t_0}\right)^{-2m-n/2} \lambda_0^2 K_0^{1/2} \quad (6b)$$

At high Reynolds numbers the terms involving the viscosity drop out and equation (6a) implies that  $n + 2m = 2$  and  $n = \alpha_H \lambda_0 K_0^{1/2} t_0$ . In addition, equation (6b) implies that  $n + 2m = 2$  and  $m = \beta_H \lambda_0 K_0^{1/2} t_0$ . The first two conditions are identical. The second two constraints can be

cast into the alternative relation that

$$\frac{n}{m} = \frac{\alpha_H}{\beta_H}. \quad (7a)$$

At high Re we know that  $n = 6/5$  when the low wavenumber portion of the spectrum varies as  $k^2$ . This implies that  $m = 2/5$  and  $\beta_H = \frac{\alpha_H}{3}$ . For a  $k^4$  low wavenumber spectrum it is expected that  $n = \frac{10}{7}$ , implying  $m = 2/7$  and  $\beta_H = \frac{\alpha_H}{5}$ . In either case we can write  $\beta_H = \frac{\alpha_H}{p+1}$ , where  $p$  is the low wavenumber exponent.

At low Reynolds numbers the terms involving viscosity dominate the right-hand side of both equations (6a) and (6b). Equation (6a) therefore implies that  $m = 1/2$  and  $n = \alpha_L \nu \lambda_0^2 t_0$  and equation (6b) implies  $m = 1/2$  and  $m = \beta_L \nu \lambda_0^2 t_0$ . As in the high Reynolds number case, the first two constraints are identical. The final two constraints produce a relation similar to equation (7a):

$$\frac{n}{m} = \frac{\alpha_L}{\beta_L}. \quad (7b)$$

At low Re we know that  $n = 3/2$  when the low wavenumber portion of the spectrum varies as  $k^2$ , since  $m = 1/2$ ,  $\beta_L = \frac{\alpha_L}{3}$ . For a  $k^4$  low wavenumber spectrum it is expected that  $n = \frac{5}{2}$ , so with  $m = 1/2$ ,  $\beta_L = \frac{\alpha_L}{5}$ . As with the high Reynolds number case this can be generally expressed as  $\beta_L = \frac{\alpha_L}{p+1}$ . Note that the value of  $m = 1/2$  implies that the lengthscales grow with classic viscous scaling with time in the low Re limit.

It is remarkable that this formulation of the problem reduces a number of seemingly disjoint asymptotic limits to two very simple expressions for the model constants (equations (7)). It is now possible to reformulate (5a) and (5b) as,

$$\frac{dK}{dt} = -\frac{1}{\tau} K \quad (8a)$$

$$\frac{d\lambda}{dt} = -\frac{1}{p+1} \frac{1}{\tau} \lambda, \quad (8b)$$

where  $p$  is the exponent of the low wavenumber portion of the spectrum and the inverse time scale is given by

$$\frac{1}{\tau} = \alpha_H \left\{ \frac{\alpha_L}{\alpha_H} \nu \lambda^2 + K^{\frac{1}{2}} \lambda \right\}. \quad (8c)$$

We write it this way to emphasize that the constant  $\alpha_H$  is arbitrary and only serves to set the ratio between  $\lambda$  and the large eddy lengthscale. For example, an obvious choice for the inverse lengthscale is that it equals the large eddy lengthscale at high Reynolds numbers ( $\lambda = \frac{\varepsilon}{K^{3/2}}$ ). Since  $\frac{1}{\tau} = \frac{\varepsilon}{K}$  (see equation (8a)) this choice implies  $\alpha_H = 1$ . We will use this value in all the following results, but emphasize that any choice is possible, and any choice has absolutely no effect on the model results.

The only parameter of real interest in this model is therefore,  $\frac{\alpha_L}{\alpha_H}$ . This ratio sets the Re number range where the model changes from high Reynolds number behavior ( $m = 2/5$  or  $m = 2/7$ ) to low Reynolds number behavior ( $m = 1/2$ ). Note that both the high and low Re limits are not affected by the value of  $\frac{\alpha_L}{\alpha_H}$  which only sets the transition point.

The simplicity of equations (8a)–(8c) and their ability to capture exactly all the asymptotic limits are a suggestions, though certainly not a proof, that this approach to modeling low and moderate Reynolds numbers is preferable to the classic  $k/\varepsilon$  approach where a functional dependence for  $C_{\varepsilon 2}$  is hypothesized.

Note that having a model that is mildly sensitive to the low wavenumber portion of the spectrum in no way invalidates the conceptual interpretation of the epsilon equation as a

model for the energy cascade. However, it is clear from the analysis of Saffman [7] and others [2, 6] that the details of the cascade (the actual value of the decay rate and therefore the constant in the epsilon equation) are influenced by the shape of the energy spectrum. The present model can take information about the spectrum shape as an input and reproduce the known cascade behaviors (decay exponents).

### 3. Intermediate range

Direct numerical simulation (DNS) data and experimental results indicate that the crossover between high and low Reynolds number decay rates occurs roughly in the range  $0.1 \leq Re_T \leq 100$ . Other works on this topic often use the Taylor microscale Reynolds number,  $Re_{\lambda_g} = \frac{\lambda_g (\frac{2}{3}K)^{1/2}}{\nu}$ , where  $\lambda_g$  is the transverse Taylor microscale (not to be confused with our inverse lengthscale,  $\lambda$ ). The relationship between the two Reynolds numbers in isotropic turbulence is  $Re_{\lambda_g} = (\frac{20}{3}Re_T)^{1/2}$ . It is probably not an accident that the crossover point occurs at a value where these two Reynolds numbers have roughly the same magnitude.

In DNS simulations and experiments it is easy to poorly estimate  $Re_T$  if the low wavenumber portion of the spectrum is not very well resolved, whereas  $Re_{\lambda_g}$  (which depends on the small scale structure) is less prone to this type of error and is the more reliable indicator. On the other hand,  $Re_T$  is the variable almost universally used in turbulence models, and we use it in this work for that reason.

Figure 1 shows the behavior of the kinetic energy power law exponent,  $n$ , as a function of the Reynolds number for the case when the low wavenumber behavior of the spectrum is  $k^2$ . The two horizontal lines are the theoretical high and low Reynolds number asymptotic limits of 1.2 and 1.5, respectively. The dotted lines are results from five different  $256^3$  LES simulations of Chasnov [12]. These simulations start with reasonable, but not Navier–Stokes turbulence, and therefore have an initial oscillatory transient that violates the high Re number upper bound. In addition, it is shown in the following section that Chasnov’s method for estimating the exponent is a consistent underestimate with the error being very large at early

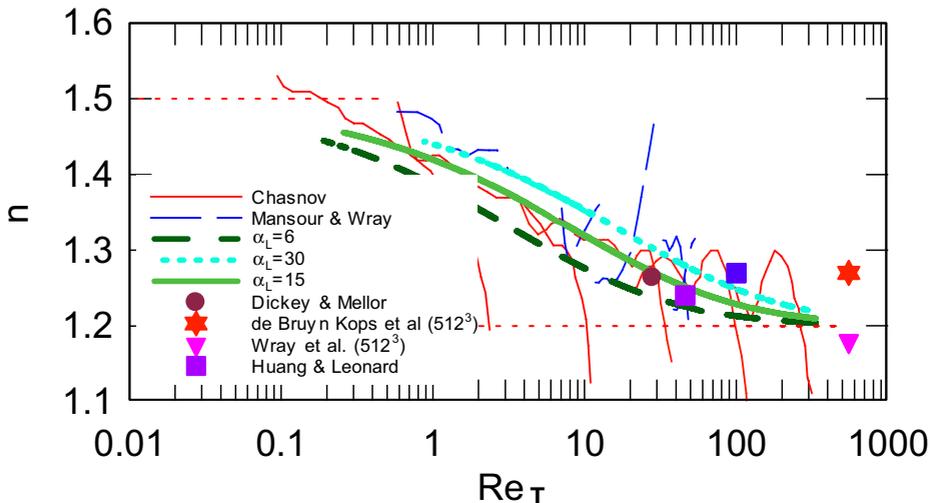


Figure 1. Power law exponent as a function of turbulent Reynolds number for a  $k^2$  low wavenumber spectrum.

times. The lowest Reynolds number case also violates the low Re number bound. We speculate that this could be due to lack of resolution. A too small box causes the magnitude of  $n$  to be too large [13–15] (approaching a value of 2).

In addition to the data of Chasnov, the five lowest Reynolds number  $256^3$  DNS simulations of Mansour and Wray [16] (thin dashed lines) are shown. Their method for calculating  $n$  is better, but the initial conditions for these simulations are still unphysical and hence results at early times are still suspect. Huang and Leonard [17] also performed  $256^3$  DNS simulations but only provide a single value for the decay exponent. The two simulations with the best resolved  $k^2$  low wavenumber portion of the spectrum are shown because the other two simulations may be affected by the small box size of the simulation.

At the upper end of the Reynolds number range the results from the  $512^3$  DNS simulation of de Bruyn Kops and Riley [13] are shown (star). This DNS is a very close approximation (in terms of spectra and Reynolds number) to the experiments of Comte–Bellot and Corrsin [18] but provides considerably more detail. Wray [19] performed a similar  $512^3$  DNS simulation (indicated by the triangle). The difference between these two simulations shows the difficulty of estimating the decay exponent, and the strong influence that the low wavenumbers have on the decay exponent (the two simulations were initialized with slightly different numerical approximations of the Comte–Bellot spectra). In addition, figure 1 shows the experiments of Dickey and Mellor [20] (circle). Many other experiments exist, but the results are usually too noisy to accurately predict a decay exponent. Finally, the thick lines represent the predictions of the proposed model. Three different curves are presented for values of  $\frac{\alpha_L}{\alpha_H}$  equal to 6, 15 and 30. We recommend the value of 15. Note that the results are not highly sensitive to this ratio and that with this one constant the functional variation of the decay exponent is well captured over its entire range.

Figure 2 is similar to figure 1, except that results for a spectrum with a  $k^4$  low wavenumber behavior. The horizontal dashed lines are the asymptotic high and low Reynolds number limits. The thin dashed lines are four  $256^3$  DNS simulations of Mansour and Wray. The four thin solid lines are some more recent  $128^3$  LES simulations of Yu *et al.* [21]. Three  $256^3$  DNS simulation results from Huang and Leonard are also included. No experimental values have been included on this plot because they would all fall under the high Re value—even if that value were adjusted downward by 6%.

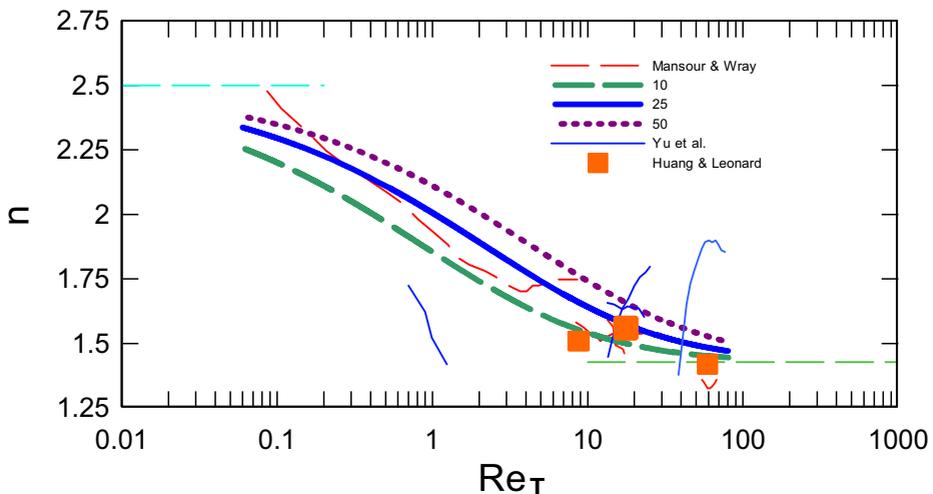


Figure 2. Power law exponent as a function of Reynolds number for a  $k^4$  low wavenumber spectrum.

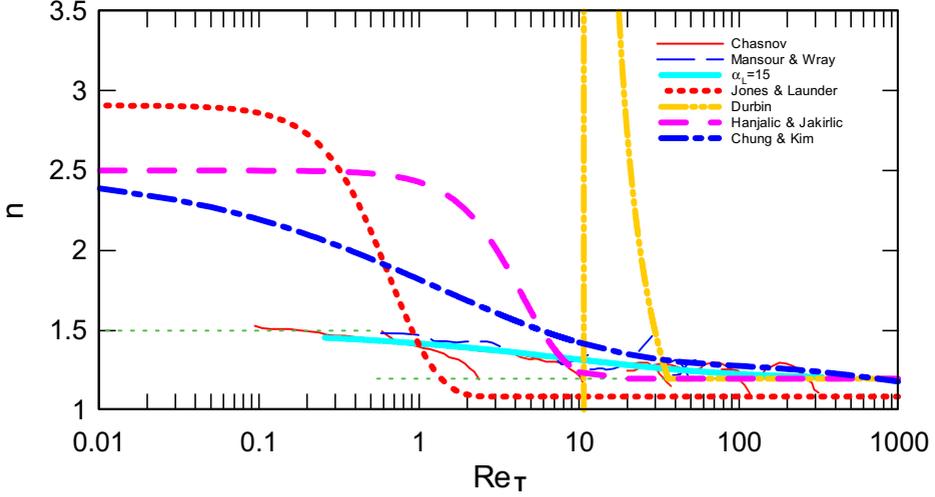


Figure 3. Comparison of low Reynolds number models for the  $k^2$  low-wavenumber spectrum.

The thick solid curve is the proposed model prediction with  $\frac{\alpha_L}{\alpha_H} = 25$ , the dashed curve is  $\frac{\alpha_L}{\alpha_H} = 10$  and the small thick dashed line is  $\frac{\alpha_L}{\alpha_H} = 50$ . A single constant captures the functional variation reasonably well over its entire range. The data do not support finding this ratio very precisely. Given the level of accuracy of the data the simple previous value of  $\frac{\alpha_L}{\alpha_H} = 15$  is probably sufficient at this time. Note that  $\frac{\alpha_L}{\alpha_H} = 15$  implies that at very low Reynolds numbers the inverse lengthscale,  $\lambda$ , is proportional to the inverse of the Taylor microscale,  $\lambda = \sqrt{\frac{\varepsilon}{K15\nu}} = \sqrt{\frac{2}{3}} \frac{1}{\lambda_g} = \sqrt{\frac{4}{3}} \frac{1}{\lambda_f}$ . The inverse lengthscale therefore has a firm physical interpretation at both high and low Re.

For comparison the models described by equations (4a)–(4d) are compared with the proposed model in figure 3 for the  $k^2$  low wavenumber spectra case. The models tend to be close to 2.5 in the low Re limit. This is due to Batchelor’s renowned ‘final period of decay’ work. Unfortunately, the applicability of this result and the utility of these models may be limited by the fact that Batchelor’s result is for a  $k^4$  low wavenumber spectrum which may be rare in practice. The model of Durbin has a decay exponent that goes to infinity around a turbulent Reynolds number of 10 and then gives a negative exponent when  $C_{\varepsilon 2} < 1$ . This is due to the fact that it is tuned for walls—not low Re, and as discussed in section 5 the two situations are really very different.

#### 4. Decay exponent estimation

The decay exponent is very difficult to determine accurately. Batchelor and other early experiments assumed that at large times  $K = K_0(1 + \frac{\varepsilon_0 t}{nK_0})^{-n} \approx Ct^{-n}$  and therefore  $n$  could be obtained from the slope of a log–log plot. Even a few current researchers such as Chasnov use this method to estimate  $n$ . However, in practice, the virtual origin cannot be neglected and doing so leads to an underestimate. Since  $\log K = \log K_0 - n \log(1 + \frac{\varepsilon_0 t}{nK_0})$  the slope on a log–log plot is really the decay exponent times a number that is less than 1,  $\frac{d \ln K}{d \ln t} = -n \frac{\varepsilon_0 e^{\ln t}/nK_0}{(1 + \varepsilon_0 e^{\ln t}/nK_0)} = -n \frac{1}{(nK_0/\varepsilon_0 t + 1)}$ . At early times (less than an eddy turnover time) this underestimate is severe. At 10 eddy turnover times (the limit of many experiments), the error is still 10% (which is significant when the values only vary between 1.2 and

1.5). Chasnov's data extends further than 10 eddy turnover times and this estimate becomes acceptable.

Later estimates attempted to account for the 'offset' by finding the correction that produced the straightest line on a log–log plot<sup>22</sup>. Mohamed and LaRue [22] show (in their figure 7) that the estimate for the decay exponent is significantly affected by differences in this assumed offset, and that very different offsets can still produce lines which continue to look very linear on a log–log plot. Looking at which line is the most linear (in a least squares sense) is unreliable since these best fits are frequently corrupted by the data at early or late times, where the turbulence is still wakes (early times) or is affected by walls (late times). For these reasons, most of the quoted values available in the literature should be used with considerable caution.

Mansour and Wray use equations (2) and (3) and the relation  $C_{\varepsilon 2} = \frac{(n+1)}{n}$  to obtain the expression

$$\frac{1}{n} = \frac{K \frac{d^2 K}{dt^2}}{\left(\frac{dK}{dt}\right)^2} - 1. \quad (9a)$$

This method performs best when data are available with very small time increments so the second derivative is accurately calculated. An alternative method used in this work is

$$\frac{1}{n} = -\frac{d}{dt} \left( \frac{K}{\frac{dK}{dt}} \right). \quad (9b)$$

In addition to these complications the low Reynolds number regime can also be strongly affected by the computational or experimental setup. Touil *et al.* [14] show that when walls (or computational periodicity) constrains the flow the low wavenumber spectrum is truncated and the decay exponent approaches  $n = 2$ . The extremely precise experimental measurements of Ling and Huang show this effect very clearly [23]. In addition, boundary layers in wind tunnel experiments can contaminate the core flow after long times or at low Reynolds numbers [24]. This remains a fundamental flow situation in which careful experiments could be very valuable.

## 5. Implications for $K/\varepsilon$ models

The inverse time constant is related to the dissipation by  $\frac{1}{\tau} = \left\{ \frac{\alpha_L}{\alpha_H} \nu \lambda^2 + K^{\frac{1}{2}} \lambda \right\} = \frac{\varepsilon}{K}$  (assuming  $\alpha_H = 1$ ). This expression can be inverted to obtain an explicit expression for the inverse lengthscale. Starting with  $K \frac{\alpha_L}{\alpha_H} \nu \lambda^2 + K^{\frac{1}{2}} \lambda - \varepsilon = 0$ , we find that only one root is physical and is given by

$$\lambda = K^{\frac{1}{2}} \left[ \left( 1 + 4 \frac{\alpha_L}{\alpha_H} \frac{1}{Re_T} \right)^{1/2} - 1 \right] / \left[ 2 \frac{\alpha_L}{\alpha_H} \nu \right]. \quad (10)$$

When the Reynolds number is large this becomes  $\lambda \approx \frac{K^{\frac{1}{2}}}{\nu Re_T} = \frac{\varepsilon}{K^{\frac{3}{2}}}$ . When the Reynolds number is small it becomes  $\lambda \approx \left( \frac{\alpha_L}{\alpha_H} \frac{K}{\varepsilon} \nu \right)^{-1/2} = \frac{\varepsilon}{K^{3/2}} \left( \frac{\alpha_H}{\alpha_L} Re_T \right)^{1/2}$ . It is convenient to write the inverse lengthscale as  $\lambda = \frac{\varepsilon}{K^{\frac{3}{2}}} f$ , where  $f = Re_T \left[ \left( 1 + 4 \frac{\alpha_L}{\alpha_H} \frac{1}{Re_T} \right)^{1/2} - 1 \right] / \left[ 2 \frac{\alpha_L}{\alpha_H} \right]$ . At high Reynolds numbers  $f \approx 1$  and at low Reynolds numbers  $f = \left( \frac{\alpha_H}{\alpha_L} Re_T \right)^{1/2}$ .

Since we have  $\varepsilon = K \left\{ \frac{\alpha_L}{\alpha_H} \nu \lambda^2 + K^{\frac{1}{2}} \lambda \right\}$ , the dissipation equation can also be derived,

$$\frac{d\varepsilon}{dt} = \frac{\alpha_L}{\alpha_H} \nu \left( 2\lambda \frac{d\lambda}{dt} K + \lambda^2 \frac{dK}{dt} \right) + \frac{3}{2} K^{\frac{1}{2}} \frac{dK}{dt} \lambda + K^{\frac{3}{2}} \frac{d\lambda}{dt}, \quad (11a)$$

which simplifies to

$$\frac{d\varepsilon}{dt} = -\frac{\alpha_L}{\alpha_H} \nu \left( \frac{2}{p+1} + 1 \right) \varepsilon \lambda^2 - \left( \frac{3}{2} + \frac{1}{p+1} \right) K^{\frac{1}{2}} \varepsilon \lambda. \quad (11b)$$

Substituting for the inverse lengthscale gives our final expression,

$$\frac{d\varepsilon}{dt} = -\frac{\varepsilon^2}{K} \left\{ \frac{\alpha_L}{\alpha_H} \left( \frac{2}{p+1} + 1 \right) \frac{1}{\text{Re}_T} f^2 + \frac{1}{2} \left( \frac{2}{p+1} + 3 \right) f \right\}. \quad (11c)$$

The dissipation constant  $C_{\varepsilon 2}$  can now be identified as the term in brackets and is given by the expression  $C_{\varepsilon 2} = \left\{ \frac{1}{2} \left( \frac{2}{p+1} + 3 \right) f + \frac{\alpha_L}{\alpha_H} \left( \frac{2}{p+1} + 1 \right) \frac{1}{\text{Re}_T} f^2 \right\}$ . At high Reynolds numbers the first term dominates and at low Reynolds numbers the second term dominates. This expression can also be written as

$$C_{\varepsilon 2} = f \left\{ 1 + \left( \frac{1}{p+1} + \frac{1}{2} \right) \left[ \left( 1 + 4 \frac{\alpha_L}{\alpha_H} \frac{1}{\text{Re}_T} \right)^{1/2} \right] \right\}. \quad (12)$$

We note that the ideas developed in section 2 cannot be applied directly to the  $k/\varepsilon$  equation system. The fundamental reason for this is that the variable  $\varepsilon$  represents two physical effects (viscous dissipation and nonlinear dissipation) at the same time. In detail, this difficulty is explained below.

If we start with the general  $k/\varepsilon$  system

$$\frac{dK}{dt} = -\varepsilon \quad (13a)$$

$$\frac{d\varepsilon}{dt} = -\left\{ \beta_L \nu \frac{\varepsilon^2}{K^3} + \beta_H \frac{\varepsilon}{K} \right\} \varepsilon \quad (13b)$$

and assume power law solutions of the form  $K = K_0(t + t_0)^{-n}$  and  $\varepsilon = \varepsilon_0(t + t_0)^{-m}$ , then the following relations are obtained:

$$K_0 n(t + t_0)^{-n-1} = (t + t_0)^{-m} \varepsilon_0 \quad (14a)$$

$$\varepsilon_0 m(t + t_0)^{-m-1} = \beta_L \nu (t + t_0)^{-3m+3n} K_0^{-3} \varepsilon_0^3 + \beta_H (t + t_0)^{-2m+n} K_0^{-1} \varepsilon_0^2. \quad (14b)$$

At high Reynolds numbers, the system works fine and we find that  $\beta_H = \frac{11}{6}$  for the  $k^2$  low wavenumber spectrum and  $\beta_H = \frac{17}{10}$  for the  $k^4$  low wavenumber spectrum. In general,  $\beta_H = \frac{3p+5}{2p+2}$ , where  $p$  is the spectrum exponent.

However, at low Reynolds numbers (the case of interest) it is found that for power law solutions to exist,  $n = 1$  and  $m = 2$  is required. This is the fully self-similar state determined by Speziale and Bernard [5] but it is not the correct power law behavior that is sought at low Reynolds numbers. This deficiency of equations (13) is, of course, corrected in the classic modeling approach by making  $\beta_H$  and  $\beta_L$  (or equivalently  $C_{\varepsilon 2}$ ) some function of the Reynolds number. But this is invariably an *ad hoc* fix that is avoiding a real issue that equations (13) are trying to point out (a single variable should not represent two very different physical processes).

The  $k/\omega$  model is traditionally derived by setting  $\omega = K/\varepsilon$ . In this case the scale transport equation becomes  $\frac{d\omega}{dt} = -(C_{\varepsilon 2} - 1)\omega^2$ . Wilcox [29] recommends a value for this constant that is equivalent to  $C_{\varepsilon 2} = \frac{11}{6}$ , which is the  $k^2$  high Re theoretical limit. Commercial CFD

literatures often recommend this model for low Re situations but what they really mean to refer to is near-wall situations.

## 6. Near-wall modifications

Near the wall (in the laminar sublayer) the viscous diffusion and dissipation terms dominate the evolution equations. The production term goes like  $y^3$  near the wall and the turbulent transport terms like  $y^2$  or higher, where  $y$  is the distance to the wall. For this reason these terms are neglected in any analysis of the near-wall model behavior. Since the viscous diffusion term does not require a model, it is the dissipation model that controls the behavior in this region.

The classic epsilon equation dissipation term ( $-C_{\varepsilon 2} \frac{\varepsilon^2}{K}$ ) goes to infinity at a no-slip wall because the turbulent kinetic energy goes to zero and the dissipation is finite. This is a disaster and low Reynolds number models that integrate up to a wall attempt to fix this problem. A very common solution is to make  $C_{\varepsilon 2}$  a function of the turbulent Reynolds number. Since the turbulent kinetic energy and hence the turbulent Reynolds number go to zero at a no-slip wall, it is simple to invent functions of the turbulent Reynolds number that force  $C_{\varepsilon 2}$  to go to zero near the wall. Equation (4b) was chosen as a representative example of these types of models.

In what follows we show that any  $C_{\varepsilon 2}$  that goes to zero at low Reynolds numbers is not a good solution to the near-wall singularity problem. These models not only give incorrect low Reynolds number predictions for isotropic decay, but are actually unstable in that limit. As mentioned earlier, the low Reynolds number decay limit is not an esoteric one and is not limited to the near-wall region. The core flow above a turbulent boundary layer is frequently very low Reynolds number decaying turbulence as well. This type of instability can cause problems in practical flow situations, and may explain why free-stream turbulence levels in RANS simulations are frequently forced to be higher than in the corresponding experiments (to keep the Re well above this instability).

Consider the classic  $k/\varepsilon$  equation system for decaying turbulence in which  $C_{\varepsilon 2}$  is understood to be a function of the turbulent Reynolds number,

$$\frac{dK}{dt} = -\varepsilon \quad (15a)$$

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2} \frac{\varepsilon^2}{K}. \quad (15b)$$

The equation for the turbulent timescale is then

$$\frac{d}{dt} \left( \frac{K}{\varepsilon} \right) = \frac{1}{\varepsilon} K_{,t} - \frac{K}{\varepsilon^2} \varepsilon_{,t} = -1 + C_{\varepsilon 2}. \quad (16)$$

As soon as  $C_{\varepsilon 2}$  drops below 1 the time scale decreases when it should be increasing linearly in time. The problem is not self-correcting. The small time scale leads to an even smaller turbulent Reynolds number (and hence smaller  $C_{\varepsilon 2}$ ). Eventually, the time scale itself,  $K/\varepsilon$ , becomes negative which is unphysical (and corresponds to either  $K$  or  $\varepsilon$  becoming negative). This analysis makes it clear that  $C_{\varepsilon 2}$  should never be less than 1 and an alternative solution for the near-wall singularity is required that does not require  $C_{\varepsilon 2}$  to go to zero. In summary, the near-wall singularity problem should never be accounted for by using functions of the Reynolds number.

Near a wall, the proposed model takes the low Re limit for  $C_{\varepsilon 2}$ ,  $C_{\varepsilon 2} = 1 + \frac{2}{p+1}$ , which is greater than 1, but which does not fix the near-wall singularity problem. How can the proposed approach address the near-wall singularity issue?

One solution (used in a few low  $\text{Re } K/\varepsilon$  models) is to note that while the Reynolds number goes to zero near a wall, the turbulent length and time scales should not go to zero, they should have viscous lower limits. With this interpretation the dissipation equation can be written as

$$\frac{d\varepsilon}{dt} = -C_{\varepsilon 2} \frac{\hat{\varepsilon}}{K} \varepsilon, \quad (17)$$

where the inverse time scale,  $\frac{\hat{\varepsilon}}{K}$ , is finite at the wall. For the same reasons outlined above, it is not sufficient for the modified dissipation,  $\hat{\varepsilon}$ , or the inverse time scale  $\frac{\hat{\varepsilon}}{K}$ , to simply be functions of the Reynolds number. Some other information about the system is necessary besides the  $\text{Re}$ .

Sometimes this variable is defined as an explicit function of the distance to the wall. However, wall distance functions are an *ad hoc* solution that is not very well defined for complex geometries. A better solution is to note that the near-wall region is a region of strong inhomogeneity, not just a region of low Reynolds number. The modified dissipation could therefore be a function of the inhomogeneity. It can be shown that at a no-slip wall  $2\nu[\nabla(K^{1/2})]^2 = \varepsilon$ . In fact, it can even be shown that  $\varepsilon - 2\nu[\nabla(K^{1/2})]^2 = O(y^2)$  in the vicinity of a no-slip wall<sup>25</sup>. The singularity can therefore be removed by defining

$$\hat{\varepsilon} = \varepsilon - 2\nu[\nabla(K^{1/2})]^2. \quad (18)$$

However, significant care is necessary while implementing this correction in a computer code, so that the singularity is also properly eliminated in the numerical approximation. The classic solution is to define the modeled epsilon transport equation to be directly for  $\hat{\varepsilon}$  rather than for  $\varepsilon$ . An alternative formulation [26] is to define,

$$\hat{\varepsilon} = \varepsilon \left/ \left( 1 + C^* \frac{\nu |\nabla(K^{1/2})|}{K} \right) \right., \quad (19)$$

where the value  $C^* = 10$  is suggested [25].

These corrections are only significant very near the wall where the inhomogeneity is large and the Reynolds number is small. This type of near-wall correction (of the time scale rather than  $C_{\varepsilon 2}$ ) does not affect any of the prior results for homogeneous decay or our conclusions about the appropriate functional form for  $C_{\varepsilon 2}$ . Figure 4 shows the modified dissipation

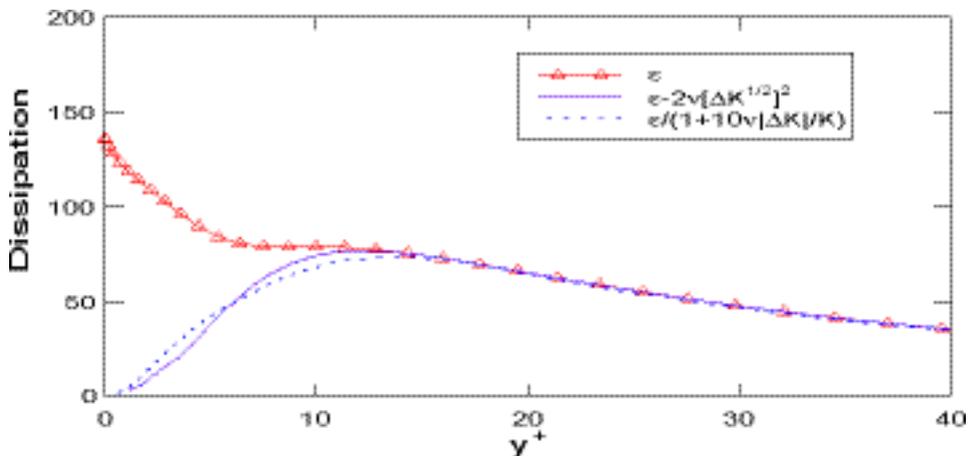


Figure 4. Comparison of dissipation and modified dissipation formulas in channel flow at  $\text{Re}_\tau = 590$ . Note the  $O(y^2)$  behavior at the wall and the restriction to the laminar sublayer.

(equations (18) and (19) and the actual dissipation near the wall using the DNS data of Moser *et al.* [27]. It is clear that the modification is limited to the laminar sublayer.

This approach to removing the singularity can be readily adapted to the inverse lengthscale equation. Normally the inverse lengthscale is also singular as the wall is approached, since  $\lambda \approx \frac{K^{1/2}}{\nu} (\frac{\alpha_L}{\alpha_H} Re_T)^{-1/2} = (\frac{\alpha_H}{\alpha_L} \frac{\varepsilon}{\nu K})^{1/2}$  near a wall. To make this nonsingular at the wall it is desired that  $\lambda \approx (\frac{\alpha_H}{\alpha_L} \frac{\hat{\varepsilon}}{\nu K})^{1/2}$  as the wall is approached. In keeping with the previous assumption of additive terms for each physical dissipation effect we add an additional term for inhomogeneous dissipation (which is exact in the limit of very strong inhomogeneity). Close to the wall, where production and turbulent transport are negligible, this results in the following equation system:

$$\frac{dK}{dt} = -\frac{1}{\tau}K - 2\nu[\nabla K^{1/2}]^2 + \nu\nabla^2 K \quad (20a)$$

$$\frac{d\lambda}{dt} = -\frac{1}{p+1} \frac{1}{\tau} \lambda + \nu\nabla^2 \lambda, \quad (20b)$$

with the inverse time scale still defined by  $\frac{1}{\tau} = \{\frac{\alpha_L}{\alpha_H} \nu \lambda^2 + K^{\frac{1}{2}} \lambda\}$ . The additional inhomogeneous term (2nd term on the right-hand side of equation (20a)) is exact near-walls and forces  $K$  to have  $O(y^2)$  behavior near the wall even when only the boundary condition  $K = 0$  is imposed. The lengthscale is now given by  $\lambda = K^{\frac{1}{2}} \{[1 + 4\frac{1}{Re} \frac{\alpha_L}{\alpha_H}]^{1/2} - 1\} / [2\frac{\alpha_L}{\alpha_H} \nu]$  which is almost the same as equation (10) but uses a modified Reynolds number,  $Re_T = \frac{K^2}{\nu\{\varepsilon - 2\nu[\nabla(K^{1/2})]^2\}}$ .

## 7. Conclusions

A new  $K/\lambda$  model has been proposed that very simply accounts for the dissipation of moderate and low Reynolds number turbulence. This model obtains the correct high and low Reynolds number asymptotic limits for any low wavenumber form of the spectra (with  $k^2$  and  $k^4$  being the most likely choices). It uses one simple constant,  $\frac{\alpha_L}{\alpha_H} = 15$  to capture the entire range of Reynolds numbers to within the scatter of the data. Unlike previous low Re models no hypothesized functionals of the Re were necessary to achieve this behavior.

A translation of this model to the more familiar  $K/\varepsilon$  framework was performed to facilitate implementation in existing codes. While the model is slightly less elegant when viewed from the  $K/\varepsilon$  framework, the proposed functional form for  $C_{\varepsilon 2}$  now has a rational physical basis, is sensitive to the low wavenumber form of the spectrum, and has the correct high and low Reynolds number limits. The paper also shows that it is not possible to perform a similar analysis starting directly from the  $K/\varepsilon$  equation system.

The difficulties that can occur when applying low Reynolds number dissipation models to the near-wall region were carefully examined. It is demonstrated that the singularity near the wall is not fundamentally a low Reynolds number issue and cannot be solved by using Reynolds number corrections. The proposed model was easily extended to the near-wall situation by incorporating an exact inhomogeneous dissipation term. This modification was additive and directly in keeping with the original philosophy of the modeling approach.

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