Newtonian Dynamics

Equations of Motion

The fundamental notions of Mechanics are the notions of state (of a closed system), time, and evolution/motion, all the three being closely related to each other. The closed system is a system which does not interact with the rest of the World. For our purposes, the two words, evolution and motion, are synonyms.

Speaking generally, by the state one means some mathematical object, $S$ (typically, a number, or a set of numbers) being a function of a real variable $t$, which is called time. The evolution/motion is understood as a change of $S$ with $t$. Finding the function $S(t)$ for a given system is the ultimate goal of Mechanics. It is achieved by knowing the state of the system $S(t_0)$ at some initial time moment $t_0$ and utilizing the laws of motion.

In the Newtonian picture of the World, any system can be viewed as consisting of point objects, particles. Each particle $i$ is characterized by a radius-vector (position) $r_i$ in a $d$-dimensional vector space. Typically, $d = 3$, since this corresponds to our real World, but often problems in $d = 3$ (we will be also writing 3D) can be reduced to 2D and 1D problems. If a system consists of $N$ particles, then its state is given by a set of $2N$ vectors, $S(t) \equiv \{r_i(t), v_i(t)\}$, $i = 1, 2, 3, \ldots, N$, where by $v_i$ we mean the derivative of $r_i$ with respect to time (the dot is a shorthand notation for time-derivative):

$$v_i(t) = \frac{d}{dt} r_i(t) \equiv \dot{r}_i(t).$$

The quantity $v_i(t)$ is called velocity of the $i$-th particle at the time $t$. The fundamental problem of Mechanics is then formulated as: Given the positions (radius-vectors) and velocities of all the particles at some moment $t_0$, find the positions and velocities of the particles at any $t > t_0$.

The main axiom of the Newtonian Mechanics (known as the Newton’s Second Law) states that the solution to the problem comes from the following system of $N$ equations of motion.

$$m_i \ddot{r}_i = \sum_{j=1}^{N} F_{ij} \quad (i = 1, 2, 3, \ldots, N),$$

where $m_i$ is some positive number characterizing the $i$-th particle, called its mass; the vector $F_{ij}$ is called the force exerted to the $i$-th particle by the $j$-th particle. The symbol $\ddot{r}_i$ stands for the second derivative of $r_i$ with respect to $t$, which is called acceleration of the $i$-th particle and often is denoted by a special letter:

$$a_i(t) \equiv \ddot{r}_i(t) \equiv \frac{d^2}{dt^2} r_i(t) = \dot{v}_i(t).$$

The force $F_{ij}$ describes the interaction of the $i$-th particle with the $j$-th one. If $F_{ij} = 0$, then the interaction is absent. In particular, if $F_{ij} = 0$ for all $j$’s, then the $i$-th particle does not interact with all the other particles, in which case Eq. (2) says that acceleration of this particle is identically equal to zero, which implies a constant velocity (the so-called Newton’s First Law).

To complete the main axiom, we need to say something about the structure of $F_{ij}$’s: Otherwise, Eq. (2) will remain quite ambiguous and pretty useless. In principle, we could simply give a number of typical examples of specific forms of $F_{ij}$’s. However, we prefer to use a different strategy of developing the theory. In the spirit of a generic axiomatic approach, we will formulate just a few properties of the forces. Amazingly, these properties will allow us to arrive at most deep fundamental theorems.
concerning the evolution.

- First, we require that $F_{ij}$ be a function of only two vector variables, $r_i$ and $r_j$. With this requirement, both mathematical and physical aspects of the theory become unambiguous: The knowledge of the initial state $S(t)$ implies the knowledge of the forces since the latter are defined by the positions of the particles, and, mathematically, the problem of finding $S(t)$ from $S(t_0)$ is a standard Cauchy problem for a system of second-order ordinary differential equations, the existence and uniqueness of the solution being guaranteed by corresponding theorems.

- Second, we require that the forces are given by the gradients of some scalar functions called \textit{interaction potentials} (the sign minus is a matter of convention):

$$F_{ij} = -\frac{\partial}{\partial r_i} U_{ij}(r_i, r_j).$$  \hspace{1cm} (4)

[The forces obeying Eq. (4) are called potential forces.] Furthermore, we postulate the following structure of $U_{ij}(r_i, r_j)$:

$$U_{ij}(r_i, r_j) = U_{ji}(r_i, r_j) \equiv U_{ij}(|r_i - r_j|).$$  \hspace{1cm} (5)

That is we require that the interaction potential between two particles is a function of only the distance between them. [Such potentials are called central potentials and corresponding forces are called central forces.]

From (4), (5) it follows that

$$F_{ij} = f_{ij}(|r_i - r_j|) (r_i - r_j), \quad f_{ij}(r) = f_{ji}(r) = -\frac{1}{r} \frac{dU_{ij}(r)}{dr},$$  \hspace{1cm} (6)

which leads to two very important relations:

$$F_{ij} \parallel (r_i - r_j),$$  \hspace{1cm} (7)

$$F_{ij} = -F_{ji},$$  \hspace{1cm} (8)

The latter relation is known as the Newton’s Third Law.

\textit{Example.} It will not be an exaggeration to say that THE most important example of the interaction potential is

$$U_{ij}(r) = \frac{\gamma_{ij}}{r},$$  \hspace{1cm} (9)

with $\gamma_{ij}$ being a constant depending on the properties of the two particles. In accordance with (7), the force corresponding to this potential is

$$F_{ij}(r) = \gamma_{ij} \frac{r}{r^3};$$  \hspace{1cm} (10)

the absolute value of the force decays as $1/r^2$ with the interparticle distance. This example covers two fundamental forces: (i) the Coulomb force and (ii) the gravitational force. In the former case, $\gamma_{ij}$ is the product of the electric charges of the two particles, while in the latter case it is minus the product of the two masses.

\textbf{Galilean Invariance}

The theory (2), (4), (5) features a remarkable property, called \textit{Galilean invariance}: The form of the equations remains the same, if instead of the variables $\{r_i\}$ one introduces the new set of variables, $\{r'_i\}$, such that

$$r'_i = r_i - v_0 t,$$  \hspace{1cm} (11)
where \( \mathbf{v}_0 \) is some constant velocity. Explicitly, Galilean invariance means

\[
m_i \ddot{\mathbf{r}}_i^t = \sum_{j=1}^{N} \frac{\partial}{\partial \mathbf{r}_i^t} U_{ij}(|\mathbf{r}_i^t - \mathbf{r}_j^t|).
\]

(12)

**Problem 1.** Make sure that this is the case.

Note that Eq. (11) leads to the following transformation of the velocities:

\[
\mathbf{v}_i^t = \mathbf{v}_i - \mathbf{v}_0.
\]

(13)

The geometrical meaning of the transformation (11) is quite simple: The radius-vector \( \mathbf{r}_i^t \) is nothing but the radius-vector of the \( i \)-th particle with respect to the new origin of coordinates that moves with respect to the old origin with a constant velocity \( \mathbf{v}_0 \). Galilean transformation does not change relative distances and, correspondingly, relative velocities.

The deep physical meaning of the Galilean invariance becomes clear from the observation that the theory (2), (4), (5) is supposed to be applicable to the whole World including all possible measuring devices which one can use to measure time and coordinates. And this leads us to the conclusion that once at the mathematical level it is impossible to tell the difference between the two sets of variables, \( \{\mathbf{r}_i\} \) and \( \{\mathbf{r}_i^t\} \), then the two sets are physically equivalent—we have no physical way to tell one from another! This property of the theory (2), (4), (5) is called *Galilean relativity*.

**Reference Frame**

After discussing Galilean relativity, and in particular realizing that all our measuring devices are supposed to obey the laws (2), we have to address the following question. When introducing Eq. (2) we were taking for granted the existence of the set of radius-vectors \( \{\mathbf{r}_i\} \). Now we know that actually there is no unique choice of the set: given one set, we immediately construct a continuum of equivalent sets by Eq. (11). The choice of coordinates becomes ambiguous, and one might suspect that the theory is logically inconsistent. In fact, it is not, but to have a consistent theory we need to add one more axiom that says: There exists at least one way of ascribing coordinates and time moments, such that the equations of motion take on the form (2).

Any particular choice of coordinates and time moments is called *reference frame*. Any reference frame in which Eq. (2) holds true is called *inertial* reference frame. With the above axiom, the consistent logics of the theory is as follows. The axiom guarantees the existence of at least one inertial reference frame. Galilean relativity then provides a continuum of other inertial reference frames, the choice of a particular one being just a matter of convenience. Finally, and probably most importantly, Eq. (2) tells us what is the practical way of choosing an inertial reference frame. Indeed, from this equation it follows that a particle which does not interact with the rest of the particles moves with a constant velocity in *any* inertial reference frame. Hence, an inertial reference frame can be associated with (formed by) a set of particles which do not interact with each other and the rest of the World, and do not move with respect to each other.

**Conservation Laws**

A conservation law states that this or that quantity does not change in time. Such a conserved quantity is called constant of motion. It turns out that any theory (2), (4), (5) has three generic constants of motion: momentum (sometimes called linear momentum), energy, and angular momentum.
**Momentum.** Summing up all the $N$ equations (2) and taking into account (8) we get

$$\frac{d}{dt} \sum_{i=1}^{N} m_i v_i = \sum_{i,j} F_{ij} = - \sum_{i,j} F_{ij} = 0 . \quad (14)$$

This means that the vector (called the *total momentum*)

$$\mathbf{P} = \sum_{i=1}^{N} m_i \mathbf{v}_i , \quad (15)$$

does not change in time. Apart from the total momentum, it is convenient to define the momentum of particle $i$ by $\mathbf{p}_i = m_i \mathbf{v}_i$; the total momentum of a system is then a sum of the momenta of all its particles.

**Center of Mass.** The conservation of momentum has an important consequence that for a special collective variable, called *center of mass* and defined as

$$\mathbf{R} = \frac{\sum_{i=1}^{N} m_i \mathbf{r}_i}{\sum_{i=1}^{N} m_i} , \quad (16)$$

the equation of motion decouples from the rest of variables, and reads:

$$\ddot{\mathbf{R}} = 0 . \quad (17)$$

That is the center of mass of an isolated system always moves with a constant velocity.

**Problem 2.** Prove Eq. (17). Note that the particular form of the denominator in Eq. (16) is not relevant for the proof, and comment on why, in your opinion, it is important to have this denominator in the definition of the center of mass. *Hint:* Study how the vector $\mathbf{R}$ changes when the positions of all the particles are shifted by one and the same vector $\mathbf{r}_0$, and note the role of denominator in this transformation.

**Angular momentum.** The quantity

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \quad (18)$$

is called *total angular momentum* of the system (with respect to the origin of the coordinates). This quantity is also conserved by any theory satisfying Eqs. (2), (4), (5). The idea of the proof is to form the vector product of $\mathbf{r}_i$ with the $i$-th equation in (2), then to sum over all $i$'s, and finally to employ the chain rule to identify time-derivatives to show that

$$\dot{\mathbf{L}} = 0 . \quad (19)$$

When proving (19), it is crucial to take into account Eq. (6), as well as some general properties of vector products. The details of the proof are as follows.

$$\sum_{i=1}^{N} m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i = \sum_{i,j} \mathbf{r}_i \times \mathbf{F}_{ij} . \quad (20)$$

First, we note that

$$\frac{d}{dt} (\mathbf{r}_i \times \mathbf{p}_i) = m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i + m_i \dot{\mathbf{r}}_i \times \dot{\mathbf{r}}_i = m_i \mathbf{r}_i \times \ddot{\mathbf{r}}_i . \quad (21)$$
Hence, we see that l.h.s. of (20) is $\mathbf{L}$, so that we just need to make sure that

$$
\sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij} = 0 .
$$

(22)

With Eq. (6) we have

$$
\sum_{ij} \mathbf{r}_i \times \mathbf{F}_{ij} = \sum_{ij} f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \mathbf{r}_i \times (\mathbf{r}_i - \mathbf{r}_j) = -\sum_{ij} f_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \mathbf{r}_i \times \mathbf{r}_j .
$$

(23)

Swapping dummy subscripts $i$ and $j$ in the r.h.s. changes the sign of the expression and thus proves (22).

If we shift the origin of the coordinate system,

$$\mathbf{r}_i \rightarrow \mathbf{r}_i + \mathbf{r}_0 ,
$$

(24)

$\mathbf{r}_0$ being the vector of the translation of the origin, then the angular momentum transforms as

$$\mathbf{L} \rightarrow \mathbf{L} + \mathbf{r}_0 \times \mathbf{P} .
$$

(25)

The fact that the new quantity is also a constant of motion is explicitly seen.

**Energy.** The quantity

$$
E = \sum_{i=1}^{N} \frac{m_i \mathbf{v}_i^2}{2} + \sum_{i<j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|)
$$

(26)

is called energy of the system; it is conserved in any theory satisfying Eqs. (2), (4), (5). To get an idea of the consequences of the energy conservation, consider separately the two terms of Eq. (26).

The first one depends only on velocities and is called kinetic energy,

$$
E_{\text{kin}} = \sum_{i=1}^{N} \frac{m_i \mathbf{v}_i^2}{2} .
$$

(27)

The second one depends only on interaction potentials and is called potential energy

$$
E_{\text{pot}} = \sum_{i<j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \equiv (1/2) \sum_{i,j} U_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) .
$$

(28)

Given the fact that kinetic energy is non-negative, the conservation of energy can lead to serious qualitative constraints on the motion of the system. For example, for a system of two particles attracting each other by a potential $U_{12}(r) < 0$, such that $U_{12}(r) \rightarrow 0$ at $r \rightarrow \infty$, the interparticle distance can diverge during the evolution only if $E \geq 0$. Otherwise, there is always an upper bound $r_*$ for the interparticle distance, given by $U_{12}(r_*) = E$.

The idea of the proof of the energy conservation is to multiply the $i$-th equation in (2) by $\dot{\mathbf{r}}_i$, sum all the equations up, and finally—identifying corresponding pieces of the sum with corresponding chain-rule expressions for time derivatives—explicitly show that

$$
\frac{dE}{dt} = 0 .
$$

(29)

The details of the proof are as follows. We have

$$
\sum_{ij} m_i \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i = \sum_{ij} \dot{\mathbf{r}}_i \cdot \mathbf{F}_{ij} .
$$

(30)
By the chain rule, we see that l.h.s of (30) is \( \dot{E}_{\text{kin}} \), and we just need to make sure that r.h.s. of (30) is \( -\dot{E}_{\text{pot}} \). First, we symmetrize r.h.s. of (30),

\[
\sum_{i,j} \dot{r}_i \cdot F_{ij} \equiv \frac{1}{2} \sum_{i,j} \left( \dot{r}_i \cdot F_{ij} + \dot{r}_j \cdot F_{ji} \right). \tag{31}
\]

Then we use (4) and (5),

\[
\sum_{i,j} \dot{r}_i \cdot F_{ij} = -\frac{1}{2} \sum_{i,j} \left( \dot{r}_i \cdot \frac{\partial U_{ij}}{\partial r_i} + \dot{r}_j \cdot \frac{\partial U_{ij}}{\partial r_j} \right) = -\frac{1}{2} \sum_{i,j} \left( \dot{r}_i \cdot \frac{\partial U_{ij}}{\partial r_i} + \dot{r}_j \cdot \frac{\partial U_{ij}}{\partial r_j} \right), \tag{32}
\]

and arrive at the chain-rule expression for the time derivative from the potential energy.

\section*{Symmetries}

By a symmetry of equations of motion one means a property of being invariant with respect to this or that transformation. The above-discussed Galilean invariance is an example of symmetry. There are other important symmetries of the theory (2), (4), (5).

\begin{itemize}

\item **Translational symmetry.** The translational symmetry is the invariance with respect to shifting all the coordinates by a constant vector: \( r_i(t) \rightarrow r_i(t) + r_0 \). 

That is if we take some solution \( \{r_i(t)\} \) of the problem (2), (4), (5) and construct a new set of functions, \( \{r_i'(t)\} \), such that \( r_i'(t) = r_i(t) - r_0 \), then this set of function will also be a solution of the problem (2), (4), (5).

\item **Rotational symmetry.** The idea of rotational symmetry is very close to that of translational symmetry. It is the invariance with respect to rotation of all the coordinates around some axis. We take some solution \( \{r_i(t)\} \) of the problem (2), (4), (5) and construct a new set of functions, \( \{r_i'(t)\} \), by rotating all radius vectors around some fixed axis by one and the same angle. The rotational symmetry means that \( \{r_i'(t)\} \) satisfies the equations (2), (4), (5).

\item **Time-translational symmetry.** By the time-translation symmetry (also known as the homogeneity of time) one means the invariance with respect to the transformation: \( t \rightarrow t + t_0 \).

That is if we take some solution \( \{r_i(t)\} \) of the problem (2), (4), (5) and construct a new set of functions, \( \{r_i'(t)\} \), such that \( r_i'(t) = r_i(t - t_0) \), then this set of function will also be a solution of the problem (2), (4), (5).

\item **Time-reversal symmetry.** Time-reversal symmetry is the invariance with respect to the transformation \( t \rightarrow -t \).

\end{itemize}
That is if \( \{r_i(t)\} \) is a solution of the problem (2), (4), (5), then \( \{r'_i(t)\} \), such that
\[
r'_i(t) = r_i(-t) ,
\]
is also a solution of this problem.

**Problem 3.** Prove the translational, time-translational, and time-reversal symmetries of the problem (2), (4), (5).

**Problem 4.** Prove the rotational symmetry of the problem (2), (4), (5). Note that you will have to deal with the rotation matrix \( R_{\alpha\beta} \) that transforms the components of the vectors (Greek letters stand for the components):
\[
(r'_i)_\alpha = \sum_{\beta=1}^{3} R_{\alpha\beta} (r_i)_\beta .
\]
(39)

A good news it that to prove the rotational invariance, one does not need to use an explicit form of the matrix \( R \). What really matters are the following properties of this matrix
\[
\sum_{\beta=1}^{3} R_{\alpha\beta} R_{\gamma\beta} = \sum_{\beta=1}^{3} R_{\beta\alpha} R_{\beta\gamma} = \delta_{\alpha\gamma} .
\]
(40)

These properties, in particular, imply that the inverse rotational transformation is given by
\[
(r_i)_\alpha = \sum_{\beta=1}^{3} R_{\beta\alpha} (r'_i)_\beta .
\]
(41)