Surface area

Recall that for a surface $z = f(x, y)$ we can calculate its surface area over a region of inputs R as

$$\iint_R \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dA.$$

If we interpret the surface in polar form instead, i.e. $z = f(x(r, \theta), y(r, \theta)) = f(r\cos(\theta), r\sin(\theta))$, then by the chain rule we have that

$$\begin{aligned}
\frac{\partial f}{\partial r} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial f}{\partial x}\left(\cos(\theta)\right) + \frac{\partial f}{\partial y}\left(\sin(\theta)\right) \\
\frac{\partial f}{\partial \theta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x}\left(-r\sin(\theta)\right) + \frac{\partial f}{\partial y}\left(r\cos(\theta)\right)
\end{aligned}$$

Now observe that

$$\begin{aligned}
\left(\frac{\partial f}{\partial r}\right)^2 &= \left(\frac{\partial f}{\partial x}\right)^2 \cos^2(\theta) + 2\cos(\theta)\sin(\theta)\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^2 \sin^2(\theta) \\
\left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)^2 &= \left(\frac{\partial f}{\partial x}\right)^2 \sin^2(\theta) - 2\cos(\theta)\sin(\theta)\frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + \left(\frac{\partial f}{\partial y}\right)^2 \cos^2(\theta).
\end{aligned}$$

So summing them gives

$$\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 \left(\cos^2(\theta) + \sin^2(\theta)\right) + \left(\frac{\partial f}{\partial y}\right)^2 \left(\sin^2(\theta) + \cos^2(\theta)\right) = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

Therefore, we find the *correct* surface area formula using polar derivatives by substituting this in:

$$\iint_R \sqrt{1 + \left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial f}{\partial \theta}\right)^2}\, dA$$

Solution to question from last week's note

Recall that we can rotate a surface $z = f(x, y)$ (about the z-axis) by reinterpreting it in polar form via $z = f(x, y) = f(r\cos(\theta), r\sin(\theta))$ and taking $g(r, \theta) = f(r\cos(\theta+\psi), r\sin(\theta+\psi))$ where ψ is the angle we rotate by. In particular, if $f(x, y) = x^2 - y^2$ is the standard hyperbolic paraboloid, then following this procedure gives $g(r, \theta) = r^2\cos(2\theta + 2\psi)$. We can use Lagrange multipliers to maximize this surface with respect to a generic elliptical constraint given by $h(x, y) = ax^2 + by^2 = c$ for positive constants a, b, c. In fact, let's interpret h in polar form and then run Lagrange using the polar coordinates: $h(r, \theta) = r^2(a\cos^2(\theta) + b\sin^2(\theta)) = c$ and

$$\begin{aligned}\nabla g(r, \theta) &= \lambda \nabla h(r, \theta)\\ \langle 2r\cos(2\theta + 2\psi), -2r^2\sin(2\theta + 2\psi)\rangle &= \lambda\langle 2r(a\cos^2(\theta) + b\sin^2(\theta)), 2r^2(b - a)\cos(\theta)\sin(\theta)\rangle,\end{aligned}$$

which along the constraint gives us three equations in four unknowns, namely r, θ, ψ, λ. However, say that we *know* g is maximized at the point $M = \left(\sqrt{\frac{c}{a+b}}, \sqrt{\frac{c}{a+b}}\right)$. corresponding to $\theta = \pi/4$. Then with this knowledge we actually have a system of three equations in three unknowns r, ψ, λ, let's try to solve for the needed angle of rotation ψ. After setting $\theta = \pi/4$ and simplifying we have equations

$$\begin{aligned}2r\cos(\pi/2 + 2\psi) &= \lambda r(a + b)\\ -2r^2\sin(\pi/2 + 2\psi) &= \lambda r^2(b - a)\\ \frac{r^2}{2}(a + b) &= c\end{aligned}$$

We see that $r = \sqrt{\frac{2c}{a+b}}$, but this is extra–we don't need this information to solve for ψ. We can divide by r and r^2 in the first and second equations respectively, since $r = 0$ corresponds to the origin which isn't a point on the ellipse. Then using a standard trig identity, we see that

$$\begin{aligned}-2\sin(2\psi) &= \lambda(a + b)\\ -2\cos(2\psi) &= \lambda(b - a)\end{aligned}$$

Squaring both equations and summing gives $4 = \lambda^2(2a^2 + 2b^2)$, so that $\lambda = \sqrt{\frac{2}{a^2+b^2}}$. Substituting this back into the first equation, we can ultimately get

$$-2\sin(2\psi) = \frac{(a + b)\sqrt{2}}{\sqrt{a^2 + b^2}} \implies \psi = \frac{1}{2}\arcsin\left(-\frac{a + b}{\sqrt{2(a^2 + b^2)}}\right).$$

Notice that if $a = b$, i.e. the elliptical constraint is actually a circular constraint, then $\psi = -\pi/4$. Of course, there was nothing special about our original choice of maximum at $\theta = \pi/4$, so you may as well say the maximum is at $\theta = \theta_0$ for some constant angle θ_0 and follow the same procedure. You will see that there is an *almost* linear relation between how θ_0 relates to ψ, with a small oscillatory term given by arcsin.