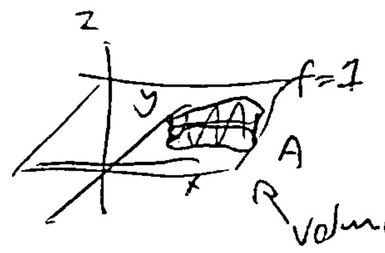
15.1 Double Integrals over Rectangles

1. What is the geometric interpretation of $\iint_D f(x,y)dA$?

Signed volume under/above the graph of $f(x,y)$ and above/below the xy-plane for inputs (x,y) in a region A.

2. In the previous question, if we set $f(x,y) = 1$, what will the double integral represent geometrically?



volume of this region A.

As a number, will equal the area of A.

3. Let R be the rectangle where $0 \le x \le 2$ and $-1 \le y \le 1$. Fill in the blanks, then compute the double integral using the *easier* order of integration.

$$\iint_R y\sin(\pi xy)dA = \int_{\boxed{-1}}^{\boxed{1}} \int_{\boxed{0}}^{\boxed{2}} y\sin(\pi xy)\ dx\ dy = \int_{\boxed{0}}^{\boxed{2}} \int_{\boxed{-1}}^{\boxed{1}} y\sin(\pi xy)\ dy\ dx$$

↓ ~~easier~~ easier

$$\int_{-1}^{1}\int_0^2 y\sin(\pi xy)\,dx\,dy$$

$$\parallel$$

$$\int_{-1}^{1}\left[\frac{-1}{\pi}\cos(\pi xy)\right]_0^2 dy$$

$$= \frac{-1}{\pi}\int_{-1}^{1} \cos(2\pi y) - 1\ dy = -\frac{1}{\pi}\left[\frac{1}{2\pi}\sin(2\pi y) - y\right]_{-1}^{1}$$

$$= -\frac{1}{\pi}\left(\left(\frac{1}{2\pi}\cancel{\sin(2\pi)}_{0} - 1\right) - \left(\frac{1}{2\pi}\cancel{\sin(-2\pi)}_{0} - (-1)\right)\right)$$

$$= \frac{2}{\pi}$$

[Ans $\frac{2}{\pi}$]

15.2 Double Integrals over General Regions

4. To compute a double integral over a more general region, we should consider the *type* of region over which we are integrating.

   - In a type I region, we have a $y=f(x)$ curve and a $y=g(x)$ curve.
     $dy\ dx$

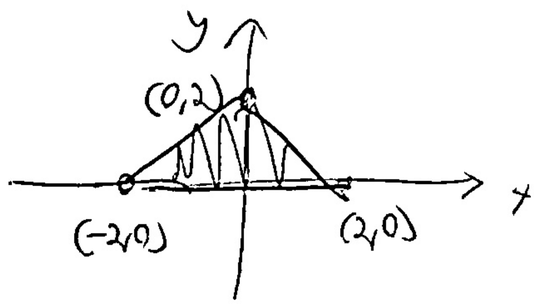
     In this case, The variable x always has constants as bounds.

   - In a type II region, we have an $x=f(y)$ curve and a $x=g(y)$ curve.
     $dx\ dy$

     In this case, the variable y always has constants as bounds.

5. Let D be the triangular region with vertices $(-2,0)$, $(0,2)$, and $(2,0)$.

   (a) Sketch the region D.



   (b) Express the region D as a type I and a type II region.

Type I: $D=\{(x,y)\in\mathbb{R}^2 \mid -2\le x\le 0,\ 0\le y\le x+2\}$

union $\to$ $\cup\ \{(x,y)\in\mathbb{R}^2 \mid 0\le x\le 2,\ 0\le y\le -x+2\}$

Type II: $D=\{(x,y)\in\mathbb{R}^2 \mid 0\le y\le 2,\ y-2\le x\le 2-y\}$

   (c) Compute $\iint_D (x+y)\,dA$, using the easiest order of integration.

type II

$$\int_0^2\int_{y-2}^{2-y}(x+y)\,dx\,dy$$

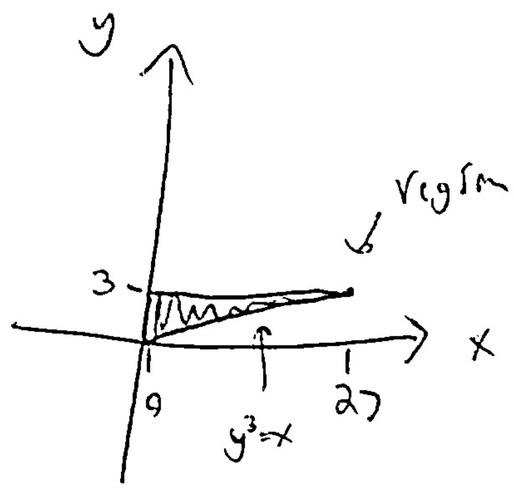
$$=\int_0^2\left[\tfrac{1}{2}x^2+yx\right]_{y-2}^{2-y}dy=2\int_0^2 2y-y^2\,dy$$

$$=2\left[y^2-\tfrac{1}{3}y^3\right]_0^2=2\left(4-\tfrac{8}{3}\right)=\boxed{\tfrac{8}{3}}$$

[Ans. $\frac{8}{3}$]

6. **(Extra question)** Evaluate $\int_0^{27}\int_{\sqrt[3]{x}}^{3} 6e^{y^4}\,dy\,dx$.

$y \Rightarrow y^3 = x$



sup order:

$$\int_0^3\int_0^{y^3} 6e^{y^4}\,dx\,dy$$

$$= 6\int_0^3 \left[xe^{y^4}\right]_0^{y^3} dy$$

$$= 6\int_0^3 y^3 e^{y^4}\,dy$$

$$= 6\left[\frac{1}{4}e^{y^4}\right]_0^3$$

[Ans. $\frac{3}{2}(e^{81}-1)$]

$$\frac{3}{2}\left(e^{3^4} - e^{0^4}\right) = \boxed{\frac{3}{2}\left(e^{81}-1\right)}$$