14.4

1. Consider the surface $z = \overset{f(x,y)}{x^3 + \ln(y)}$ $\quad \nabla f(x,y) = \langle 3x^2, \frac{1}{y} \rangle$

   (a) Find the equation of the plane tangent to this surface at the point $P(1,1,1)$.

$$z - 1 = 3(1)^2(x-1) + \frac{1}{(1)}(y-1)$$

$$\Downarrow$$

$$0 = 3x + y - z - 3$$

   (b) Use your answer from part (a) to find the linearization for the function $f(x,y) = x^3 + \ln(y)$ at the point P.

$$L_P(x,y) = z = 3x + y - 3$$

   (c) Use your answer from part (b) to approximate $f(1.1, 0.9)$. Use a calculator to compare it with the exact value of $f(1.1, 0.9)$. Is it a good approximation?

$$L_P(1.1, 0.9) = 3(1.1) + (0.9) - 3 = 3.3 + 0.9 - 3$$
$$= 1.2$$

$$f(1.1, 0.9) = (1.1)^3 + \ln(0.9) \approx 1.22564\ldots$$

   (d) Compare the values of Δz and dz when (x,y) changes from $(1,1)$ to $(1.1, 0.9)$. How is this comparison related to whether the approximation is good or not?

$$\Delta z = f(1.1, 0.9) - f(1,1) \approx 0.22564$$
$$dz = L_P(1.1, 0.9) - L_P(1,1) = 1.2 - 1 = 0.2$$

we can see how close the linearization is to Δz in relative terms.

[Ans:
(a) $z - 1 = 3(x-1) + 1(y-1)$, (b) $L(x,y) = 3(x-1) + (y-1) + 1$, (c) 1.2, (d) 0.2]

14.5 The Chain Rule

$\frac{\partial x}{\partial t} = \frac{1}{2\sqrt{t}}$ $\frac{\partial y}{\partial t} = -\sin(t)$

2. Given $z = 2^{x^2+y}$ and $x = \sqrt{t}$, $y = \cos(t)$, find $\frac{\partial z}{\partial t}$ for $t = 1$. $\rightarrow$ $x(1) = \sqrt{1} = 1$, $y(1) = \cos(1)$

$$\frac{\partial z}{\partial t} = \left(\ln(2)\, 2^{x^2+y}\right) \cdot \left[2x \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\right] = \ln(2)\, 2^{x^2+y}\left[\frac{\cancel{2}x}{\cancel{2}\sqrt{t}} - \sin(t)\right]$$

at $t = 1$: $\ln(2) \cdot 2^{1+\cos(1)}\left(\frac{1}{\sqrt{1}} - \sin(1)\right)$

[Ans. $(2^{1+\cos(1)})(\ln(2))(1 - \sin(1))$]

$f(x,y,z)$ at level surface

3. An ellipsoid is given by the equation $\frac{x^2}{2} + y^2 + \frac{z^2}{4} = 1$. Find the equation of the tangent plane to this surface at $P(1, 0.5, -1)$. Try plotting the ellipsoid with the tangent plane on a 3D graphing calculator (like GeoGebra) when you are done!

$$\nabla f(x,y,z) = \left\langle x, 2y, \frac{z}{2} \right\rangle$$

so $\vec{n}$ at P is $\left\langle 1, 1, -\frac{1}{2} \right\rangle$ and point is $(1, 0.5, -1)$

then tangent plane is

$$\ell = 1(x-1) + 1(y-0.5) - \frac{1}{2}(z+1)$$

$$\Downarrow$$

$$z + 1 = 2(x-1) + 2(y-0.5)$$

[Ans. $z + 1 = 2(x - 1) + 2(y - 0.5)$]

14.6 Directional Derivatives and Gradient Vector

4. Consider the function $f(x, y, z) = e^{x^2+y^2-2z^2}$.

   (a) Find the rate of change of the function f at the point $P(1, 2, 3)$ in the direction *toward* the point $(2, 3, 4)$.

   vector is $\vec{v} = \langle 1,1,1 \rangle \Rightarrow |\vec{v}| = \sqrt{3}$

   $$\nabla f(x,y,z) = \langle 2x e^{x^2+y^2-2z^2}, 2y e^{x^2+y^2-2z^2}, -4z e^{x^2+y^2-2z^2} \rangle$$

   directional derivative: $\nabla f(1,2,3) \cdot \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{3}}\left(2e^{1+4-18} + 4e^{1+4-18} - 12e^{1+4-18}\right) = \frac{-6}{e^{13}\sqrt{3}}$

   (b) In which (unit) direction does the function f *decrease* most rapidly at the point P?

   in the direction $\frac{\nabla f(1,2,3)}{|\nabla f(1,2,3)|}$ = i.e. $\frac{\langle -2e^{-13}, -4e^{-13}, 12e^{-13} \rangle}{\sqrt{(2e^{-13})^2 + (4e^{-13})^2 + (-12e^{-13})^2}}$

   decrease

   $\uparrow$ max value

   $\|$

   $\frac{1}{\sqrt{164}}\langle -2, -4, 12 \rangle$

   (c) Find the maximum rate of change of f at P.

   $$|\nabla f(1,2,3)| = e^{-13}\sqrt{164}$$

[Ans: (a) $\frac{-6}{e^{13}\sqrt{3}}$, (b) $\frac{1}{\sqrt{164}}\langle -2, -4, 12 \rangle$, (c) $e^{-13}\sqrt{164}$]