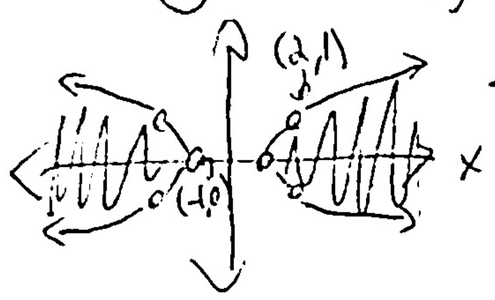
14.1 Functions of several variables

1. Find and sketch the domains of the following functions.

(a) $f(x,y) = \ln(\underbrace{x^2 - 3y^2 - 1}_{\text{argument must be } > 0})$

so $D = \{(x,y) \mid x^2 > 3y^2 + 1\}$

$x^2 - 3y^2 > 1$



not "between" hyperbola $x^2 - 3y^2 = 1$

[Ans: $D = \{(x,y) | x^2 > 3y^2 + 1\}$]

(b) $f(x,y) = \dfrac{\sqrt{y - x^2}}{\underbrace{x^2 - y^2}_{\text{can't equal } 0}}$ can be ≥ 0 $\Rightarrow$ $y \geq x^2$

$\Downarrow$

$y \neq \pm x$

$D = \{(x,y) \mid y \geq x^2, x \neq \pm y\}$

not be on or above parabola $y = x^2$ and not on lines $y = x$ or $y = -x$

[Ans: $D = \{(x,y) | y \geq x^2, x \neq \pm y\}$]

14.2 Limits and continuity

2. Find the limit if it exists or show that the limit does not exist.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^4 + y^6}$ get two paths that don't give same limit

path 1 $\rightarrow$ $\boxed{1}$ $\displaystyle\lim_{\substack{(x=y^3/2)\\ y\to 0}} \frac{x^2y^3}{x^4+y^6} = \lim_{y\to 0} \frac{(y^3)y^3}{(y^6)+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \boxed{\frac{1}{2}}$

path 2 $\rightarrow$ $\boxed{2}$ $\displaystyle\lim_{\substack{(x=0)\\ y\to 0}} \frac{x^2y^3}{x^4+y^6} = \lim_{y\to 0} \frac{(0)y^3}{(0)+y^6} = \lim_{y\to 0} 0 = \boxed{0}$

so $\displaystyle\lim_{(x,y)\to(0,0)}$ DNE.

(b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4 + y^4}$ (Hint: Since the function is bounded, use Squeeze Theorem).

note that $-|x| \leq x \leq |x|$ $\cdot \left(\dfrac{y^4}{x^4+y^4}\right)$

(since $0 \leq \dfrac{y^4}{x^4+y^4}$)

$\Downarrow$

$-|x| \leq -\dfrac{|x|y^4}{x^4+y^4} \leq \dfrac{xy^4}{x^4+y^4} \leq \dfrac{|x|y^4}{x^4+y^4} \leq |x|$

since $\dfrac{y^4}{x^4+y^4} \leq 1$

by squeeze theorem

some path

$\displaystyle\lim_{\substack{y=f(x)\\ x\to 0}} -|x| \leq \lim_{\substack{y=f(x)\\ x\to 0}} \frac{xy^4}{x^4+y^4} \leq \lim_{\substack{y=f(x)\\ x\to 0}} |x|$

[Ans: (a) DNE, (b) 0]

$\Downarrow$

$0 \leq \displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^4+y^4} \leq 0$

14.3 Partial derivatives

3. Find the first partial derivatives of the following functions.

(a) $f(x, y) = x^3 + 2x^2y^3 + xy^4$

$f_x = 3x^2 + 4xy^3 + y^4$

$f_y = 6x^2y^2 + 4xy^3$

(b) $g(\rho, \theta) = \sin(\rho \cos\theta)$ using chain rule

$g_\rho = \cos(\rho\cos\theta)\cdot\cos\theta$

$g_\theta = \cos(\rho\cos\theta)\cdot\rho\cdot -\sin\theta$

(c) $f(x, y) = x^y$

$f_x = yx^{y-1}$

$f_y = \ln(x)x^y$

4. Find all second partial derivatives of the function $f(x, y) = \cos(x^2y)$ and confirm Clairaut's Theorem.

i.e. $f_{xy} = f_{yx}$

$f_x = -\sin(x^2y)\cdot 2xy \xrightarrow{\text{product rule}} f_{xy} = -\sin(x^2y)[2x] - 2xy\cos(x^2y)[x^2]$

$f_{xx} \overset{\text{product rule}}{=} -\sin(x^2y)[2y] - 2xy\cos(x^2y)[2xy]$

equal ✓ Clairaut's Theorem

$f_y = -\sin(x^2y)\cdot x^2 \xrightarrow{\text{product rule}} f_{yx} = -x^2\cos(x^2y)\cdot\underset{2xy}{[y]} - [2x]\sin(x^2y)$

$f_{yy} = -x^2\cos(x^2y)\cdot x^2$

5. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $e^z = xyz$. implicit z and xy.

$$\frac{\partial}{\partial x}\left[e^z = xyz\right] \Rightarrow \frac{\partial z}{\partial x}e^z = yz + xy\frac{\partial z}{\partial x}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}}$$

$$\frac{\partial}{\partial y}\left[e^z = xyz\right] \Rightarrow \frac{\partial z}{\partial y}e^z = xz + xy\frac{\partial z}{\partial y}$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}}$$

[Ans. $\frac{\partial z}{\partial x} = \frac{yz}{e^z - xy}, \frac{\partial z}{\partial y} = \frac{xz}{e^z - xy}$]

[Question 3. Ans:
(a) $f_x = 3x^2 + 4xy^3 + y^4$, $f_y = 6x^2y^2 + 4xy^3$
(b) $g_\rho = \cos(\theta)\cos(\rho\cos\theta)$, $g_\theta = -\rho\sin(\theta)\cos(\rho\cos\theta)$
(c) $f_x = yx^{y-1}$, $f_y = x^y\ln(x)$]

[Question 4. Ans:
$f_{xx} = -4x^2y^2\cos(x^2y) - 2y\sin(x^2y)$,
$f_{xy} = f_{yx} = -2x^3y\cos(x^2y) - 2x\sin(x^2y)$,
$f_{yy} = -x^4\cos(x^2y)$]