16.7 Surface Integrals

1. Find the flux of the vector field $\vec{F}(x,y,z) = x\vec{i} + y\vec{j} + z^2\vec{k}$, across the sphere with radius 1 and center at the origin, oriented outwards.

   [The parametrization of the sphere (in spherical coordinates) is $\vec{r}(\theta,\phi) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$ with domain $D = \{(\theta,\phi) | 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$.]

S

$x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$, $z = \cos\phi$

$\rightarrow$ so $\vec{F}(\vec{r}(\theta,\phi)) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos^2\phi\rangle$

$\vec{r}_\theta = \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle$

$\vec{r}_\phi = \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle$

$\vec{r}_\theta \times \vec{r}_\phi = \langle -\sin^2\phi\cos\theta, -\sin^2\phi\sin\theta, -\cos\phi\sin\phi\rangle$

(careful, this is actually the negative orientation, note the minus sign and the fact S is closed) [so we'll use $\vec{r}_\phi \times \vec{r}_\theta$ instead]

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}(\vec{r})\cdot(\vec{r}_\phi \times \vec{r}_\theta)\, dA$$

$$= \int_0^{2\pi}\int_0^{\pi} \langle \sin\phi\cos\phi, \sin\phi\sin\theta, \cos^2\phi\rangle \cdot \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \cos\phi\sin\phi\rangle \, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin^3\phi\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1} + \cos^3\phi\sin\phi \, d\phi\, d\theta = 2\pi\int_0^{\pi} \sin\phi(\underbrace{\sin^2\phi}_{1-\cos^2\phi} + \cos^3\phi)\, d\phi$$

let $u = \cos\phi \rightarrow du = -\sin\phi\, d\phi \Rightarrow$

$$= -2\pi\int_1^{-1} 1 - u^2 + u^3\, du$$

$$= 2\pi\int_{-1}^{1} 1 - u^2 + u^3\, du$$

$$= 2\pi\left[u - \frac{1}{3}u^3 + \frac{1}{4}u^4\right]_{-1}^{1}$$

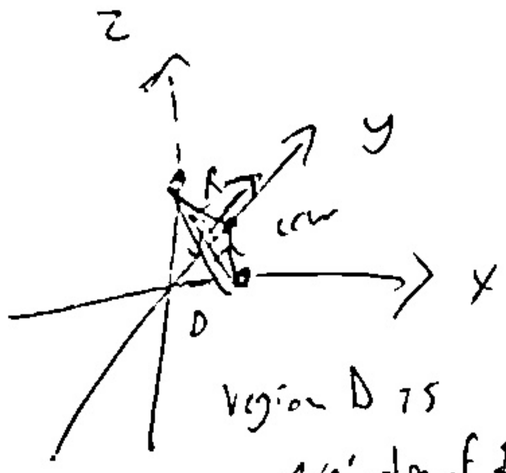
$$= 2\pi\left(2 - \frac{2}{3}\right) = \frac{8\pi}{3}$$

[Ans. $\frac{8\pi}{3}$]

[Can also use Divergence Theorem see the last page!]

16.8 Stokes' Theorem

2. Use Stokes' Theorem to evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = (x+y^2)\vec{i}+(y+z^2)\vec{j}+(z+x^2)\vec{k}$ and C is a triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ oriented counterclockwise as viewed from from above (ie. the region enclosed by C has normal vector pointing away from the xy-plane).



Region D is projection of S onto xy-plane

triangle C bounds S on piece of plane $x+y+z=1$

So normal vector w/ $z=f(x,y)=1-x-y$ should be $\langle 1,1,1\rangle$

(use parametrization $\vec{r}(x,y) = \langle x, y, 1-x-y\rangle$ for example)

$\vec{F} = \langle x+y^2, y+z^2, z+x^2\rangle$

Then by Stokes' Theorem:

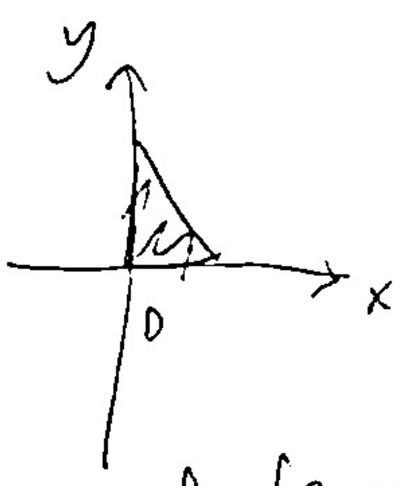
$$\text{curl}(\vec{F}) = \left\langle \frac{\partial}{\partial y}(z+x^2) - \frac{\partial}{\partial z}(y+z^2),\ \frac{\partial}{\partial z}(x+y^2) - \frac{\partial}{\partial x}(z+x^2),\ \frac{\partial}{\partial x}(y+z^2) - \frac{\partial}{\partial y}(x+y^2)\right\rangle$$

$$= \langle -2(z), -2x, -2y\rangle$$

on S $= (1-x-y)$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}(\vec{F})\cdot dS$$

$$=$$



$$\iint_D \langle -2(1-x-y), -2x, -2y\rangle \cdot \langle 1,1,1\rangle\, dA$$

$$\|$$

$D = \begin{Bmatrix} 0 \le x \le 1 \\ 0 \le y \le 1-x \end{Bmatrix}$

$$\int_0^1\int_0^{1-x} -2 + 2x + 2y - 2x - 2y \; dy\, dx$$

$$= -2\underbrace{\int_0^1\int_0^{1-x} dy\, dx}_{\text{area of D is } \frac{1}{2}} \ldots$$

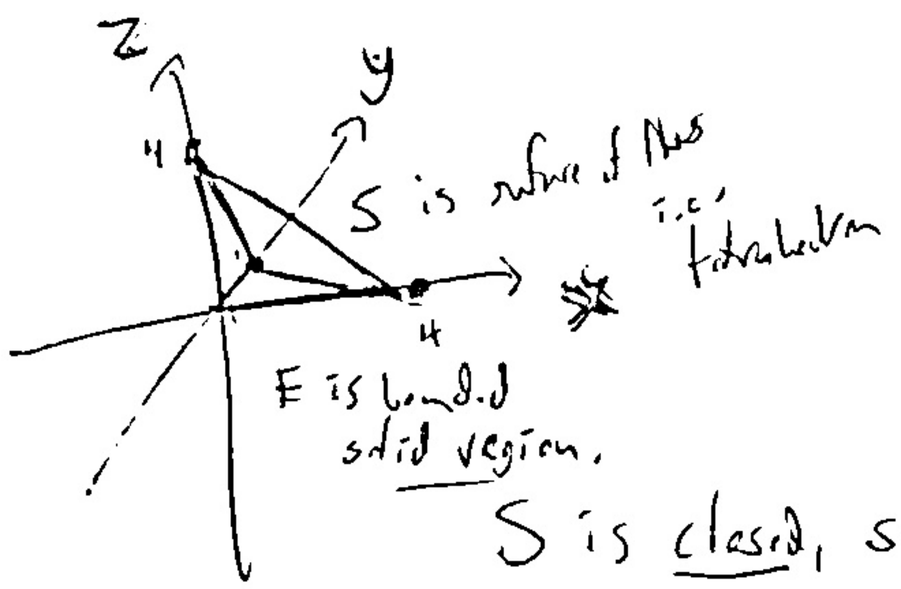
$$-2\cdot\frac{1}{2} = \boxed{-1}$$

[Ans. -1]

16.9 Divergence Theorem

3. Calculate the flux of $\vec{F}$ across the surface S, where $\vec{F} = x^2y\vec{i} + xy^2\vec{j} + 4xyz\vec{k}$ and S is the surface of the tetrahedron bounded by the coordinate planes and the plane $x + 4y + z = 4$.

has points
- $(4,0,0)$
- $(0,1,0)$
- $(0,0,4)$



$\vec{F} = \langle x^2y, xy^2, 4xyz \rangle$

$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(4xyz) = 2xy + 2xy + 4xy$
$= 8xy$

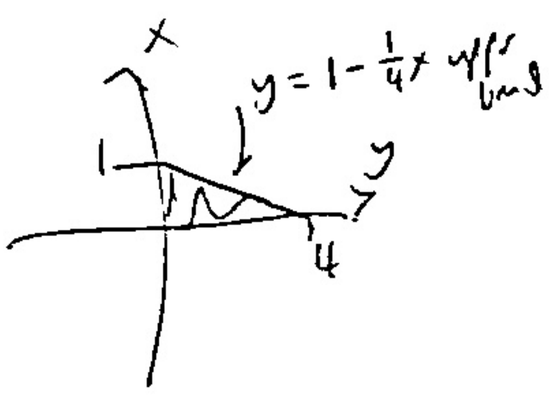
S is closed, so can use divergence theorem:

$$\iint_S \vec{F} \cdot dS = \iiint_E \operatorname{div}(\vec{F})\, dV$$

$$= \int_0^4 \int_0^{1-\frac{1}{4}x} \int_0^{4-4y-x} 8xy \, dz\, dy\, dx$$

$4-4y-x = z$ plane upper bnd

projection onto xy-plane



$$= 8\int_0^4 x\left(\int_0^{1-\frac{1}{4}x} y(4-4y-x)\, dy\right) dx$$

$$= 8\int_0^4 x\left[2y^2 - \frac{4}{3}y^3 - \frac{xy^2}{2}\right]_0^{1-\frac{1}{4}x} dx$$

$$= \frac{8}{16}\int_0^4 x\left(\left(1-\frac{1}{4}x\right)^2 - \frac{2}{3}\left(1-\frac{1}{4}x\right)^3\right) dx$$

[Ans. $\frac{64}{15}$]

$$= \frac{16}{3}\int_0^4 x - \frac{3}{4}x^2 + \frac{3}{16}x^3 - \frac{1}{64}x^4 \, dx$$

$$= \frac{16}{3}\left[\frac{x^2}{2} - \frac{1}{4}x^3 + \frac{3}{64}x^4 - \frac{1}{5 \cdot 64}x^5\right]_0^4 = \frac{16}{3}\left(8 - 16 + 12 - \frac{16}{5}\right)$$

$$= \frac{16}{3}\left(\frac{4}{5}\right) = \frac{64}{15}$$

#1 w/ <u>divergence theorem</u>. (since S is <u>closed</u>)

sphere radius 2

$F = \langle x, y, z^2 \rangle$

$$\iint_S \vec{F} \cdot dS = \iiint_E \operatorname{div}(\vec{F})\, dV$$

solid ball radius 1

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)$$

$= 1 + 1 + 2z$ then switch to spherical

$= 2 + 2\rho\cos\phi$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (2 + 2\rho\cos\phi)\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= 4\pi \int_0^{\pi}\int_0^1 \rho^2 \sin\phi + \rho^3 \sin\phi\cos\phi \, d\rho\, d\phi$$

let $u = \cos\phi \rightarrow du = -\sin\phi\, d\phi$

$$= -4\pi \int_1^{-1}\int_0^1 \rho^2 + \rho^3 u \, d\rho\, du$$

$$= 4\pi \int_{-1}^{1} \left[\frac{1}{3}\rho^3 + \frac{1}{4}\rho^4 u\right]_0^1 du$$

$$= 4\pi \int_{-1}^{1} \frac{1}{3} + \frac{1}{4}u \, du$$

$$= 4\pi \left[\frac{1}{3}u + \frac{1}{8}u^2\right]_{-1}^{1}$$

$$= 4\pi\left(\frac{2}{3}\right) = \frac{8\pi}{3}$$