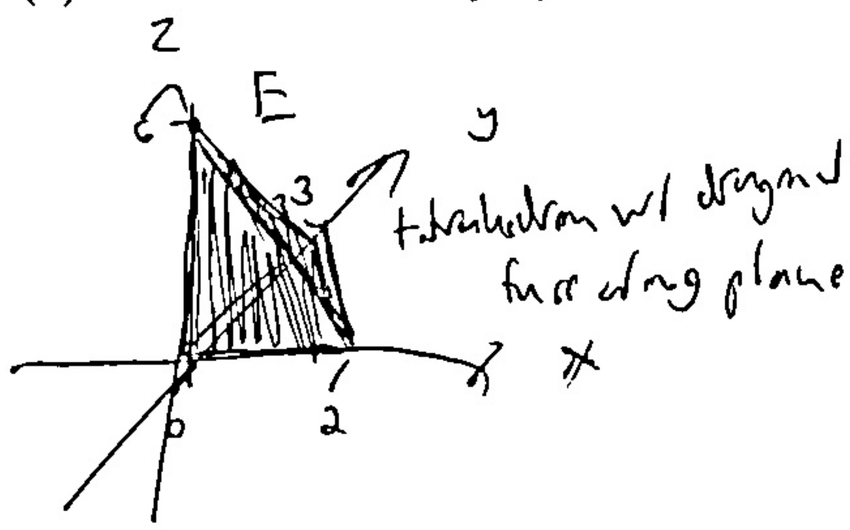
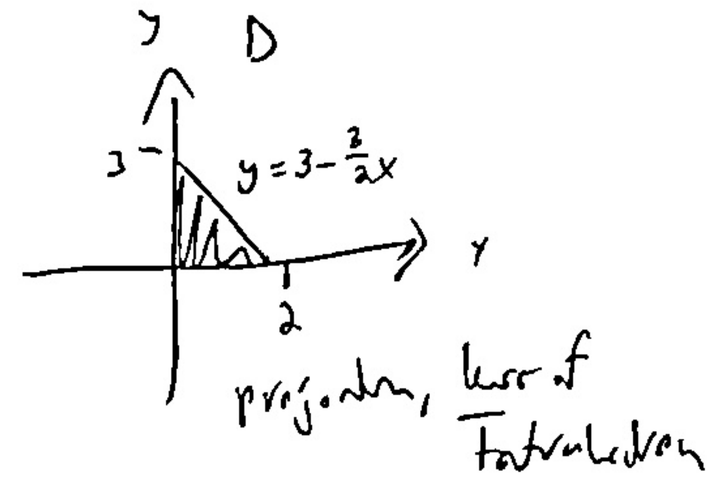
15.6 Triple Integrals

1. Given the solid region $E = \{(x, y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x, 0 \le z \le \underbrace{6 - 3x - 2y}_{z=\text{ plane}}\}$

   (a) Sketch E and its projection D.





   (b) The solid region E is represented as a type I solid. Express the solid region as a type II solid and as a type III solid. $dz\,dy\,dx$ e.g. bounds depends on xy

$u_1(y,z) \le x \le u_2(y,z)$ $\qquad$ $u_1(x,z) \le y \le u_2(x,z)$

$\Downarrow$ solve plane eq for x $\qquad$ $\Downarrow$ solve plane eq for y

e.g.
$$\begin{cases} 0 \le x \le 2 - \frac{2}{3}y - \frac{1}{3}z \\ 0 \le z \le 6 - 2y \\ 0 \le y \le 3 \end{cases}$$

$$\begin{aligned} 0 &\le y \le 3 - \tfrac{3}{2}x - \tfrac{1}{2}z \\ 0 &\le z \le 6 - 3x \\ 0 &\le x \le 2 \end{aligned}$$

   (c) Find the volume of E using a triple integral.

$$\int_0^2 \int_0^{3-\frac{3}{2}x} \int_0^{6-3x-2y} 1 \; dz\, dy\, dx$$

$$= \int_0^2 \int_0^{3-\frac{3}{2}x} (6 - 3x - 2y)\, dy\, dx$$

$$= \int_0^2 \left[(6-3x)y - y^2\right]_0^{3-\frac{3}{2}x} dx$$

$$= \int_0^2 2\left(3 - \tfrac{3}{2}x\right)^2 - \left(3 - \tfrac{3}{2}x\right)^2 dx$$

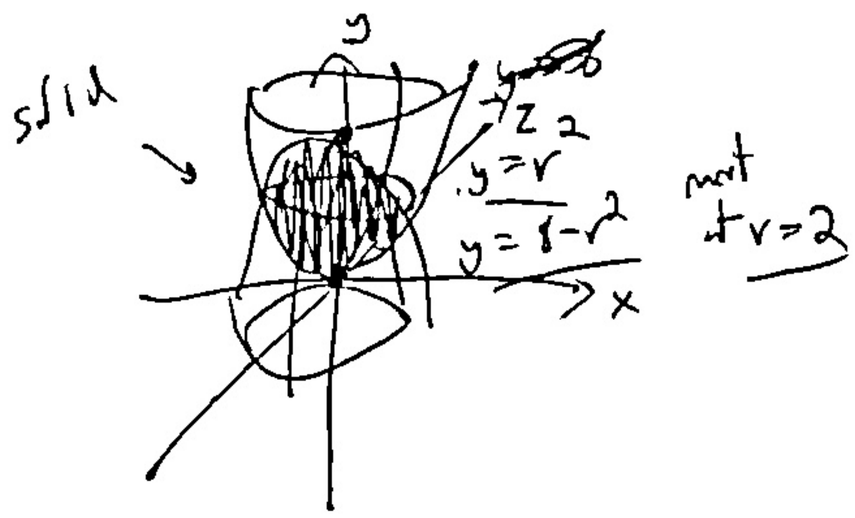
[Ans. 6 units^3]

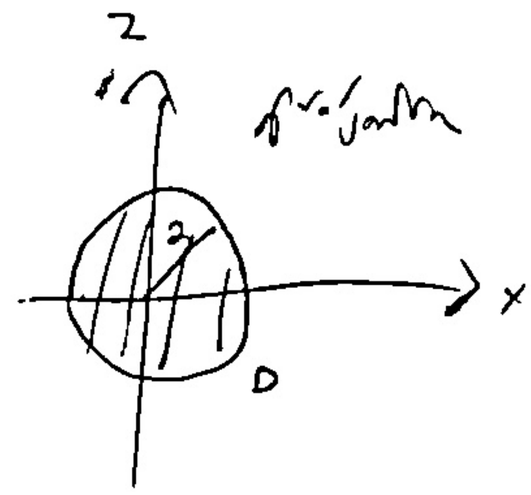
$$= \int_0^2 \left(3 - \tfrac{3}{2}x\right)^2 dx$$

$$= \left[\tfrac{1}{3} \cdot \tfrac{2}{3}\left(3 - \tfrac{3}{2}x\right)^3\right]_0^2 = -\left(-\frac{2 \cdot 3^3}{3^2}\right) = \underline{6\ u^3}$$

2. Use a triple integral to find the volume of the solid enclosed by the paraboloids $y = 8 - x^2 - z^2$ and $y = x^2 + z^2$. (Hint: Observe that in this case the projection is not on the xy-plane).

   (a) Sketch the two paraboloids, the solid and its projection on the appropriate coordinate plane.





   (b) Set up a triple integral that evaluates the volume of the solid and express its region in rectangular coordinates.

$$\iint_D \left(\int_{x^2+z^2}^{8-x^2-z^2} 1 \, dy \right) dA$$

using polar change

$x = r\cos\theta$

$z = r\sin\theta$

$$2\int_0^{2\pi}\int_0^2 (4 - r^2)\, r \, dr \, d\theta$$

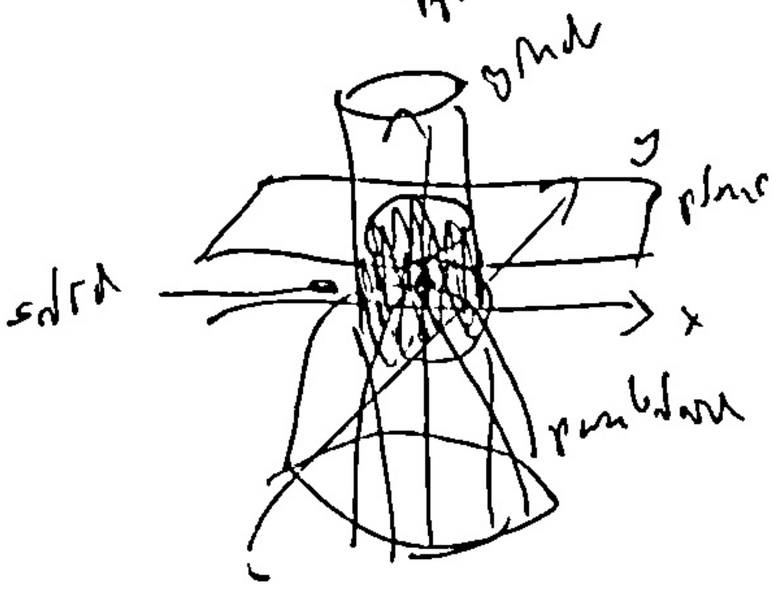
$$= 4\pi\left[2r^2 - \frac{1}{4}r^4\right]_0^2 = 4\pi(8-4) = 16\pi \text{ units}^3$$

[Ans. 16π units^3]

15.7 Triple Integrals in Cylindrical Coordinates

3. Evaluate $\iiint_E z\, dV$, where E is the solid inside the cylinder $x^2 + y^2 = 1$, under the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$.

upper bound $\quad$ lower bound $= 1 - r^2$ $\quad$ $r = 1$ $\quad$ θ anything



using cylindrical coordinates

$0 \le \theta \le 2\pi$

$0 \le r \le 1$

$1 - r^2 \le z \le 4$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\int_0^{2\pi}\int_0^1\int_{1-r^2}^{4} z \; r\,dz\,dr\,d\theta$$

$$= 2\pi\int_0^1 r\left[\tfrac{1}{2}z^2\right]_{1-r^2}^{4} dr = \pi\int_0^1 16r - r^5 + 2r^3 - r\, dr$$

$$= \pi\left[\tfrac{15}{2}r^2 - \tfrac{1}{6}r^6 + \tfrac{1}{2}r^4\right]_0^1$$

$$= \pi\left(\frac{45 - 1 + 3}{6}\right)$$

$$= \frac{47\pi}{6}$$

[Ans. $\frac{16\pi}{3}$ ~~$-\frac{\sqrt{2}}{2}$~~]

4. Evaluate $\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^{\sqrt{x^2+y^2}} x\, dz\, dy\, dx$ using cylindrical coordinates.

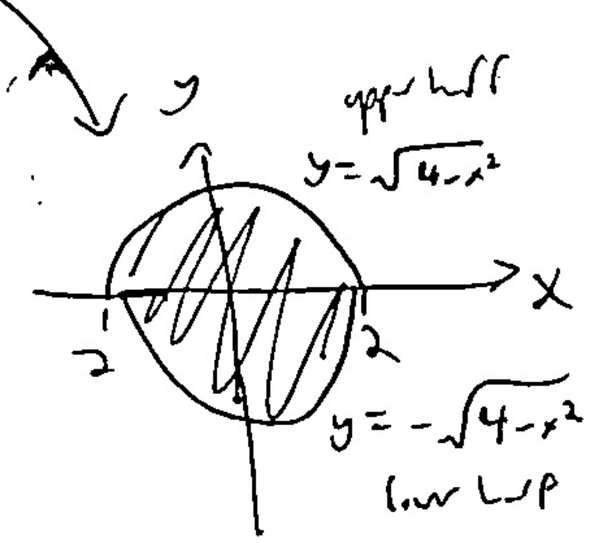
letting $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

new bounds are

$0 \le z \le r$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

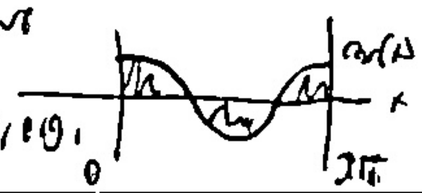


since other integrals don't depend on θ

$$\int_0^{2\pi}\int_0^2\int_0^r (r\cos\theta)\, r\,dz\,dr\,d\theta = \underbrace{\left(\int_0^{2\pi}\cos\theta\, d\theta\right)}_{=0}\underbrace{\left(\int_0^2\int_0^r r^2\,dz\,dr\,d\theta\right)}_{=\text{something}}$$

so whole thing is 0.

even also concave over below, θ.



[Ans. 0 ☺]