Consider $f(x,y) = \frac{xy}{x^2+y^2}$, which is defined everywhere except the origin. So then we could ask if

$$\lim_{(x,y)\to(0,0)} f(x,y)$$

exists. Let's make a change of coordinates to polar form via $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Doing so gives

$$f(r,\theta) = \frac{(r\cos(\theta))(r\sin(\theta))}{(r\cos(\theta))^2 + (r\sin(\theta))^2} = \frac{r^2\cos(\theta)\sin(\theta)}{r^2(\cos^2(\theta)+\sin^2(\theta))} = \cos(\theta)\sin(\theta) \qquad (\text{for } r \neq 0)$$

So in actuality the function only depends on θ, i.e. its level curves are rays from the origin given by fixing θ. These rays all meet at the origin, so picking any two rays $\theta = \theta_1$ and $\theta = \theta_2$ such that $f(r,\theta_1) \neq f(r,\theta_2)$ proves that the limit at the origin does not exist. For a concrete example, let $\theta_1 = 0$ and $\theta_2 = \pi/4$. Then

$$\lim_{r\to 0} f(r,0) = \lim_{r\to 0}\cos(0)\sin(0) = \lim_{r\to 0} 0 = 0$$
$$\lim_{r\to 0} f(r,\pi/4) = \lim_{r\to 0}\cos(\pi/4)\sin(\pi/4) = \lim_{r\to 0}\frac{1}{2} = \frac{1}{2}$$

so $0 \neq \frac{1}{2}$ means the limit of f at the origin does not exist. These two angles correspond to the substitutions $y = 0$ and $y = x$. In general, a ray from the origin given by angle θ from the positive x-axis is described by (half of) the line $y = \tan(\theta)x$. (Can you see why?) We can make the relevant substitution to get

$$f(x,\tan(\theta)x) = \frac{x(\tan(\theta)x)}{x^2 + (\tan(\theta)x)^2} = \frac{x^2\tan(\theta)}{x^2(1+\tan^2(\theta))} = \frac{\tan(\theta)}{1+\tan^2(\theta)} \qquad (\text{for } x \neq 0)$$

Since $f(x,\tan(\theta)x) = f(r,\theta)$ due to how everything was set up, we've recovered a trigonometric identity!

$$\cos(\theta)\sin(\theta) = \frac{\tan(\theta)}{1+\tan^2(\theta)}$$

On the other hand, consider $g(x,y) = \frac{xy^4}{x^4+y^4}$. We could use the squeeze theorem to show that the limit of g at the origin is 0, or we could try a substitution into polar form and see what happens.

$$g(r,\theta) = \frac{(r\cos(\theta))(r\sin(\theta))^4}{(r\cos(\theta))^4 + (r\sin(\theta))^4} = \frac{r^5\cos(\theta)\sin^4(\theta)}{r^4(\cos^4(\theta)+\sin^4(\theta))} = \frac{r\cos(\theta)\sin^4(\theta)}{\cos^4(\theta)+\sin^4(\theta)} \qquad (\text{for } r \neq 0)$$

Any path toward the origin must have $r \to 0$, and regardless of how θ changes, we see that $\lim_{r\to 0} g(r,\theta) = 0$. Therefore $\lim_{(x,y)\to(0,0)} g(x,y) = 0$ is shown. Let's look at one last example where things are messier.

Consider $h(x,y) = \frac{x^2y^3}{x^4+y^6}$. We can evaluate $\lim_{(x,y)\to(0,0)} h(x,0) = 0$ and $\lim_{(x,y)\to(0,0)} h(x,x^{2/3}) = 1/2$, so then $\lim_{(x,y)\to(0,0)} h(x,y)$ does not exist in general. Let's substitute to polar form:

$$h(r,\theta) = \frac{(r\cos(\theta))^2(r\sin(\theta))^3}{(r\cos(\theta))^4 + (r\sin(\theta))^6} = \frac{r^5\cos^2(\theta)\sin^3(\theta)}{r^4(\cos^4(\theta)+r^2\sin^6(\theta))} = \frac{r\cos^2(\theta)\sin^3(\theta)}{\cos^4(\theta)+r^2\sin^6(\theta)} \qquad (\text{for } r \neq 0)$$

It's tempting to make the same claim that letting $r \to 0$ shows $\lim_{(x,y)\to(0,0)} h(x,y) = 0$, but we know this is false since we know a path on which the limit is $1/2$. The difference here is that if our limit path has $\theta \to \pi/2$ (or $\theta \to 3\pi/2$) as it goes to the origin, then the $\cos^4(\theta)$ in the denominator goes to 0 along with $r \to 0$, and we actually have an indeterminate form $0/0$. But $\cos^4(\theta) \neq 0$ for any other θ, so the limit paths which give a limit value other than 0 have to be pretty specific. Contrast this with the situtaion for the first function $f(x,y)$ that we considered. More can be said, but it's beyond our scope. Take a peek if you're curious: `https://arxiv.org/pdf/1209.0363.pdf`