# Rotating a hyperbolic paraboloid

Consider the function $f(x, y) = 2xy$. Let us show that $f$ is the standard hyperbolic paraboloid $g(x, y) = x^2 - y^2$ rotated by an angle of $\pi/4$, or $45°$. We will use these trig identities along the way:

- $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$
- $2\cos(\theta)\sin(\theta) = \sin(2\theta)$
- $\cos(\theta + \pi/2) = \sin(\theta)$

Let's rewrite $g(x, y)$ as a function of $r$ and $\theta$ using the polar change of coordinates:

$$\begin{aligned} g(r, \theta) &= (r\cos(\theta))^2 - (r\sin(\theta))^2 \\ &- r^2(\cos^2(\theta) - \sin^2(\theta)) \\ &= r^2\cos(2\theta). \end{aligned}$$

We can rotate $g$ by $\pi/4$; all we do is evaluate

$$\begin{aligned} g(r, \theta + \pi/4) &= r^2\cos(2(\theta + \pi/4)) \\ &= r^2\cos(2\theta + \pi/2) \\ &= r^2\sin(2\theta) \\ &= 2r^2\cos(\theta)\sin(\theta) = 2(r\cos(\theta))(r\sin(\theta)). \end{aligned}$$

Then we may undo the polar coordniate change to get $f(x, y) = 2xy$.

# Optimizing hyperbolic paraboloid with elliptical constraint

This is precisely what was done on Quiz 5, using $f(x, y) = xy$. (The scaling factor 2 in the above section doesn't really change the underlying geometry) Let's do it the "cleanest" way with any nondegenerate ellipse, i.e. a constraint of the form $g(x, y) = ax^2 + by^2 = c$ where $a, b, c$ are all positive constants.

$$\nabla f(x, y) = \lambda\nabla g(x, y) \implies \langle y, x\rangle = \lambda\langle 2ax, 2by\rangle.$$

So we have a system of three equations: $y = 2\lambda ax$, $x = 2\lambda by$ and $ax^2 + by^2 = c$. Multiplying the first equation by $by$ and the second equation by $ax$ gives us the equality $by^2 = 2\lambda abxy = ax^2$. Then substituting into the third equation gives $ax^2 + ax^2 = c \implies x^2 = \frac{c}{2a}$ and therefore $y^2 = \frac{c}{2b}$. It follows that the four points to check are

$$\left(\sqrt{\frac{c}{2a}}, \sqrt{\frac{c}{2b}}\right), \left(\sqrt{\frac{c}{2a}}, -\sqrt{\frac{c}{2b}}\right), \left(-\sqrt{\frac{c}{2a}}, \sqrt{\frac{c}{2b}}\right), \left(-\sqrt{\frac{c}{2a}}, -\sqrt{\frac{c}{2b}}\right)$$

The maximum of $f$ subject to the elliptical constraint is then $\frac{c}{2\sqrt{ab}}$ and the minimum is $\frac{-c}{2\sqrt{ab}}$.

# What's the connection?

Try optimizing the standard hyperbolic paraboloid given by $f(x, y) = x^2 - y^2$ with a generic elliptical constraint, and see what happens. What about a hyperbolic paraboloid $f(x, y) = y^2 - x^2$ which has been rotated by $\frac{\pi}{2}$ radians? You might notice that somehow the minima/maxima points rotate along with the hyperbolic paraboloid, but it is *not* a linear relationship. For a challenge, note that the point $M = \left(\sqrt{\frac{c}{a+b}}, \sqrt{\frac{c}{a+b}}\right)$ is on the ellipse and corresponds to the angle $\theta = \frac{\pi}{4}$, try adapting the work from both of the above parts to find an angle $\phi$ by which $M$ is a maximum of the standard hyperbolic paraboloid rotated by $\phi$, subject to the elliptical constraint. Better yet, you can see that any $\theta$ corresponds to a single point on an ellipse via the parametrization $\vec{r}(\theta) = \langle\sqrt{a}\cos(\theta), \sqrt{b}\sin(\theta)\rangle$. Try answering the same question for the point $M = \vec{r}(\theta_0)$ given by some fixed $\theta_0$ between 0 and $\frac{\pi}{2}$. I might post a solution for the first case $\theta_0 = \frac{\pi}{4}$.