CHAPTER I

APPROXIMATE METHODS

I.1 TIME-DEPENDENT PERTURBATION THEORY

Now consider the case that the full Hamiltonian $\hat{H}$ can be separated into two components — a time-independent piece $\hat{H}_0$ for which exact solutions are available and a time dependent component $\hat{V}(t)$ which is in some sense small – *e.g.*

$$\frac{\left\langle \phi|\hat{V}(t)|\psi \right\rangle}{\left\langle \phi|\hat{H}_0|\psi \right\rangle} << 1 \quad . \tag{1.1}$$

In this case one can use the methods of time dependent perturbation theory, which we now develop.

TDPT and the Interaction Representation

An exact form for the time development operator is, of course, known (*cf.* sect. I.2)

$$\hat{U}(t,0) = e^{-i(\hat{H}_0+\hat{V})t}\theta(t) \quad . \tag{1.2}$$

If $\hat{V}$ is small, it is natural to attempt some sort of Taylor series expansion of the exponential. However, this is not directly feasible since in general $\hat{H}_0$ and $\hat{V}$ do not commute. The solution is provided by going to the "interaction representation." That is, if $|\psi_S(t)\rangle$ defines the usual state in the Schrödinger representation, the corresponding interaction representation state is defined to be

$$|\psi_I(t)\rangle \equiv e^{i\hat{H}_0 t}|\psi_S(t)\rangle \quad . \tag{1.3}$$

We introduce a time development operator in the interaction representation as

$$|\psi_I(t)\rangle = \hat{U}_I(t,0)|\psi_I(0)\rangle \tag{1.4}$$

in analogy to

$$|\psi_S(t)\rangle = \hat{U}_S(t,0)|\psi_S(0)\rangle \quad . \tag{1.5}$$

Since

$$i\frac{\partial}{\partial t}|\psi_S(t)\rangle = \hat{H}|\psi_S(t)\rangle \tag{1.6}$$

we find that $\hat{U}_S(t,0)$ obeys the differential equation

$$i\frac{\partial}{\partial t}\hat{U}_S(t,0) = \hat{H}\hat{U}_S(t,0) \quad . \tag{1.7}$$

Similarly we have

$$\begin{aligned} i\frac{\partial}{\partial t}|\psi_I(t)\rangle &= i\frac{\partial}{\partial t}e^{i\hat{H}_0 t}|\psi_S(t)\rangle = e^{i\hat{H}_0 t}\left(-\hat{H}_0 + i\frac{\partial}{\partial t}\right)|\psi_S(t)\rangle \\ &= e^{i\hat{H}_0 t}\hat{V}\, e^{-i\hat{H}_0 t}|\psi_I(t)\rangle \end{aligned} \tag{1.8}$$

so that

$$i\frac{\partial}{\partial t}\hat{U}_I(t,0) = \hat{V}_I(t)\hat{U}_I(t,0) \quad . \tag{1.9}$$

where we have defined

$$\hat{V}_I(t) \equiv e^{i\hat{H}_0 t}\hat{V}e^{-i\hat{H}_0 t} \tag{1.10}$$

We can convert Eq. 4.9 to an integral equation via

$$\hat{U}_I(t,0) = \mathbf{1} - i\int_0^t dt'\, \hat{V}_I(t')\hat{U}_I(t',0) \quad . \tag{1.11}$$

with the unit operator being required by the boundary condition $\hat{U}_I(t,0) \underset{t\to 0^+}{\longrightarrow} \mathbf{1}$. Eq. 4.11 can then be solved by successive iteration, yielding an expansion in "powers" of the interaction potential $\hat{V}_I$.

$$\begin{aligned} \hat{U}_I(t,0) &= \mathbf{1} - i\int_0^t dt_1\hat{V}_I(t_1) + (-i)^2\int_0^t dt_2\int_0^{t_2} dt_1\hat{V}_I(t_2)\hat{V}_I(t_1) + \dots \\ &= \sum_{n=0}^{\infty}\hat{U}_I^{(n)}(t,0) \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} \hat{U}_I^{(0)}(t,0) &= \mathbf{1} \\ \hat{U}_I^{(1)}(t,0) &= -i\int_0^t dt_1\hat{V}_I(t_1) \\ &\vdots \\ \hat{U}_I^{(n)}(t,0) &= (-i)^n\int_0^t dt_n\int_0^{t_n} dt_{n-1}\dots\int_0^{t_2} dt_1\hat{V}_I(t_n)\hat{V}_I(t_{n-1})\dots\hat{V}_I(t_1) \quad . \end{aligned} \tag{1.13}$$

Note the time ordering here

$$t \geq t_n \geq t_{n-1}\dots \geq t_1 \geq 0 \quad . \tag{1.14}$$

This ordering is very important, as the $\hat{V}_I(t)$ do *not* commute.

The relationship between $\hat{U}_I(t,0)$ and its Schrödinger counterpart $\hat{U}_S(t,0)$ is given via

$$\begin{aligned} \hat{U}_I(t,t')|\psi_I(t')\rangle &= |\psi_I(t)\rangle = e^{i\hat{H}_0 t}|\psi_S(t)\rangle \\ &= e^{i\hat{H}_0 t}\hat{U}_S(t,t')|\psi_S(t')\rangle = e^{i\hat{H}_0 t}\hat{U}_S(t,t')e^{-i\hat{H}_0 t'}|\psi_I(t')\rangle \quad . \end{aligned} \tag{1.15}$$

Since $|\psi(t')>$ is arbitrary, we require

$$\hat{U}_I(t,t') = e^{i\hat{H}_0 t}\hat{U}_S(t,t')e^{-i\hat{H}_0 t'} \tag{1.16a}$$

or

$$\hat{U}_S(t,t') = e^{-i\hat{H}_0 t}\hat{U}_I(t,t')e^{i\hat{H}_0 t'} \tag{1.16b}$$

so that we can write a corresponding perturbation series for $\hat{U}_S(t,t')$

$$\hat{U}_S(t,t') = \sum_{n=0}^{\infty}\hat{U}_S^{(n)}(t,t') = \sum_{n=0}^{\infty} e^{-iH_0 t}\hat{U}_I^{(n)}(t,t')e^{i\hat{H}_0 t'} \tag{1.17}$$

with

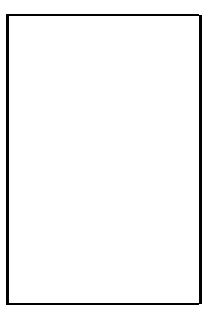
$$\begin{aligned} \hat{U}_S^{(0)}(t,t') &= e^{-i\hat{H}_0(t-t')} \\ \hat{U}_S^{(1)}(t,t') &= -i\int_{t'}^{t} dt_1\, e^{-i\hat{H}_0(t-t_1)}\hat{V}(t_1)\, e^{-i\hat{H}_0(t_1-t')} \\ &\ldots \end{aligned} \tag{1.18}$$

Transition Amplitude and Feynman Diagrams

We can also generate a series representation for the *amplitude* to make a transition from state $|i\rangle$ at $t=0$ to state $|f\rangle$ at time t. We have (hereafter we shall remain in the Schrödinger representation but drop the subscript S)

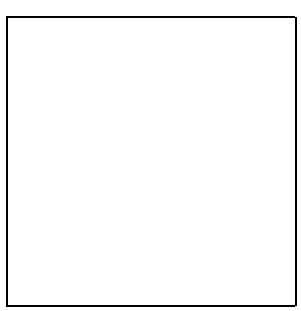
$$\text{Amp}_{fi}(t) = \left\langle f|\hat{U}(t,0)|i\right\rangle = \sum_{n=0}^{\infty}\text{Amp}_{fi}^{(n)}(t) = \sum_{n=0}^{\infty}\left\langle f\left|\hat{U}^{(n)}(t,0)\right|i\right\rangle \quad , \tag{1.19}$$

It is extremely easy and useful to employ diagrams in order to give a pictorial history of the time evolution. We associate a line



with the propagator $e^{-iE_a(t_2-t_1)}$ (1.20)

and a "vertex"



with $(-i)\,\langle b|V|a\rangle\, dt_1$. (1.21)

When an internal line a joins two vertices we speak of intermediate state $|a\rangle$, while i and f designate "external" lines. Each diagram represents an independent route by which the system can evolve from state $|i\rangle$ at time 0 to state $|f\rangle$ at time t. Each route has its own corresponding amplitude and the total amplitude is given by the sum over all possible routes

$$\text{Amp}_{\text{tot}} = \sum_{\text{route}} \text{Amp (route)} \quad . \tag{1.22}$$

(Note the similarity here to the Feynman path integral in the sum over *all* trajectories.) We have then, assuming $\hat{V}$ to be independent of time,

$$\begin{aligned}
\text{Amp}_{fi}^{(0)}(t) &= \langle f|i\rangle\, e^{-iE_i t} \\
\text{Amp}_{fi}^{(1)}(t) &= -i \int_0^t dt_1 e^{-iE_f(t-t_1)} \left\langle f|\hat{V}|i\right\rangle e^{-iE_i t_1} \\
&= -\frac{i}{i(E_f - E_i)} e^{-iE_f t} \left(e^{i(E_f - E_i)t} - 1\right) \left\langle f|\hat{V}|i\right\rangle \\
&= \frac{1}{E_f - E_i} \left(e^{-iE_f t} - e^{-iE_i t}\right) \left\langle f|\hat{V}|i\right\rangle \\
\text{Amp}_{fi}^{(2)}(t) &= (-i)^2 \sum_a \int_0^t dt_2 \int_0^{t_2} dt_1 e^{-iE_f(t-t_2)} \left\langle f|\hat{V}|a\right\rangle \\
&\quad \times e^{-iE_a(t_2-t_1)} \left\langle a|\hat{V}|i\right\rangle e^{-iE_i t_1}
\end{aligned} \tag{1.23}$$

etc., and may associate a "Feynman diagram" with each order of perturbation theory, as shown in Figure I.3.

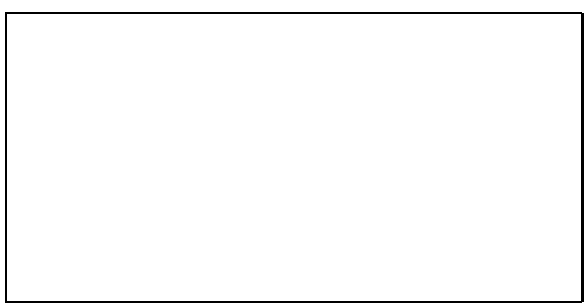


Fig. 1.3: Lowest order diagrams for the transition amplitude $\text{Amp}_{fi}(t)$.

TDPT and Frequency Space

It is also possible (and useful) to derive the time-dependent perturbation expansion using frequency space methods. We write the time development operator (assuming $\hat{V}$ is time-independent) as

$$\begin{aligned}
\hat{U}(t,0)\theta(t) = e^{-i(\hat{H}_0+\hat{V})t}\theta(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \frac{i}{\omega - \hat{H}_0 - \hat{V} + i\epsilon} \\
&\equiv \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} \hat{\mathcal{K}}(\omega)
\end{aligned} \tag{1.24}$$

where we have defined the "full propagator" in frequency space as

$$\hat{\mathcal{K}}(\omega) = \frac{i}{\omega - \hat{H}_0 - \hat{V} + i\epsilon} \quad . \tag{1.25}$$

We may generate a perturbative expansion by use of the operator identity

$$\frac{1}{\hat{A} - \hat{B}} = \frac{1}{\hat{A}} + \frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}} + \frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}} + \dots \tag{1.26}$$

[Note: The proof of Eq. 4.26 is provided by multiplication by $\hat{A} - \hat{B}$

$$\begin{aligned}(\hat{A} - \hat{B})&\left[\frac{1}{\hat{A}} + \frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}} + \frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}} + \dots\right] \\ &= \mathbf{1} - \hat{B}\frac{1}{\hat{A}} + \hat{B}\frac{1}{\hat{A}} - \hat{B}\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}} + \hat{B}\frac{1}{\hat{A}}\hat{B}\frac{1}{\hat{A}} - \dots = \mathbf{1}]\end{aligned}$$

The full propagator can then be written as

$$\begin{aligned}-i\hat{\mathcal{K}}(\omega) &= \frac{1}{\omega - \hat{H}_0 - \hat{V} + i\epsilon} = \frac{1}{\omega - \hat{H}_0 + i\epsilon} + \frac{1}{\omega - \hat{H}_0 + i\epsilon}\hat{V}\frac{1}{\omega - \hat{H}_0 + i\epsilon} \\ &+ \frac{1}{\omega - \hat{H}_0 + i\epsilon}\hat{V}\frac{1}{\omega - \hat{H}_0 + i\epsilon}\hat{V}\frac{1}{\omega - \hat{H}_0 + i\epsilon} + \dots \\ &= -i\left(\hat{K}^{(0)}(\omega) + \hat{K}^{(0)}(\omega)(-i\hat{V})\hat{K}^{(0)}(\omega)\right. \\ &\left.+\hat{K}^{(0)}(\omega)(-i\hat{V})\hat{K}^{(0)}(\omega)(-i\hat{V})\hat{K}^{(0)}(\omega) + \dots\right)\end{aligned} \tag{1.27}$$

where

$$\hat{K}^{(0)}(\omega) = \frac{i}{\omega - \hat{H}_0 + i\epsilon} \tag{1.28}$$

is the free propagator. Returning to a real time rather than frequency representation we have

$$\begin{aligned}\hat{U}(t,0) &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\hat{\mathcal{K}}(\omega) \\ &= \hat{U}^{(0)}(t,0) - i\int_0^t dt_1\hat{U}^{(0)}(t,t_1)\hat{V}\hat{U}^{(0)}(t_1,0) \\ &+ (-i)^2\int_0^t dt_2\int_0^{t_2} dt_1\hat{U}^{(0)}(t,t_2)\hat{V}\hat{U}^{(0)}(t_2,t_1)\hat{V}\hat{U}^{(0)}(t_1,0) + \dots\end{aligned} \tag{1.29}$$

where

$$\hat{U}^{(0)}(t,0) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{i}{\omega - \hat{H}_0 + i\epsilon} = \theta(t)e^{-i\hat{H}_0 t} \tag{1.30}$$

is the free propagator. [Eq. 4.29 follows via, *e.g.*

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\frac{1}{\omega-\hat{H}_0+i\epsilon}\hat{V}\frac{1}{\omega-\hat{H}_0+i\epsilon} &= \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\int_{-\infty}^{\infty}dt_1 e^{-i\omega(t-t_1)}\\ &\times e^{-i\omega' t_1}\frac{1}{\omega-\hat{H}_0+i\epsilon}\hat{V}\frac{1}{\omega'-\hat{H}_0+i\epsilon} = \int_{-\infty}^{\infty}dt_1\hat{U}^{(0)}(t,t_1)\hat{V}\hat{U}^{(0)}(t_1,0)\\ &= \int_0^t dt_1 e^{-i\hat{H}_0(t-t_1)}\hat{V}\,e^{-i\hat{H}_0 t_1}\end{aligned} \tag{1.31}$$

and similarly for the other terms.] Then for a transition from state $|i\rangle$ at time $t=0$ to state $|f\rangle$ at (later) time t we have

$$\begin{aligned}\mathrm{Amp}_{fi}(t) = \left\langle f\left|\hat{U}(t,0)\right|i\right\rangle &= i\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{-i\omega t}\Bigg\{\delta_{fi}\frac{1}{\omega-E_i+i\epsilon}\\ &+\frac{1}{\omega-E_f+i\epsilon}\left\langle f|\hat{V}|i\right\rangle\frac{1}{\omega-E_i+i\epsilon}+\sum_a\frac{1}{\omega-E_f+i\epsilon}\left\langle f|\hat{V}|a\right\rangle\\ &\times\frac{1}{\omega-E_a+i\epsilon}\left\langle a|\hat{V}|i\right\rangle\frac{1}{\omega-E_i+i\epsilon}+\ldots\Bigg\}\\ &= \delta_{fi}e^{-iE_i t}+\frac{\left\langle f|\hat{V}|i\right\rangle}{E_f-E_i+i\epsilon}\left(e^{-iE_f t}-e^{-iE_i t}\right)+\ldots\end{aligned} \tag{1.32}$$

as found previously (*cf.* Eq. 4.23).

TDPT and Path Integrals

It is particularly instructive to derive the time-dependent perturbation expansion within the path integral formalism, where we have

$$\begin{aligned}\left\langle x'\left|\hat{U}(t,0)\right|x\right\rangle &= \int\mathcal{D}\left[x(t)\right]\exp i\int_0^t dt'\left(\frac{1}{2}m\dot{x}^2(t')-V(x(t'))\right)\\ &= \int\mathcal{D}\left[x(t)\right]\exp i\int_0^t dt'\frac{1}{2}m\dot{x}^2(t')\\ &\times\left(1-i\int_0^t dt_1 V\left(x(t_1)\right)+\frac{1}{2!}\left(-i\int_0^t dt_1 V\left(x(t_1)\right)\right)^2+\ldots\right)\quad .\end{aligned} \tag{1.33}$$

Note that here since we are dealing only with *classical* quantities — no operators — we do not have to worry about lack of commutativity and can expand the exponential involving the potential straightforwardly. The first term in the expansion is recognized to be the free particle propagator

$$\left\langle x'\left|\hat{U}^{(0)}(t,0)\right|x\right\rangle = \int\mathcal{D}\left[x(t)\right]\exp i\int_0^t dt'\frac{1}{2}m\dot{x}^2(t')\quad . \tag{1.34}$$

For the term linear in the potential, we interchange orders of integration over time t_1 and paths $x(t)$ yielding

$$-i\int_0^t dt_1 \int \mathcal{D}\left[x(t)\right] V\left(x(t_1)\right) \exp i \int_0^t \frac{1}{2} m\dot{x}^2(t')dt' \quad , \tag{1.35}$$

and in the path integration we separate the paths into two pieces:

i) a path which begins at x at time $t=0$ and connects in all possible ways with point $x''=x(t_1)$ at time t_1; and

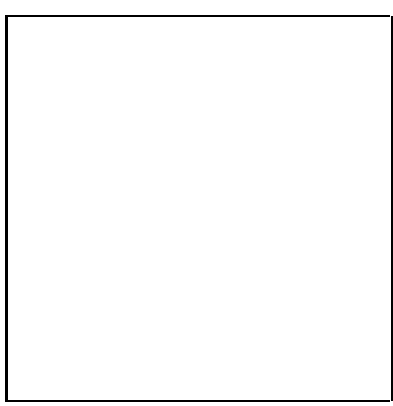
ii) a path which begins at $x''=x(t_1)$ at time t_1 and connects in all possible ways with point x' at time t.

Of course, x'' is not fixed but must take on all possible values, so that the linear term becomes

$$\begin{aligned} &-i\int_0^t dt_1 \int_{-\infty}^{\infty} dx'' \left\langle x' \left| \hat{U}^{(0)}(t,t_1) \right| x'' \right\rangle V(x'') \left\langle x'' \left| \hat{U}^{(0)}(t_1,0) \right| x \right\rangle \\ &\quad = \text{(by completeness)} \; -i\int_0^t dt_1 \left\langle x' \left| \hat{U}^{(0)}(t,t_1)\hat{V}\,\hat{U}^{(0)}(t_1,0) \right| x \right\rangle \quad . \end{aligned} \tag{1.36}$$

Likewise we can analyze the quadratic term. In this case, however, we must divide the time integration into two regions depending on whether $t_1 > t_2$ or $t_1 < t_2$:

$$\begin{aligned} &\frac{(-i)^2}{2!}\int_0^t dt_2 \int_0^{t_2} dt_1 \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dx''' \left\langle x' \left| \hat{U}^{(0)}(t,t_2) \right| x''' \right\rangle V(x''') \\ &\quad \times \left\langle x''' \left| \hat{U}^{(0)}(t_2,t_1) \right| x'' \right\rangle V(x'') \left\langle x'' \left| \hat{U}^{(0)}(t_1,0) \right| x \right\rangle \\ &\quad + \frac{(-i)^2}{2!}\int_0^t dt_2 \int_{t_2}^{t} dt_1 \int_{-\infty}^{\infty} dx'' \int_{-\infty}^{\infty} dx''' \left\langle x' \left| \hat{U}^{(0)}(t,t_1) \right| x'' \right\rangle V(x'') \\ &\quad \times \left\langle x'' \left| \hat{U}^{(0)}(t_1,t_2) \right| x''' \right\rangle V(x''') \left\langle x''' \left| \hat{U}^{(0)}(t_2,0) \right| x \right\rangle \quad . \end{aligned} \tag{1.37}$$



$$\int_0^t dt_2 \int_{t_2}^t dt_1 \equiv \int_0^t dt_1 \int_0^{t_1} dt_2$$

Fig. I.4: The t_1, t_2 integration region: Either integration can be performed first, provided the limits indicated above are used.

In the second term we can change the order of integration (*cf.* Figure I.4), and if we now interchange the identities of the variables $t_2 \leftrightarrow t_1$ the second term in Eq. 4.37 is seen to be identical to the first, cancelling the 2!. The quadratic piece of the expansion becomes then

$$(-i)^2 \int_0^t dt_2 \int_0^{t_2} dt_1 \left\langle x' \left| \hat{U}^{(0)}(t,t_2)\hat{V}(t_2)\,\hat{U}^{(0)}(t_2,t_1)\hat{V}(t_1)\,\hat{U}^{(0)}(t_1,0) \right| x \right\rangle \tag{1.38}$$

and we recognize the general form of the propagator to be

$$\begin{aligned} \left\langle x' \left| \hat{U}(t,0) \right| x \right\rangle = \Big\langle x' \Big| \hat{U}^{(0)}(t,0) - i \int_0^t dt_1 U^{(0)}(t,t_1)\hat{V}(t_1)\hat{U}^{(0)}(t_1,0) \\ + (-i)^2 \int_0^t dt_2 \int_0^{t_2} dt_1 \hat{U}^{(0)}(t,t_2)\hat{V}(t_2)\hat{U}^{(0)}(t_2,t_1)\hat{V}(t_1)\hat{U}^{(0)}(t_1,0) + \ldots \Big| x \Big\rangle \end{aligned} \tag{1.39}$$

which is identical in form to the perturbative expansion derived by the conventional technique—Eq. 4.18.

I.5 PROPAGATOR AS THE GREEN'S FUNCTION

Most readers have no doubt already have recognized the propagator as the Green's function for the Schrödinger equation, satisfying

$$\left(-\frac{1}{2m}\vec{\nabla}_2^2 + V(\vec{x}_2) - i\frac{\partial}{\partial t_2} \right) D_F\left(\vec{x}_2, t_2; \vec{x}_1, t_1\right) = -i\delta^3(\vec{x}_2 - \vec{x}_1)\delta(t_2 - t_1) \tag{5.1}$$

with the condition that

$$D_F\left(\vec{x}_2, t_2; \vec{x}_1, t_1\right) = 0 \quad \text{for} \quad t_2 < t_1 \quad . \tag{5.2}$$

This assertion is easily checked using the representation in terms of a sum over a complete set of states:

$$D_F\left(\vec{x}_2, t_2; \vec{x}_1, t_1\right) = \begin{cases} \sum_n \psi_n(\vec{x}_2)\psi_n^*(\vec{x}_1)e^{-iE_n(t_2-t_1)} & t_2 > t_1 \\ 0 & t_2 < t_1 \end{cases} \tag{5.3}$$

Here $\psi_n(\vec{x})$ is a solution to the time-independent Schrödinger equation

$$\left(-\frac{\vec{\nabla}^2}{2m} + V(\vec{x}) \right) \psi_n(\vec{x}) = E_n \psi_n(\vec{x}) \tag{5.4}$$

so that

$$\left(-\frac{1}{2m}\vec{\nabla}_2^2 + V(\vec{x}_2) - i\frac{\partial}{\partial t_2} \right) \sum_n \psi_n(\vec{x}_2)\psi_n^*(\vec{x}_1)e^{-iE_n(t_2-t_1)} = 0 \quad \text{if} \quad t_2 > t_1 \quad . \tag{5.5}$$

Also for $t_2 < t_1$ Eq. 5.1 is obviously satisfied, so that we need only verify the normalization associated with the step function. If we perform a time integration of Eq. 5.1 over the interval

$$t_1 - \epsilon < t_2 < t_1 + \epsilon \tag{5.6}$$

and subsequently take the limit as $\epsilon \to 0$, then on the right-hand side we have

$$\lim_{\epsilon \to 0} -i \int_{t_1-\epsilon}^{t_1+\epsilon} dt_2 \delta^3(\vec{x}_2 - \vec{x}_1)\delta(t_2 - t_1) = -i\delta^3(\vec{x}_2 - \vec{x}_1) \tag{5.7}$$

while on the left-hand side

$$\begin{aligned}
&\lim_{\epsilon \to 0} \int_{t_1-\epsilon}^{t_1+\epsilon} dt_2 \left(-\frac{\vec{\nabla}_2^2}{2m} + V(\vec{x}_2) - i\frac{\partial}{\partial t_2} \right) D_F\left(\vec{x}_2, t_2; \vec{x}_1, t_1\right) \\
&= \lim_{\epsilon \to 0} \Bigg[2\epsilon \left(-\frac{\vec{\nabla}_2^2}{2m} + V(\vec{x}_2) \right) D_F(\vec{x}_2, \bar{t}; \vec{x}_1, t_1) \\
&\qquad - iD_F(\vec{x}_2, t_1 + \epsilon; \vec{x}_1, t_1) + iD_F(\vec{x}_2, t_1 - \epsilon; \vec{x}_1, t_1) \Bigg] \\
&= -iD_F(\vec{x}_2, t_1; \vec{x}_1, t_1)
\end{aligned} \tag{5.8}$$

where $t_1 - \epsilon < \bar{t} < t_1 + \epsilon$ and we have used the mean value theorem. Finally, because of the completeness of the set of states n we have

$$-iD_F(\vec{x}_2, t_1; \vec{x}_1, t_1) = -i \sum_n \psi_n(\vec{x}_2)\psi_n^*(\vec{x}_1) = -i\delta^3(\vec{x}_2 - \vec{x}_1) \tag{5.9}$$

so that our assertion—Eq. 5.1—is verified. This identification of the propagator with the Green's function will prove to be useful in the relativistic problem also, as we shall see later.

With this result a perturbation expansion is easily derived. If we define the free propagator $D_F^{(0)}(\vec{x}_2, t_2; \vec{x}_1, t_1)$ via

$$\left(-\frac{\vec{\nabla}_2^2}{2m} - i\frac{\partial}{\partial t_2} \right) D_F^{(0)}(\vec{x}_2, t_2; \vec{x}_1, t_1) = -i\delta^3(\vec{x}_2 - \vec{x}_1)\delta(t_2 - t_1) \tag{5.10}$$

then it is easy to see that an expression which satisfies the strictures required of the full

propagator is†

$$\begin{aligned} D_F(\vec{x}_2, t_2; \vec{x}_1, t_1) &= D_F^{(0)}(\vec{x}_2, t_2; \vec{x}_1, t_1) \\ &\quad - i \int d^4x_3\, D_F^{(0)}(\vec{x}_2, t_2; \vec{x}_3, t_3) V(\vec{x}_3) D_F(\vec{x}_3, t_3; \vec{x}_1, t_1) \end{aligned} \tag{5.11}$$

which may be solved by iteration to yield the usual perturbative expansion

$$\begin{aligned} D_F(\vec{x}_2, t_2; \vec{x}_1, t_1) &= D_F^{(0)}(\vec{x}_2, t_2; \vec{x}_1, t_1) \\ &\quad - i \int d^4x_3\, D_F^{(0)}(\vec{x}_2, t_2; \vec{x}_3, t_3) V(\vec{x}_3) D_F^{(0)}(\vec{x}_3, t_3; \vec{x}_1, t_1) \\ &\quad + \ldots \end{aligned} \tag{5.12}$$

† Note that

$$\begin{aligned} &\left(-\frac{\vec{\nabla}_2^{\,2}}{2m} - i\frac{\partial}{\partial t_2} \right) D_F(\vec{x}_2, t_2; \vec{x}_1, t_1) = -i\delta^3(\vec{x}_2 - \vec{x}_1)\delta(t_2 - t_1) \\ &\quad - \int d^4x_3 \delta^3(\vec{x}_2 - \vec{x}_3)\delta(t_2 - t_3) V(\vec{x}_3) D_F(\vec{x}_3, t_3; \vec{x}_1, t_1) \\ &\quad = -i\delta^3(\vec{x}_2 - \vec{x}_1)\delta(t_2 - t_1) - V(\vec{x}_2) D_F(\vec{x}_2, t_2; \vec{x}_1, t_1) \end{aligned}$$