Physics 615

Homework Set #2

2. Propagator for the Linear Potential

In order to gain experience dealing with the Feynman path integral, consider a particle of mass m moving in one dimension under the influence of a constant gravitational acceleration g. The corresponding potential energy is

$$V(x) = mgx$$

and the Lagrangian becomes

$$L = \frac{1}{2}m\dot{x}^2 - mgx \quad .$$

The propagator for such a situation is exactly calculable and can be evaluated in at least two different ways:

i) Solve for the classical trajectory $x_{\rm cl}(t)$ which obeys

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_{\rm cl}} = \frac{\partial L}{\partial x_{\rm cl}}$$

and calculate the classical action

$$S[x_{\rm cl}(t)] = \int_{t_1}^{t_2} dt'\, L(x_{\rm cl}(t'), \dot{x}_{\rm cl}(t'))$$

for a path satisfying the boundary conditions

$$x_{\rm cl}(t_1) = x_1 \qquad x_{\rm cl}(t_2) = x_2 \quad .$$

Show that

$$S[x_{\rm cl}(t)] = \frac{m}{2}\,\frac{(x_2 - x_1)^2}{t_2 - t_1} - \frac{mg}{2}(t_2 - t_1)(x_2 + x_1) - \frac{mg^2}{24}\,(t_2 - t_1)^3 \quad .$$

Now imagine performing the path integral by expanding about this trajectory

$$x(t) = x_{\rm cl}(t) + \delta x(t) \quad .$$

Demonstrate that

$$D_F(x_2, t_2; x_1, t_1) = J(t_2 - t_1) \exp(iS[x_{\rm cl}(t)]) \quad .$$

Here

$$\begin{aligned} J(t_2 - t_1) &= \int \mathcal{D}\left[\delta x(t)\right] \exp\left(i \int_{t_1}^{t_2} dt' \frac{1}{2} m (\delta \dot{x}(t'))^2\right) \\ &= D_F^{(0)}(0, t_2; 0, t_1) \end{aligned} \tag{1}$$

where $D_F^{(0)}$ is the free propagator. However, we already know the form of the free propagator, yielding

$$J(t_2 - t_1) = \sqrt{\frac{m}{2\pi i (t_2 - t_1)}} \quad .$$

Thus the propagator for the linear potential takes the form

$$D_F(x_2, t_2; x_1, t_1) = \sqrt{\frac{m}{2\pi i (t_2 - t_2)}} \exp(iS[x_{\rm cl}(t)]) \quad .$$

ii) Evaluate the path integral by breaking the paths into infinitesimal time slices and performing successive integrations, *i.e.*

$$\begin{aligned} D_F(x_2, t_2; x_1, t_1) &= \lim_{n \to \infty} \left(\frac{m}{2\pi i \epsilon}\right)^{n/2} \prod_{i=1}^{n-1} \left(\int_{-\infty}^{\infty} dx_i\right) \\ &\quad \times \exp i \left[\frac{m}{2} \sum_{j=1}^{n-1} \frac{(x_j - x_{j-1})^2}{\epsilon} - \sum_{j=1}^{n-1} \epsilon m g x_j\right] \quad . \end{aligned} \tag{2}$$

Demonstrate that the form after k integrations is

$$\begin{aligned} \left(\frac{m}{2\pi i (k+1)\epsilon}\right)^{1/2} \quad & \exp\left[\frac{im}{2\epsilon(k+1)} (x_{k+1} - x_0)^2 - i\epsilon m g \frac{k}{2} (x_{k+1} + x_0)\right. \\ & \left. - i \frac{\epsilon^3}{24} k(k+1)(k+2) m g^2\right] \end{aligned} \tag{3}$$

and that this expression reduces to the previously derived result in the limit that $k \to \infty$.

3. Time Evolution of the Harmonic Oscillator

A particle of mass m moving in the harmonic oscillator potential $V(x) = \frac{1}{2}m\omega^2 x^2$ is prepared at time $t = 0$ in the state

$$< x|\psi(t=0) >= N \exp(-\frac{1}{2}m\omega x^2)[(1-i)\sqrt{m\omega}x + 2i]$$

where N is a normalization constant.

i) Find N so that the wavefunction is normalized to unity.

ii) Determine $< x|\psi(t) >$ by expanding in terms of harmonic oscillator eigenstates.

iii) Determine $< x|\psi(t) >$ via use of the propagator:

$$< x|\psi(t) >= \int_{-\infty}^{\infty} dx' D_F(x,t;x',0) < x'|\psi(0) >$$

and compare your answer with that obtained in part ii).

iv) Suppose at time t an energy measurement is made. What are the possible outcomes and with what probability is each such outcome realized?

v) Calculate $< x(t) >$.