Physics 615

Homework Set #1

1. Wave Packet Spreading: A Paradox

It was demonstrated in lecture using the identity

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial z}\right) z^{-1/2} \exp\left(-\frac{x^2}{4z}\right) = 0$$

that a Gaussian wavepacket

$$\psi(x, t=0) = \left(2\pi\sigma^2\right)^{-1/4} \exp\left(-\frac{x^2}{4\sigma^2}\right)$$

evolves in time via

$$\psi(x,t) = \xi^{-1/2} \left(2\pi\sigma^2\right)^{-1/4} \exp\left(-\frac{x^2}{4\sigma^2\xi}\right)$$

where

$$\xi = 1 + i\frac{t}{2m\sigma^2} \quad .$$

Then

$$|\psi(x,t)|^2 = \left(2\pi\sigma^2(t)\right)^{-1/2} \exp\left(-\frac{x^2}{2\sigma^2(t)}\right)$$

where

$$\sigma^2(t) = \sigma^2 + \frac{t^2}{4m^2\sigma^2} \quad .$$

i) Show that

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, |\psi(x,t)|^2 &= 1 \\ \int_{-\infty}^{\infty} dx\, x^2\, |\psi(x,t)|^2 &= \sigma^2(t) \end{aligned} \tag{1}$$

so that the wavepacket remains normalized to unity but has a width

$$\sigma(t) = \sqrt{\sigma^2 + \frac{t^2}{4m^2\sigma^2}}$$

which evolves with time. This is simply the usual "spreading" of a quantum mechanical wave packet.

ii) Derive the time evolution of the Gaussian wavepacket without exploiting the identity by using a power series expansion

$$\begin{aligned}\psi(x,t) &= e^{-iH_0(x)t}\psi(x,0) \\ &= (2\pi\sigma^2)^{-1/4}\sum_{\ell=0}^{\infty}\frac{1}{\ell!}\left(i\frac{t}{2m}\right)^{\ell}\frac{\partial^{2\ell}}{\partial x^{2\ell}}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{x^2}{4\sigma^2}\right)^{n} \quad .\end{aligned} \tag{2}$$

iii) Now suppose that

$$\psi(x,0) = N\begin{cases}\exp\left(-\frac{a^2}{a^2-x^2}\right) & |x|<a \\ 0 & |x|>a\end{cases}$$

where N is a normalization constant. Although this functional form may look a bit strange, a little thought should convince you that the wavefunction and all its derivatives are continuous at any point on the real line. However, it is easy to see that

$$e^{-iH_0(x)t}\psi(x,0) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(i\frac{t}{2m}\right)^{n}\frac{\partial^{2n}}{\partial x^{2n}}\psi(x,0)$$

vanishes for all time if $|x| \geq a$ since

$$\frac{\partial^{2n}}{\partial x^{2n}}0 = 0 \quad .$$

Hence, this type of wavepacket apparently does not undergo spreading. Is this assertion correct? If not, where have we made an error in our analysis and what does the actual time evolved wavefunction look like?