Perturbative Series for $s$-Scattering Length

The right-hand side of the integral equation for the pseudo-potential,
\begin{equation}
\tilde{U}(q) = U(q) - \int U(q - q_1) G^{(0)}(q_1) \tilde{U}(q_1) \, d^3q_1/(2\pi)^3, \tag{1}
\end{equation}
can be expanded into a perturbative series (the utility of which will be discussed later). To this end, consider a diagrammatic series shown in Fig. 1. The rules of interpreting these diagrams are as follows. Each line denotes a certain function of the momentum associated with the line: The bold dashed line in the left-hand side stands for a certain function $\tilde{F}(q)$, thin dashed lines in the right-hand side stand for functions $F$ of corresponding momenta, and solid horizontal lines denote certain functions $Q$ of corresponding momenta. Each separate diagram is understood as an integral over all the (independent) momenta, the integrand being a product of all the functions denoted with lines. By independent momenta we mean the momenta satisfying the following momentum-conservation constraints. For each of the vertices but the leftmost one, the total momentum (the sum of all the incoming momenta minus the sum of all the outgoing moments) equals zero. The leftmost vertex is special: the sum of its incoming momenta has to be equal to $q$. Note also that $q$ is a fixed external parameter, not an integration variable. Without loss of generality, the set of independent momenta over which the integration is performed can be chosen as the momenta of all the solid lines. In accordance with the above rules/definitions, the equation behind behind Fig. 1 reads
\begin{align}
\tilde{F}(q) &= F(q) + \int F(q - q_1) Q(q_1) F(q_1) \, d^3q_1/(2\pi)^3 \\
&\quad + \int F(q - q_1) Q(q_1) F(q_1 - q_2) Q(q_2) F(q_2) \, d^3q_1 \, d^3q_2/(2\pi)^3 + \cdots. \tag{2}
\end{align}

Now consider all the diagrams in the right-hand side of Fig. 1 starting with the second one and observe that, if the leftmost dashed line and the leftmost solid line are detached, the rest will be nothing but the whole right-hand side of Fig. 1. This immediately leads us to the diagrammatic equality shown in Fig. 2. Such a procedure of diagrammatic summation (the so-called Dyson summation) is widely used in diagrammatic techniques. The analytic equation (Dyson-type equation) behind Fig. 2 is
\begin{equation}
\tilde{F}(q) = F(q) + \int F(q - q_1) Q(q_1) \tilde{F}(q_1) \, d^3q_1/(2\pi)^3. \tag{2}
\end{equation}
Figure 1: Generic diagrammatic series relevant to perturbative expansion of the pseudo-potential.

Figure 2: Dyson-type summation of the diagrammatic series of Fig. 1.
Equation (2) coincides with (1) if we set $\tilde{F} = \tilde{U}$ and $Q = -G(0)$. This means that Eq. (1) can be represented as the diagrammatic series of Fig. 1. The physical meaning of this expansion is clear from its structure. It is the perturbative expansion of the pseudo-potential in powers of the amplitude (i.e. strength) of the potential $U$. In particular, the first term of the series corresponds to the Born approximation.

In the context of calculation of the $s$-scattering length $a$ by diagrammatic Monte Carlo, the convenience of working with the infinite diagrammatic expansion Fig. 1 rather than just two terms of Fig. 2 comes from the fact that, with the perturbative expansion, we have an explicit series for $a$, see Fig. 3, allowing us to directly and readily simulate $a$ in any dimensions, without bothering about (the systematic error of) histogramming the function $\tilde{U}(q)$.

A downside of the perturbative expansion Fig. 3 is the issue of convergence: The convergence takes place only at appropriately small amplitude of $U$. A good news, however, is that, at larger $U$’s the series can be re-summed. For example, Lindelöf re-summation method can be used. The method is very simple. It amounts to multiplying each order-$n$ term of the series by the factor $\exp(-\epsilon n \ln n)$. Normally, this results in convergence of the series for any $\epsilon > 0$. Then the results obtained for finite $\epsilon$’s are extrapolated to the $\epsilon \to 0$ limit.