Diagrammatics for Fermions

1 Grassmann Variables. Fermionic Coherent States

1.1 Super-Numbers and Functions

Grassmann algebra is a set of super-numbers; a super-number is a polynomial, with c-number coefficients, formed out of anti-commutative objects (of an arbitrary nature), called generators (denoted in this section with Greek letters). Apart from anti-commutativity of the generators,

\[ \theta \eta = -\eta \theta , \]

the rules of addition and multiplication are standard. With these rules, one can add (subtract) and multiply super-numbers.

If \( f(x) \) is an analytic function of the complex variable \( x \), then, for any super-number \( a \), (the value of) the super-function \( f(a) \) is obtained by substituting \( a \) for \( x \) in the Taylor expansion for \( f(x) \). As a result we always get a super-number—i.e. a polynomial, rather than an infinite series—due to the nilpotency of the generators, implied by (1):

\[ \theta \theta = -\theta \theta = 0 . \]

For example, \( e^\eta = 1 + \eta \); similarly, \( e^{\eta \theta} = 1 + \eta \theta \).

A very important for our purposes sub-set of super-numbers consists of even-power polynomials of generators. Each of such numbers commutes with any super-numer, and thus is algebraically identical to a c-number. Correspondingly, any function of such numbers inherits functional properties of of its c-number counterpart. For example, \( e^{\eta \theta + \lambda \mu} = e^{\eta \theta} e^{\lambda \mu} \).

1.2 Grassmann Integration/Differentiation

Now we introduce an operation which, depending on the context, is called either integration, or differentiation, because its properties, as well as field-theoretical applications, are reminiscent of those of conventional integration and differentiation. For definiteness, below we use the term integral, with corresponding notation.
The integral of a super-function over a super-variable $\theta$ (here $\theta$ is understood as a dummy generator) is defined formally by the following three rules

$$\int 1 \, d\theta = 0, \quad \int \theta \, d\theta = 1, \quad \int [af(\theta) + bg(\theta)] \, d\theta = a \int f(\theta) \, d\theta + b \int g(\theta) \, d\theta,$$

(3)

where $a$ and $b$ are $\theta$-independent super-functions.

**Problem 1.** Make sure that the axioms (3) allow one to unambiguously integrate any super-function.

As far as the analogy with usual numbers is concerned, one can notice that the integration over $\theta$ is in fact analogous to *differentiation* with respect to $\theta$. That is why the procedure defined by Eq. (3) is also referred to as differentiation. The term “integration” is suggested by the analogy between functional-integral representation for bosons and its fermionic counterpart based on the rules (3).

It is worth noting that any super-function $f$ (of the generator $\theta$ and, possibly, other super-variables) is uniquely represented in the form

$$f = f_0 + f_1 \theta,$$

(4)

where

$$f_1 = \int f \, d\theta, \quad \int f_0 \, d\theta = 0.$$

(5)

**Problem 2.** Prove the above statement. Trace its relationship with the formula (shift substitution)

$$\int f(\theta) \, d\theta = \int f(\theta + g) \, d\theta,$$

(6)

valid for any $\theta$-independent super-function $g$ such that $g^2 = 0$ and $g\theta = -\theta g$.

With the representation (4)-(5) one can readily see that, for any $\theta$-independent $a$,

$$\int f(\theta) a \, d\theta = \left[ \int f(\theta) \, d\theta \right] \tilde{a},$$

(7)

where $\tilde{a}$ is obtained from $a$ by changing the signs of all the odd-order terms in its polynomial of generators.
Problem 3. Derive Eq. (7).

For practical applications, the following observation proves useful. If the “differential” \( d\theta \) is formally treated as an anti-commuting (with all the generators) quantity, then the integral \( \int f(\theta) a \, d\theta \) can be evaluated in two steps: (i) Swap the differential \( d\theta \) with \( a \), thereby converting the latter into \( \tilde{a} \) by anti-commuting \( d\theta \) with generators in the explicit expression for \( a \),

\[
\int f(\theta) a \, d\theta \rightarrow \int f(\theta) d\theta \tilde{a} .
\]  

(ii) Apply the formal rule (in our case, with \( b = \tilde{a} \))

\[
\int f(\theta) \, d\theta b = \left[ \int f(\theta) \, d\theta \right] b .
\]  

A multiple integral is defined as the iterated integral, from left to right with respect to the string of differentials; the same quantity is also referred to as partial derivative, where the differentiation is performed in the right-to-left order:

\[
\frac{\partial^n f}{\partial \theta_n \ldots \partial \theta_2 \partial \theta_1} = \int f \, d\theta_1 d\theta_2 \cdots d\theta_n = \int \left[ \cdots \left[ \int f(\theta_1) \, d\theta_2 \right] \cdots \right] d\theta_n .
\]  

(10)

There is an obvious direct analog of (4)-(5):

\[
f = f_0 \theta_n \cdots \theta_2 \theta_1 + f_1 ,
\]  

where

\[
f_0 = \int f \, d\theta_1 d\theta_2 \cdots d\theta_n , \quad \int f_1 \, d\theta_1 d\theta_2 \cdots d\theta_n = 0 .
\]  

Problem 4. Prove that the trick of treating differentials as anti-commuting variables—see Eqs. (8)-(9)—can be generalized to multiple integrals, with the requirement that differentials anti-commute not only with generators, but also with each other.

Linear substitution. Given \( n \) (dummy or actual) independent generators of a Grassmann algebra, \( \theta_1 \theta_2 \cdots \theta_n \), and a c-number \( n \times n \) matrix \( A \), such that \( \det A \neq 0 \), we can construct \( n \) linear-independent anti-commuting variables/super-numbers,

\[
\lambda_j = \sum_{k=1}^{n} A_{jk} \theta_k \quad (j = 1, 2, \ldots, n) ,
\]  

(13)
and use them as new generators, that, in particular, can be used for constructing integrals. Equation (13) then can be viewed as a linear substitution in the integral $\int f(\{\lambda_i\}) \, d\lambda_1 \cdots d\lambda_n$. The substitution rule is readily established from (11)-(12). It reads

$$\int f(\{\lambda_j\}) \, d\lambda_1 \cdots d\lambda_n = [\det A]^{-1} \int f(\{\lambda_j(\{\theta_k\})\}) \, d\theta_1 \cdots d\theta_n. \quad (14)$$

**Problem 5.** Derive this relation.

**Problem 6.** Prove that for any $c$-number $n \times n$ matrix $A$,

$$\int \exp \left[ \sum_{j,k} \eta_j A_{jk} \theta_k \right] \prod_{s=1}^n d\theta_s d\eta_s = \det A. \quad (15)$$

*Hint:* Use the substitution $\theta_j = \sum_{k=1}^n (A^{-1})_{jk} \lambda_k$.

**Problem 7.** Prove that for any $c$-number $n \times n$ matrix $A$,

$$\int \exp \left[ \sum_{j,k} \eta_j A_{jk} \theta_k + \sum_{j=1}^n (\eta_j f_j + g_j \theta_j) \right] \prod_{s=1}^n d\theta_s d\eta_s = (\det A) e^{-\sum_{j,k} g_j (A^{-1})_{jk} f_k}, \quad (16)$$

provided $f$’s and $g$’s are independent of $\theta$’s and $\eta$’s super-variables. *Hint:* In accordance with (6), one can shift the variables:

$$\theta_j \to \theta_j + p_j, \quad \eta_j \to \eta_j + q_j, \quad (17)$$

where $p$’s and $q$’s are independent of $\theta$’s and $\eta$’s. The rest is almost identical to (11.31)–(11.33), with a simplification that in view of independence of $\theta$’s and $\eta$’s, the shifts $p_j$ and $q_j$ can be chosen independently, without any constraint on the form of the matrix $A$.

### 1.3 Fermionic Coherent States

The notion of Hilbert space can be generalized to Hilbert super-space by allowing multiplication of vectors by super-numbers. In a Hilbert super-space, it is possible to construct coherent states for Fermionic annihilation operators, $\{a_j\}, \ j = 1, 2, \ldots, n$ (for clarity, we work with a finite number of
single-particle modes). In quantum statistics, fermionic coherent states play the same role as bosonic coherent states: they allow one to represent traces as integrals.

In a Hilbert space, an important role is played by conjugation of complex numbers. Hence, we need to introduce the procedure of conjugation of super-numbers. To this end, we consider a Grassmann algebra with $2n$ generators. We pair the generators, so that any generator $\theta_j$ has its counterpart, which we denote as $\bar{\theta}_j$. For generators, the conjugation procedure is then defined as replacing a generator with its counterpart. The definition is straightforwardly generalized to the polynomials of generators, i.e., super-numbers of a general form, in which case, apart from replacing generators with their counterparts, we also complex-conjugate the $c$-number coefficients of the polynomials.

As far as the commutativity of the super-numbers with operators is concerned, this is a matter of convention. We choose the option when super-numbers commute with operators.

The fermionic coherent state is defined by the following expression

$$|\theta_1, \theta_2, \ldots, \theta_n\rangle = e^{\sum_{j=1}^{n} \theta_j a_j^\dagger} |\text{Vac}\rangle = \prod_{j=1}^{n} e^{\theta_j a_j^\dagger} |\text{Vac}\rangle = \prod_{j=1}^{n} (1 + \theta_j a_j^\dagger) |\text{Vac}\rangle,$$

where $|\text{Vac}\rangle$ is the fermionic groundstate. [Note that the commutativity of different pairs $\theta_j a_j^\dagger$ significantly simplifies the structure of the expressions.]

The crucial property of coherent states is that they are the eigenstates of annihilation operators. Namely,

$$a_j |\theta_1, \theta_2, \ldots, \theta_n\rangle = \theta_j |\theta_1, \theta_2, \ldots, \theta_n\rangle,$$

$$\langle \theta_1, \theta_2, \ldots, \theta_n | a_j^\dagger = \langle \theta_1, \theta_2, \ldots, \theta_n | \bar{\theta}_j .$$

**Problem 8.** Make sure that multiplying a coherent state by some Grassmann generator $\lambda$ (different from all the $\theta$’s) yields a new coherent state, with Grassmann eigenvalues opposite to the original ones.
The inner product of two coherent states is
\[
\langle \eta_1, \eta_2, \ldots, \eta_n | \theta_1, \theta_2, \ldots, \theta_n \rangle = \prod_{j=1}^{n} (1 + \bar{\eta}_j \theta_j) = e^{\sum_{j=1}^{n} \bar{\eta}_j \theta_j}. \tag{22}
\]

The closure relation has the form
\[
\int e^{-\sum_{j=1}^{n} \bar{\theta}_j \theta_j} | \theta_1, \theta_2, \ldots, \theta_n \rangle \langle \theta_1, \theta_2, \ldots, \theta_n | d\bar{\theta}_1 d\theta_1 \ldots d\bar{\theta}_n d\theta_n = \hat{1}, \tag{23}
\]
leading, in particular, to the following expression for the trace of an arbitrary operator \(A\):
\[
\text{Tr } A = \int e^{-\sum_{j=1}^{n} \bar{\theta}_j \theta_j} \langle -\theta_1, -\theta_2, \ldots, -\theta_n | A | \theta_1, \theta_2, \ldots, \theta_n \rangle d\bar{\theta}_1 d\theta_1 \ldots d\bar{\theta}_n d\theta_n, \tag{24}
\]

**Problem 9.** Prove (20)-(21), (22), (23), and (24). *Hint:* Commutativity of pairs \(\theta_j \theta_j, \theta_j a_j^\dagger, \bar{\theta}_j a_j,\) and \(d\bar{\theta}_j d\theta_j\) with the rest of the expression is very helpful.

## 2 Diagrammatic Technique for Normal Fermionic Systems

In this section, we derive Matsubara diagrammatic technique for normal (i.e., without Cooper pairing) fermionic systems. Qualitatively, the fermionic case is very close to the bosonic one, the only difference being sign factors coming from anti-commutativity of fermionic field operators. This sign factors are naturally captured by Grassmann fields used for the functional-integral representation of the theory.

### 2.1 Grassmann Functional Integral

First, we introduce a generating functional \(Q[\lambda, \bar{\lambda}]\), of two conjugate Grassmann auxiliary fields \(\lambda(X)\) and \(\bar{\lambda}(X)\), such that all the correlators (11.12) are generically expressed as variational derivatives of \(Q[\lambda, \bar{\lambda}]\) at \(\lambda(X) \equiv 0\):
\[
K_s(X_1, X_2, \ldots, X_s) = \left. \frac{\delta^s Q[\lambda, \bar{\lambda}]}{\delta \bar{\lambda}_1(X_1) \delta \bar{\lambda}_2(X_2) \ldots \delta \bar{\lambda}_s(X_s)} \right|_{\lambda \equiv 0}, \tag{25}
\]
where

\[
\tilde{\lambda}_j(X_j) = \begin{cases} 
\bar{\lambda}(X_j), & \text{if } \tilde{\Psi}_j(X_j) = \Psi(X_j), \\
\lambda(X_j), & \text{if } \tilde{\Psi}_j(X_j) = \hat{\Psi}(X_j). 
\end{cases}
\]  

(26)

For all practical purposes, Grassmann fields can be viewed as discrete Grassmann variables in discretized space-time, \( \lambda(X) \equiv \lambda(\tau_j, r_s) \). In discretized space-time, Grassmann variational derivative reduces to Grassmann partial derivative:

\[
\frac{\delta}{\delta \bar{\lambda}(X)} \equiv (\varepsilon \nu)^{-1} \frac{\partial}{\partial \lambda(\tau_j, r_s)}. 
\]  

(27)

Here \( \varepsilon \) is an infinitesimal time step, and \( \nu \) is an infinitesimal volume.

The generating functional can be explicitly expressed as

\[
Q[\lambda, \bar{\lambda}] = \text{Tr} \left[ T_\tau e^{-\int_0^\beta \tilde{H}_\tau d\tau} \right] = \text{Tr} e^{-\beta \tilde{H}} - \int [\bar{\lambda}(\tau, r) \hat{\psi}(r) + \lambda(\tau, r) \hat{\psi}^+(r)] d^d r. 
\]  

(28)

**Problem 10.** Prove that the functional \( Q[\lambda, \bar{\lambda}] \) (28) features property (25)–(26).

Now we develop a representation of \( Q[\lambda, \bar{\lambda}] \) in terms of the integral over Grassmann fields. Analogously to bosonic case, the derivation is based on the coherent states. Fermionic analogs of the bosonic coherent-state relations read

\[
\hat{\psi}(r) |\psi\rangle = \psi(r) |\psi\rangle, \quad \langle \psi | \hat{\psi}^+(r) = \langle \psi | \bar{\psi}(r), 
\]  

(29)

\[
\langle \psi_1 | \psi_2 \rangle = e^{\int \bar{\psi}_1 \psi_2 d^d r}, 
\]  

(30)

\[
\hat{1} = \int e^{-\int \hat{\psi} \psi d^d r} |\psi\rangle \langle \psi | \mathcal{D} \bar{\psi} \mathcal{D} \psi. 
\]  

(31)

The space is assumed to be discretized: \( \psi(r) \equiv \psi(r_s) \), and

\[
\int \bar{\psi}_1(r) \psi_2(r) d^d r = \nu \sum_s \bar{\psi}_1(r_s) \psi_2(r_s), \quad \mathcal{D} \bar{\psi} \mathcal{D} \psi \equiv \prod_s d\bar{\psi}(r_s) d\psi(r_s). 
\]  

(32)

In the field notation, Eq. (33) acquires the form

\[
\text{Tr} A = \int e^{-\int \hat{\psi} \psi d^d r} \langle -\psi | A |\psi\rangle \mathcal{D} \bar{\psi} \mathcal{D} \psi, 
\]  

(33)
Then, following in the footsteps of the bosonic derivation, we arrive at the functional-integral representation of the generating functional

\[ Q[\lambda, \bar{\lambda}] = \frac{\int e^{\tilde{S}[\psi]} D\bar{\psi} D\psi}{\int e^{S[\psi]} D\bar{\psi} D\psi}, \quad (34) \]

\[ S[\psi] = \int_0^\beta d\tau \left\{ \int \frac{\partial \bar{\psi}}{\partial \tau} \psi d^d r - H[\bar{\psi}, \psi] \right\}, \quad (35) \]

\[ \tilde{S}[\psi] = S[\psi] + \int_0^\beta d\tau \int d^d r (\bar{\lambda}\psi + \bar{\psi}\lambda). \quad (36) \]

with \( \psi \equiv \psi(\tau, r), \bar{\psi} \equiv \bar{\psi}(\tau, r) \), with the essentially discrete variable \( \tau \) being defined—precisely as in the bosonic case—on a limitingly large number, \( n \), of time slices:

\[ \frac{\partial \bar{\psi}}{\partial \tau} \equiv [\bar{\psi}(\tau_{j+1}) - \bar{\psi}(\tau_j)]/\varepsilon, \quad \int d\tau (\ldots) \equiv \varepsilon \sum_{j=1}^{n} (\ldots), \quad \varepsilon = \beta/n, \quad \tau_j = j\varepsilon. \quad (37) \]

\[ D\bar{\psi} D\psi \equiv \prod_{j,s} d\bar{\psi}(\tau_j, r_s) d\psi(\tau_j, r_s). \quad (38) \]

Equation (33) dictates a specific to fermions “boundary condition” \( \bar{\psi}(\tau_{n+1}) \equiv -\bar{\psi}(\tau_1) \). As in the bosonic case, the Hamiltonian is assumed to be written in the normal form—creation operators always left to the annihilation ones. Correspondingly, in all the pseudo-continuous functional expressions, the fields \( \psi(\tau) \) and \( \bar{\psi}(\tau) \) are understood as \( \psi(\tau_j) \) and \( \bar{\psi}(\tau_{j+1}) \), respectively. A specific detail associated with the Grassmann character of the fields \( \lambda \) and \( \bar{\lambda} \) is the strict order of \( \lambda \)’s and \( \psi \)’s in Eq. (36), cf. Problem 8.

Fermionic analogs of Eqs. (11.25)–(11.26) read

\[ \langle T \hat{\Psi}(X_1) \cdots \hat{\Psi}(X_n) \hat{\bar{\Psi}}(X_{n+1}) \cdots \hat{\bar{\Psi}}(X_{n+m}) \rangle = \ldots = \langle \psi(X_1) \cdots \psi(X_n) \bar{\psi}(X_{n+1}) \cdots \bar{\psi}(X_{n+m}) \rangle, \quad (39) \]

where

\[ \langle \psi(X_1) \cdots \psi(X_n) \bar{\psi}(X_{n+1}) \cdots \bar{\psi}(X_{n+m}) \rangle = \int \psi(X_1) \cdots \psi(X_n) \bar{\psi}(X_{n+1}) \cdots \bar{\psi}(X_{n+m}) e^{S[\psi]} D\bar{\psi} D\psi / \int e^{S[\psi]} D\bar{\psi} D\psi. \quad (40) \]
2.2 Wick’s Theorem. Diagrammatic Technique

In the case of an ideal system, i.e. bi-linear Hamiltonian, the generating functional is readily obtained with Eqs. (15) and (16). One uses shift substitutions (6) for the fields $\psi$ and $\bar{\psi}$, in the action $\tilde{S}$ in Eq. (34), to factor out the dependence on $\lambda$ and $\bar{\lambda}$. The result is

$$Q^{(0)}[\lambda, \bar{\lambda}] = e^{-\int \lambda(X_1) G^{(0)}(X_1, X_2) \lambda(X_2) dX_1 dX_2}, \quad (41)$$

with the function $G^{(0)}(X_1, X_2)$ defined by

$$\left(\frac{\partial}{\partial \tau} + H^{(0)}_{r_1}\right) G^{(0)}(\tau, r_1, r_2) = -\delta(r_1 - r_2) \delta(\tau), \quad G^{(0)}(\tau + \beta) = -G^{(0)}(\tau). \quad (42)$$

**Problem 11.** Derive the result (41)-(42) by the shift substitution

$$\psi \rightarrow \psi + f, \quad \bar{\psi} \rightarrow \bar{\psi} + g, \quad (43)$$

where the functions $f$ and $g$ are $\beta$-anti-periodic Grassmann fields defined (independently) by the equations

$$\frac{\partial f}{\partial \tau} + H^{(0)} f = \lambda \quad [f(\tau + \beta) = -f(\tau)], \quad (44)$$

$$\frac{\partial g}{\partial \tau} - [H^{(0)}]^\dagger g = -\bar{\lambda} \quad [g(\tau + \beta) = -g(\tau)]. \quad (45)$$

Trace the origin of the requirement of the anti-periodicity. Make sure that the result remains valid for a non-Hermitian operator $H^{(0)}$, with $[H^{(0)}]^\dagger$ in the equation for $g$.

Similar to the bosonic case, Eq. (41) implies Wick’s theorem. The new aspect here is that each term of the Wick’s expansion comes with a proper sign. Namely [we use a shorthand notation $\psi_s \equiv \psi(X_s)$ and $\bar{\psi}_{s'} \equiv \bar{\psi}(X_{n+s})$],

$$K_{1,1',2,2',...,n,n'} \equiv \langle \psi_1 \psi_{1'} \bar{\psi}_2 \bar{\psi}_{2'} \cdots \psi_n \bar{\psi}_{n'} \rangle \equiv \sum_{\{s_j\}} (-1)^{P(\{s_j\})} \langle \psi_{s_1} \bar{\psi}_{s'_1} \rangle \langle \psi_{s_2} \bar{\psi}_{s'_2} \rangle \cdots \langle \psi_{s_n} \bar{\psi}_{s'_n} \rangle, \quad (46)$$

where $P(\{s_j\})$ is the parity of the permutation $\{s_j\} = [s_1, s_2, \ldots, s_n]$. An easy way to prove (46) is to check that the term $\langle \psi_1 \psi_{1'} \rangle \langle \psi_2 \psi_{2'} \rangle \cdots \langle \psi_n \psi_{n'} \rangle$ enters the Wick’s expansion with the sign plus, and then make an observation that any elementary reconnection (by which we mean swapping conjugate pairs for two fields, $\psi_i$ and $\psi_j$) changes the sign of the expression.
From (46) we see that \( K_{1,1',2,2',...,n,n'} \) can be represented as a determinant of the \( n \times n \) matrix \( \mathcal{M} \) of single-particle correlators

\[
K_{1,1',2,2',...,n,n'} = \det \mathcal{M}, \quad \mathcal{M}_{ij} = \langle \psi_i \overline{\psi}_{j'} \rangle, \quad i,j = 1,2,...,n. \quad (47)
\]

Apart from being useful for application, the compact representation (47) for the correlator \( K_{1,1',2,2',...,n,n'} \) suggests an alternative (and elegant) proof of the fermionic Wick’s theorem. Namely, a direct derivation of (47) based on the definition (25) and the formula (41).

**Problem 12.** Consider the proof with all the details: Trace both ways of arriving at the fermionic Wick’s theorem and the relation (47).

Note that the l.h.s. of Eq. (46) has a canonical form when each field \( \psi_j \) is followed by some conjugate field \( \psi_{j'} \). If a correlator has a non-canonical form, it can always be re-written in the canonical form, at the expense of an extra sign factor depending on the parity of corresponding permutations of fields.

With the Wick’s theorem, the construction of diagrammatic technique for fermions literally follows all the steps we did for bosons, replacing classical fields with Grassmann ones, and, in particular, \( \psi^* \to \bar{\psi} \). The pair-interaction term now becomes

\[
H_{\text{pert}}[\psi, \bar{\psi}] = \frac{1}{2} \int \mathcal{U}_{\alpha\beta}(r_1 - r_2) \bar{\psi}_\beta(r_2) \psi_\alpha(r_1) \psi_\beta(r_1) \bar{\psi}_\alpha(r_2) d\mathbf{r}_1 d\mathbf{r}_2, \quad (48)
\]

where we also restore spin subscripts.

The primary diagrammatic rules are the same as in the bosonic case, apart from the sign factor for each diagram, and different analytic expression for the propagator. The sign factor requires special attention. The main concern here is the issue of factorability. In the bosonic case, the crucial tricks of diagrammatic technique—cancellation of disconnected diagrams and Dyson summation—were based of factorability of global symmetry factors. For these tricks to apply, the fermionic sign factor should be a product of independent sign factors for each disconnected and reducible element. Fortunately, this proves to be the case, as we see below. Hence, all the diagrammatic tricks introduced for bosons also work for fermions.

Let us establish an explicit expression for the sign factor of a diagram. First, note that topologically any diagram for a correlator—the latter is assumed to be written in a canonical form \( \langle \psi_1 \overline{\psi}_{1'} \psi_2 \overline{\psi}_{2'} \cdots \psi_n \overline{\psi}_{n'} \rangle \)—consists of a certain number, \( N_{\text{loop}} \), of closed loops of propagators and exactly \( n \) open
propagator chains, the $j$-th chain connecting some field $\bar{\psi}_j'$ to the field $\psi_j$. The open chains define naturally the permutation $\{s_j\} = [s_1, s_2, \ldots, s_n]$, a generalization of the permutation we introduced in connection with Eq. (46). It then turns out that the sign of the diagram is a product of two signs associated with the parity of the permutation and the number of closed loops:

$$\text{diagram sign} = (-1)^{P(\{s_j\}) + N_{\text{loop}}}.$$  \hspace{1cm} (49)

To prove Eq. (49), one observes that the above-introduced elementary reconnection of Wick's pairs—changing the sign factor of the average—changes either the parity of the permutation, or the parity of the number of the loops, so that the r.h.s. of (49) follows the sign of the diagram.

**Problem 13.** Prove this.

To complete the proof, we need to make sure that there is no extra sign minus in front of the r.h.s. of (49). This can be done by considering one particular diagram of a simple structure. The natural choice is a diagram in which each propagator chain consists of one propagator connecting $\bar{\psi}_j'$ to $\psi_j$, while the fields originating from the interaction elements are paired locally, forming single-propagator loops $\langle \bar{\psi}_\alpha(X) \psi_\beta(X) \rangle$, as, e.g., in the first diagram in Fig. 11.12. Each such loop comes with the sign minus since the field $\bar{\psi}_\alpha(X)$ is to the left from the filed $\psi_\beta(X)$.

The permutation sign factor deals only with the main part of the diagram, involving the external vertices. It is also worth noting that it is absent in the case of single-particle Green’s function, and thus is irrelevant for the Dyson summation. The desired factorization of the diagram sign in the case of disconnected and reducible diagrams is then obvious.

### 2.3 Symmetry Breaking in the Cooper Channel: Specifics of Diagrammatic Rules, Anomalous Propagators, and Skeleton Series

Explicit symmetry breaking in the Cooper channel is achieved by adding the following two terms to the system’s Hamiltonian (the factor $-1/2$ is a matter of convenience):

$$-\frac{1}{2} \sum_{\alpha, \beta} \int d^d r_1 d^d r_2 \left[ \eta_{\alpha\beta}(r_1, r_2) \hat{\psi}_\alpha^\dagger(r_1) \hat{\psi}_\beta^\dagger(r_2) + \eta_{\alpha\beta}^*(r_1, r_2) \hat{\psi}_\beta(r_2) \hat{\psi}_\alpha(r_1) \right],$$  \hspace{1cm} (50)
where, without loss of generality (by fermionic symmetry),

\[ \eta_{\beta\alpha}(\mathbf{r}_2, \mathbf{r}_1) = -\eta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2). \]  

(51)

The details of the dependence of the function \( \eta_{\beta\alpha}(\mathbf{r}_2, \mathbf{r}_1) \) on the distance \( r = |\mathbf{r}_1 - \mathbf{r}_2| \) are not important. The function simply has to decay with \( r \) at a certain microscopic scale. What matters is the dependence on the direction \( \hat{r} = (\mathbf{r}_1 - \mathbf{r}_2)/r \), which, on one hand, is supposed to respect the rotational symmetry of the problem (say, isotropy in a continuous space, or \( \pi/2 \)-rotation symmetry on a square/cubic lattice), and, on the other hand, needs to conspire with the symmetry of the spin dependence to obey Eq. (51).

The diagrammatic elements representing symmetry-breaking terms (50) are shown in Fig. 1. The direction indicated by the central arrow on a dotted line is not redundant: It is associated with the order of fermionic operators and thus is relevant to the sign of the diagram. The new elements enter diagrammatic expansion along with the elements arising from interaction,
following the diagrammatic rules implied by the Wick’s theorem. Graphically, the generalization of diagrammatic technique is straightforward. What requires a special discussion is the rule for determining the global sign of the diagram.

The sign. Let us generalize the notion of the closed loops and propagator chains by treating the dotted lines on the same footing as particle propagators. Then, the previously formulated rule defining the sign of the diagram is modified by an extra factor \((-1)^I\), where \(I\) is the total number of inversions of the dotted propagators. The number \(I\) is defined as follows: (a) For each closed loop \(\mathcal{L}\), fix one of the two possible directions (which one of the two does not matter since the difference in \(I\) between the two choices is an even number) and count all the dotted propagators in the loop that have the direction opposite to the direction of the loop. This yields the number \(I_{\mathcal{L}}\) for the given loop. (b) For each propagator chain \(\mathcal{C}\), define the direction of the chain as the direction from the \(\bar{\psi}\)-end to the \(\psi\)-end. Define the number \(I_{\mathcal{C}}\) as the number of all the dotted propagators in the chain that have the direction opposite to the direction of the chain. Then \(I\) is equal to the sum of all \(\mathcal{L}\)’s and \(I_{\mathcal{C}}\)’s. It is readily seen that—up to possible universal factor \((-1)\) the absence of which will be shown later—the formulated rule properly captures the sign of the diagram. Indeed, as we already discussed, the sign change can be traced by elementary reconnections. Now observe that an elementary reconnection involves one and only one of the following
three options: (a) change in the parity of the number of loops by 1, or (b) a permutation of two chains, or (c) change in the parity of inversions. [The option (c) takes place if and only if the elementary reconnection deals with the two ends of one and the same dotted propagator.] To make sure that there is no extra universal factor (-1), we need to consider some reference diagram. As before, the reference diagram involves simple factorization of interaction elements and the pairwise factorization of adjacent elements of the correlator (47), with no global sign minus coming from those groups. As far as the elements of the Fig. 1 are concerned, we group them into pairs [clearly, this is always possible for the expansion of either the partition function, or the correlator (47)] shown in Fig. 2.

With the terms (50) added to the Hamiltonian, anomalous propagators appear. We confine ourselves to single-particle anomalous propagators \[X \equiv (r, \tau, \alpha)\]

\[
\begin{align*}
-F_{\text{in}}(X_1, X_2) &= \langle T_\tau \hat{\bar{\psi}}(X_1) \hat{\Psi}(X_2) \rangle \equiv \langle \bar{\psi}(X_1) \bar{\psi}(X_2) \rangle, \\
-F_{\text{out}}(X_1, X_2) &= \langle T_\tau \hat{\Psi}(X_2) \hat{\bar{\psi}}(X_1) \rangle \equiv \langle \psi(X_2) \bar{\psi}(X_1) \rangle, \\
F_{\text{out}}(X_1, X_2) &= [F_{\text{in}}(X_1, X_2)]^*. 
\end{align*}
\]

Corresponding diagrammatic elements are shown in Fig. 3. The order of operators in Eqs. (52)–(53) matters [cf. Eq. (51)] because of the antisymmetric properties:

\[F_{\text{in}}(X_2, X_1) = -F_{\text{in}}(X_1, X_2), \quad F_{\text{out}}(X_2, X_1) = -F_{\text{out}}(X_1, X_2),\]

and that is why, speaking generally, we have to specify directions of the anomalous lines (with central arrows in corresponding elements, cf. Fig. 1). Those directions are also crucial for unambiguously formulate the sign rule for diagrammatic expansion of anomalous propagators, see Fig. 4. The rule is essentially the same as for normal propagators, provided the inversions on the propagator chain connecting the two ends of the anomalous propagator are defined with respect to the direction specified on corresponding bold line. The proof is almost the same as before, with the only new detail to consider in terms of checking the absence of the universal factor (-1) is the sign of diagram elements shown in Fig. 4.

Observing that the sign rule is factorable in terms of disconnected diagrammatic pieces, we conclude that the cancellation of the disconnected pieces between numerator and denominator (partition function) takes place, so that the expansions for normal and anomalous correlators consist of connected diagrams only. At this point we can simplify the notation by observing that changing the direction of any dotted line, on one hand, leads to a
Figure 3: Anomalous propagators.

Figure 4: First terms of the expansion of anomalous propagators.

Figure 5: Gor'kov-Nambu equations.
new type of Wick’s pairing (the absence of disconnected pieces is crucial for this statement!) and, on the other hand, does not change the value of the diagram. Hence, we can combine both cases into one element without the arrow with simultaneously eliminating the factor $1/2$. More importantly, with this notation, we do not need to count the number of inversions. Instead, we adopt the following obvious rule guaranteeing that the number of inversions is automatically zero. For each closed loop, we make sure that the directions of all dotted lines are the same within the given loop (the factor $1/2$ being replaced with $1$ to take into account the opposite direction of each dotted line). All the dotted lines along each chain are processed similarly: All their directions are supposed to coincide with the direction of the chain.

*Gor’kov-Nambu equations. Skeleton technique.* The next step is to introduce Gor’kov-Nambu equations, Fig. 5 (analogous to the Beliaev-Dyson equations for bosons). The normal and anomalous self-energies share the symmetry properties with corresponding normal and anomalous propagators:

\begin{align}
\Sigma_{11}^*(X_1, X_2) &= \Sigma_{11}(X_2, X_1), \\
\Sigma_{20}^*(X_1, X_2) &= \Sigma_{02}(X_1, X_2) = -\Sigma_{02}(X_2, X_1).
\end{align}

(56) \hspace{1cm} (57)

Then, we can introduce skeleton diagrammatic technique build on the exact normal and anomalous single-particle propagators. With the boldification, two related subtleties arise concerning the directions of the anomalous propagators and the sign rule. As opposed to changing the direction of the original dotted line of Fig. 1, changing the direction of the internal anomalous propagator of the skeleton technique is merely conventional: It does not involve alternative pattern of Wick’s pairing. Hence, for each anomalous propagator in a closed loop, only one of the two possible directions should be adopted, both being equivalent, provided the sign rule includes the factor $(-1)^I$, with $I$ the parity of the inversions of the directions of anomalous propagators. In a direct analogy with the previously discussed optimization of the notation for dotted propagators (except for the symmetry factor 2 irrelevant here), one can simply omit the direction arrows on the anomalous propagators, unambiguously restoring the proper direction by the same rule we used for the arrow-free dotted lines. This option appears to be convenient for analytic treatments. For Diagrammatic Monte Carlo, on the other hand, when one traces only the change of sign/phase of the diagrammatic expression upon an elementary update, it may prove more convenient to ascribe a specific direction for each anomalous propagator the moment it is being introduced.