Problem 19.

Consider the heat equation
\[ u_t = u_{xx}, \quad (1) \]
\( u = u(x, t), \ x \in [0, 1], \) with the boundary conditions (note the derivative in the first one)
\[ u_x(0, t) = 0, \quad u(1, t) = 0, \quad (2) \]
and the initial condition
\[ u(x, 0) = 1. \quad (3) \]

(a) Make sure that the boundary conditions are canonical.
(b) Construct the orthonormal basis of the eigenfunctions of the Laplace operator in the space of functions obeying the boundary conditions (2).
(c) Find the solution \( u(x, t) \) of the problem (1)-(3) in the form of the Fourier series in terms of the constructed ONB.

Solution

(a)
To make sure that the boundary conditions are canonical we need to check that (i) the functions satisfying (2) form a vector space, and (ii) for any two functions \( f \) and \( g \), satisfying (2), the following is true:
\[ \int_0^1 f^* g_{xx} \, dx = \int_0^1 f_{xx}^* g \, dx. \quad (4) \]
The point (i) is straightforward. Let us check the point (ii). Doing the integral by parts, we get
\[ \int_0^1 f^* g_{xx} \, dx = f^* g_{x} \bigg|_0^1 - \int_0^1 f_{xx}^* g_x \, dx = \]
\[ = f^* g_x \big|_0^1 - f_{xx}^* g \big|_0^1 + \int_0^1 f_{xx}^* g \, dx = \int_0^1 f_{xx}^* g \, dx, \quad (5) \]
because \( f_x(0) = g_x(0) = 0 \) and \( f(1) = g(1) = 0. \)

(b)
We have to solve the problem
\[ e''(x) = \lambda e(x), \quad e'(0) = 0, \quad e(1) = 0, \quad e(1) = 0, \quad (6) \]
to find eigenfunctions $e(x)$’s with corresponding eigenvalues $\lambda$’s. We start
with exploring the case
1.) $\lambda > 0$.
Two independent solutions are found by exponential substitution (see, e.g.,
the section of the Lecture Notes dealing with damped harmonic oscillator),
so that the general solution is

$$e(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x},$$

where $A$ and $B$ are some constants. From the boundary conditions we get

$$e'(0) = 0 \quad \Rightarrow \quad A\sqrt{\lambda} - B\sqrt{\lambda} = 0 \quad \Rightarrow \quad A = B.$$ 

$$e(1) = 0 \quad \Rightarrow \quad Ae^{\sqrt{\lambda}} + Be^{-\sqrt{\lambda}} = 0 \quad \Rightarrow \quad A\left(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}\right) = 0.$$ 

Since the sum of exponentials in the brackets is always positive, there is only
trivial solution $A = B = 0$.

Now we consider the case
2.) $\lambda > 0$.
Here we have

$$e''(x) = 0 \quad \Rightarrow \quad e(x) = Cx + D,$$

where $C$ and $D$ are constants. From the boundary conditions we readily
find that $C = D = 0$. That is we get only a trivial solution.

Finally, we consider the case
3.) $\lambda < 0$.
Here the solution (see, e.g., the section of the Lecture Notes dealing with
harmonic oscillator) is

$$e(x) = A\cos(\sqrt{-\lambda}x + \varphi_0),$$

where $A$ and $\varphi_0$ are some constants. From the first boundary condition we get

$$e'(0) = 0 \quad \Rightarrow \quad \sin(\varphi_0) = 0 \quad \Rightarrow \quad \varphi_0 = \pi n,$$

where $n$ is some integer, which without loss of generality can be set equal
to zero (because the only difference is the sign of the solution, which is
irrelevant). We thus have

$$e(x) \propto \cos(\sqrt{-\lambda}x),$$
and from the second boundary condition we find
\[ e(1) = 0 \Rightarrow \cos \sqrt{-\lambda} = 0 \Rightarrow \sqrt{-\lambda} = \pi/2 + \pi m , \]
where \( m \) is some integer. It turns out that negative \( m \)'s do not yield independent solutions and thus should be excluded. Indeed, let \( m < 0 \). Then,
\[
\cos[(\pi/2 + \pi m)x] = \cos[(\pi/2 - \pi |m|)x] = \\
\cos[(\pi|m| - \pi/2)x] = \cos[(\pi/2 + \pi(|m| - 1))x],
\]
and we conclude that for any \( m < 0 \) the solution coincides with the solution for positive \( m_1 = |m| - 1 \).

Our (normalized) eigenvectors thus are
\[ e_m(x) = \sqrt{2} \cos[(1/2 + m)\pi x], \quad m = 0, 1, 2, \ldots . \]

Corresponding eigenvalues are
\[ \lambda_m = -(1/2 + m)^2 \pi^2 , \quad m = 0, 1, 2, \ldots . \]

(c)
We look for the solution in the form of the Fourier series:
\[ u(x, t) = \sum_{m=0}^{\infty} u_m(t)e_m(x) . \]

Plugging this into the equation (1) and doing inner products with basis vectors yields
\[
\sum_{m=0}^{\infty} [\dot{u}_m(t) - \lambda_m u_m(t)] e_m(x) = 0 \Rightarrow \dot{u}_m(t) = \lambda_m u_m(t) .
\]

And then
\[ u_m(t) = u_m(0)e^{\lambda_m t} , \]
where \( u_m(0) \) is found from the initial condition:
\[
u_m(0) = \langle e_m | u(t = 0) \rangle = \int_0^1 e_m(x)u(x, t = 0)dx = \\
= \sqrt{2} \int_0^1 \cos[(1/2 + m)\pi x]dx = \frac{\sqrt{2}\sin[(1/2 + m)\pi]}{(1/2 + m)\pi} = \frac{(-1)^m\sqrt{2}}{(1/2 + m)\pi} .
\]
The final result thus reads
\[
    u(x, t) = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m + 1/2} e^{-(m+1/2)^2\pi^2 t} \cos((m + 1/2)\pi x) .
\] (6)

**Problem 20.**

Find the solution \( u(x), x \in [0, 1] \)—in the form of the Fourier series in terms of the eigenfunctions of the Laplace operator—of the stationary heat equation
\[
    u_{xx} + f(x) = 0 ,
\] (7)
\[
    f(x) = \begin{cases} 
    1, & x \in [0, 0.5) , \\
    0, & x \in [0.5, 1] ,
    \end{cases}
\] (8)
\[
    u(0) = u(1) = 0 .
\] (9)

**Solution**

(a) We have to solve the problem
\[
    e''(x) = \lambda e(x) , \quad e(0) = e(1) = 0 ,
\]
to find eigenfunctions \( e(x) \)'s with corresponding eigenvalues \( \lambda \)'s. We start with exploring the case
1.) \( \lambda > 0 \).

Two independent solutions are found by exponential substitution (see, e.g., the section of the Lecture Notes dealing with damped harmonic oscillator), so that the general solution is
\[
    e(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} ,
\]
where \( A \) and \( B \) are some constants. From the boundary conditions we get
\[
    e(0) = 0 \quad \Rightarrow \quad A + B = 0 \quad \Rightarrow \quad B = -A .
\]
\[
    e(1) = 0 \quad \Rightarrow \quad Ae^{\sqrt{\lambda}} + Be^{-\sqrt{\lambda}} = 0 \quad \Rightarrow \quad A \left( e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}} \right) = 0 .
\]
Since the difference of exponentials in the brackets is always positive, there is only trivial solution $A = B = 0$.

Now we consider the case

2.) $\lambda > 0$.

Here we have

$$e''(x) = 0 \quad \Rightarrow \quad e(x) = Cx + D,$$

where $C$ and $D$ are constants. From the boundary conditions we readily find that $C = D = 0$. That is we get only a trivial solution.

Finally, we consider the case

3.) $\lambda < 0$.

Here the solution (see, e.g., the section of the Lecture Notes dealing with harmonic oscillator) is

$$e(x) = A \sin(\sqrt{-\lambda} x + \varphi_0),$$

where $A$ and $\varphi_0$ are some constants. From the first boundary condition we get

$$e(0) = 0 \quad \Rightarrow \quad \sin(\varphi_0) = 0 \quad \Rightarrow \quad \varphi_0 = \pi n,$$

where $n$ is some integer, which without loss of generality can be set equal to zero (because the only difference is the sign of the solution, which is irrelevant). We thus have

$$e(x) \propto \sin(\sqrt{-\lambda} x),$$

and from the second boundary condition we find

$$e(1) = 0 \quad \Rightarrow \quad \sin \sqrt{-\lambda} = 0 \quad \Rightarrow \quad \sqrt{-\lambda} = \pi m,$$

where $m$ is some integer. The $m = 0$ yields only a trivial solution. For $m < 0$ we get the same solution as for $|m| > 0$. We thus conclude that our (normalized) basis is

$$e_m(x) = \sqrt{2} \sin m\pi x, \quad m = 1, 2, 3, \ldots.$$

Corresponding eigenvalues are

$$\lambda_m = -m^2 \pi^2, \quad m = 1, 2, 3, \ldots.$$
We look for the solution in the form of the Fourier series:

\[ u(x) = \sum_{m=1}^{\infty} u_m e_m(x) . \]

Plugging this into the equation (7) and doing inner products with basis vectors yields

\[
\sum_{m=0}^{\infty} \lambda_m u_m e_m(x) + f(x) = 0 \quad \Rightarrow \quad \lambda_m u_m + \langle e_m | f \rangle = 0 .
\]

Hence,

\[
u_m = -\lambda_m^{-1} \langle e_m | f \rangle .\]

\[
\langle e_m | f \rangle = \int_0^1 e_m(x) f(x) dx = \sqrt{2} \int_0^{1/2} \sin m\pi x dx = -\frac{\sqrt{2}}{m\pi} \left(1 - \cos \frac{m\pi}{2}\right) .
\]

Now we note that

\[
\cos \frac{m\pi}{2} = \begin{cases} 0, & m = 2n + 1, \ n = 0,1,2,\ldots , \\ (-1)^n, & m = 2n, \ n = 1,2,3,\ldots , \end{cases}
\]

and that

\[
1 - (-1)^n = \begin{cases} 2, & n = 2k + 1, \ k = 0,1,2,\ldots , \\ 0, & n = 2k, \ k = 1,2,3,\ldots , \end{cases}
\]

and get the final answer in the form

\[
u(x) = \frac{2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sin[(2n + 1)\pi x]}{(2n + 1)^3} + \frac{1}{2\pi^3} \sum_{k=0}^{\infty} \frac{\sin[2(2k + 1)\pi x]}{(2k + 1)^3} .
\]

**Problem 21.**

The wavefunction \( \psi(x,t) \) of a 1D quantum particle living on a ring of circumference \( L \) obeys Schrödinger equation

\[
i\hbar \psi_t = -\frac{\hbar^2}{2m} \psi_{xx}, \quad x \in [0, L],
\]

\( (10) \)
with periodic boundary conditions (because of the ring topology):

$$\psi(0, t) = \psi(L, t) \quad \psi_x(0, t) = \psi_x(L, t) . \quad (11)$$

At $t = 0$ the wavefunction is

$$\psi(x, 0) = \begin{cases} 
(2/L)^{1/2}, & x \in [0, L/2], \\
0, & x \in (L/2, L). 
\end{cases} \quad (12)$$

Solve for $\psi(x, t)$. Do not forget that in contrast to heat and wave equations, the function $\psi$ is complex.

*Solution.*

We start with re-scaling coordinate, time, and the function $\psi$ to get rid of all the dimensional constants in the equations and boundary conditions. First, we introduce re-scaled coordinate

$$x' = x/L ,$$

so that we get a convenient interval

$$x' \in [0, 1] .$$

Since

$$\frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial x'} ,$$

the Schrödinger equation now reads

$$i\psi_t = -\frac{\hbar}{2mL^2} \psi_{xx'} .$$

Now we introduce re-scaled time

$$t' = \gamma t , \quad \frac{\partial}{\partial t} = \gamma \frac{\partial}{\partial t'} ,$$

which allows us to get rid of the pre-factor in the r.h.s. of the equation, by choosing

$$\gamma = \frac{\hbar}{2mL^2} .$$
Finally, we introduce re-scaled \( \psi \)-function

\[
\psi' = \sqrt{\frac{L}{2}\psi}
\]

to simplify the form of the initial condition.

Now we work with only the re-scaled variables, and thus can omit the primes. The problem to solve is:

\[
i\psi_t = -\psi_{xx}, \quad x \in [0, 1], \tag{13}
\]

\[
\psi(0, t) = \psi(1, t) \quad \psi_x(0, t) = \psi_x(1, t),
\]

\[
\psi(x, 0) = \begin{cases} 
1, & x \in [0, 1/2], \\
0, & x \in (1/2, 1).
\end{cases}
\]

(a) The basis.
We have to solve the problem

\[
e''(x) = \lambda e(x), \quad e(0) = e(1), \quad e'(0) = e'(1),
\]

for the eigenfunctions \( e(x) \)'s with corresponding eigenvalues \( \lambda \)'s. We start with exploring the case

1. \( \lambda > 0 \).
Two independent solutions are found by exponential substitution (see, e.g., the section of the Lecture Notes dealing with damped harmonic oscillator), so that the general solution is

\[
e(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x},
\]

where \( A \) and \( B \) are some constants. From the boundary conditions we get

\[
e(0) = e(1) \quad \Rightarrow \quad A + B = Ae^{\sqrt{\lambda}} + Be^{-\sqrt{\lambda}},
\]

\[
e'(0) = e'(1) \quad \Rightarrow \quad A - B = Ae^{\sqrt{\lambda}} - Be^{-\sqrt{\lambda}}.
\]

Re-writing the two conditions as

\[
B \left( 1 - e^{-\sqrt{\lambda}} \right) = A \left( e^{\sqrt{\lambda}} - 1 \right)
\]

\[
B \left( 1 - e^{-\sqrt{\lambda}} \right) = -A \left( e^{\sqrt{\lambda}} - 1 \right),
\]
and noticing that the expressions in brackets are always positive, we see that we get only a trivial solution $A = B = 0$. Now we consider the case 2.) $\lambda > 0$.
Here we have

$$e''(x) = 0 \quad \Rightarrow \quad e(x) = Cx + D,$$

where $C$ and $D$ are constants. From the boundary conditions we readily find that $C = 0$, but $D$ can be arbitrary. Hence, we get a non-trivial (normalized) solution

$$e_0(x) = 1.$$

Finally, we consider the case 3.) $\lambda < 0$.
Here the solution can be written either in trigonometric or in the exponential form (see the Lecture Notes: the section dealing with harmonic oscillator). We will be using the exponential form, since it is more convenient for expanding the complex-valued functions (but if you prefer the trigonometric form, you can use it as well). So we write the general solution as

$$e(x) = Ae^{i\sqrt{|\lambda|}x} + Be^{-i\sqrt{|\lambda|}x},$$

and immediately realize that for this solution to be a periodic function of $x$ with period 1, the phase of the complex exponentials should vary by a multiple of $2\pi$ when $x$ goes from 0 to 1. And this defines the eigenvalues:

$$\lambda_m = -(2\pi m)^2, \quad m = 1, 2, 3, \ldots.$$

In fact we see that for each $\lambda < 0$ their exist two linear independent solutions, which means that corresponding vector sub-space is two-dimensional and we have an extra freedom of choosing the two basis vectors. For example, we can use sines and cosines:

$$e_m^{(s)}(x) \propto \sin 2\pi mx, \quad e_m^{(c)}(x) \propto \cos 2\pi mx, \quad m = 1, 2, 3, \ldots.$$

But it is more convenient to use exponentials:

$$e_m^{(+)}(x) \propto e^{i2\pi mx}, \quad e_m^{(-)}(x) \propto e^{-i2\pi mx}, \quad m = 1, 2, 3, \ldots.$$

The exponential eigenfunctions are conveniently labeled with integer $m$’s (including $m = 0$ that also covers the case $\lambda = 0$). The complete list of (normalized) eigenfunctions and eigenvalues then reads

$$e_m(x) = e^{i2\pi mx}, \quad \lambda_m = -(2\pi m)^2, \quad m = 0, \pm 1, \pm 2, \pm 3, \ldots.$$
Now we look for the solution in the form of the Fourier series:

$$\psi(x,t) = \sum_{m=-\infty}^{\infty} \psi_m(t)e_m(x).$$

Plugging this into the equation (13) and doing inner products with basis vectors yields

$$\sum_{m=-\infty}^{\infty} \left[i\dot{\psi}_m(t) + \lambda_m \psi_m(t)\right]e_m(x) = 0 \Rightarrow i\dot{\psi}_m(t) = -\lambda_m \psi_m(t).$$

And then (note that the differential equation we get is the same as the one we were solving for the motion of a particle in magnetic field)

$$\psi_m(t) = \psi_m(0)e^{i\lambda_m t},$$

where $\psi_m(0)$ is found from the initial condition:

$$\psi_m(0) = \langle e_m|\psi(t = 0)\rangle = \int_0^1 e_m^*(x)\psi(x,t = 0)dx = \int_0^{1/2} e^{-i2\pi mx} dx.$$

For $m = 0$, which is a special case, the integration yields $\psi_0(0) = 1/2$. For $m \neq 0$, we have

$$\int_0^{1/2} e^{-i2\pi mx} dx = \frac{i}{2\pi m} \left[e^{-i\pi m} - 1\right] = \frac{i}{2\pi m} [(-1)^m - 1].$$

We see that for even $m$’s all $\psi_m$’s are zero, while for the odd $m$’s we have ($m = 2n + 1$):

$$\psi_{2n+1} = \frac{-i}{\pi(2n + 1)}, \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots.$$

We thus get

$$\psi(x,t) = \frac{1}{2} - \frac{i}{\pi} \sum_{n=-\infty}^{\infty} \frac{1}{2n + 1} e^{-i[2\pi(2n+1)]^2 t} e^{i2\pi(2n+1)x}.$$

Finally, we return to the original coordinate, time, and $\psi$ by simply replacing

$$x \rightarrow x/L, \quad t \rightarrow \frac{ht}{2mL^2}, \quad \psi \rightarrow \psi \sqrt{2/L}.$$

The result is:

$$\psi(x,t) = \frac{1}{\sqrt{2L}} - \frac{i}{\pi \sqrt{L/2}} \sum_{n=-\infty}^{\infty} \frac{1}{2n + 1} e^{-i[2\pi(2n+1)]^2 ht/2mL^2} e^{i2\pi(2n+1)x/L}.$$