Topic 6
Estimation

Correction Pages
Recall that the basic structure of the required confidence interval is:

Lower limit = \( \text{point estimate} - \text{multiple} \times \text{SE of point estimate} \)
Upper limit = \( \text{point estimate} + \text{multiple} \times \text{SE of point estimate} \)

**Point Estimate of \( \mu \) is the Sample Mean** \( \bar{X}_{n=30} \)

\[
\bar{X}_{n=30} = \frac{\sum_{i=1}^{n} X_i}{n} = 0.51
\]

**The Standard Error of \( \bar{X}_n \) is** \( \sigma / \sqrt{n} \)

\[
SE(\bar{X}_{n=30}) = \sqrt{\text{variance}(\bar{X}_{n=30})} = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{0.25}}{\sqrt{30}} = 0.0913
\]

**The Confidence Coefficient**

For a 95% confidence interval, this number will be the 97.5th percentile of the Normal (0,1) distribution. See the table on page 19 and locate that value is 1.96.

<table>
<thead>
<tr>
<th>Desired Confidence Level</th>
<th>Value of Confidence Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>.95</td>
<td>1.96</td>
</tr>
</tbody>
</table>

Here \( 1.96 = (1-.05/2)100\text{th} = 97.5\text{th percentile of the Normal}(0,1) \text{ distribution} \)

**Putting this all together** –
Lower limit = \( \text{point estimate} - \text{multiple} \times \text{SE of point estimate} \)
\[
= 0.51 - (1.96) \times (0.0913) \\
= 0.33
\]

Upper limit = \( \text{point estimate} + \text{multiple} \times \text{SE of point estimate} \)
\[
= 0.51 + (1.96) \times (0.0913) \\
= 0.69
\]
7. Normal: Confidence Interval for $\mu$ when $\sigma^2$ is Unknown

When $\sigma^2$ is not known, the computation of a confidence interval for the mean $\mu$ is not altered much.

- We simply replace the confidence coefficient from the $N(0,1)$ with one from the appropriate Student’s t-Distribution (the one with df = n-1)

- We replace the (now unknown) standard error with its estimate. The latter looks nearly identical except that it utilizes “s” in place of “$\sigma$”

- Recall

\[
\sigma = \sqrt{\frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}}
\]

- Thus,

| Confidence Interval for $\mu$ in two settings of a sample from a Normal Distribution |
|-----------------------------|-----------------------------|
| $\sigma^2$ is KNOWN         | $\sigma^2$ is NOT Known     |
| $\bar{X} \pm (z_{1-\alpha/2})(\sigma/\sqrt{n})$ | $\bar{X} \pm (t_{n-1;1-\alpha/2})(s/\sqrt{n})$ |
1. Obtain the point estimate of $\sigma^2$. It is the sample variance $S^2$

To get the sample variance $S^2$, we will need to compute the sample mean first.

\[
\overline{X} = \frac{\sum_{i=1}^{n} X_i}{n} = 352.5 \quad \text{and} \quad S^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2}{n-1} = 3.67
\]

2. Determine the correct chi square distribution to use.

It has df = (4-1) = 3.

3. Obtain the correct multipliers.

Because the desired confidence level is 0.95, we set $0.95 = (1-\alpha)$. Thus $\alpha = 0.05$

For a 95% confidence level, the percentiles we want are
(i) $(\sigma/2)_{100^{th}} = 2.5^\text{th}$ percentile
(ii) $(1 - \alpha/2)_{100^{th}} = 97.5^\text{th}$ percentile

From Rosner Table 6 on page 758, use the row for degrees of freedom=3

(i) $\chi^2_{df=3,.025} = 0.216$

(ii) $\chi^2_{df=3,.975} = 9.35$

4. Put it all together, obtain

(i) Lower limit = \[
\frac{(n-1)S^2}{\chi^2_{1-\alpha/2}} = \frac{(3)(3.67)}{9.35} = 1.178
\]

(ii) Upper limit = \[
\frac{(n-1)S^2}{\chi^2_{\alpha/2}} = \frac{(3)(3.67)}{0.216} = 50.97
\]
Solution for a 99% Confidence Interval for $\mu_d$

**Step 1** – Point Estimate of $\mu_d$ is the Sample Mean $\bar{d}_{n=30}$

$$\bar{d}_{n=30} = \frac{\sum_{i=1}^{n} d_i}{n=30} = 0.51$$

**Step 2** – The Estimated Standard Error of $\bar{d}_n$ is $S_{\bar{d}}/\sqrt{n}$

$$S\hat{E}(\bar{d}_{n=30}) = \sqrt{\text{variance}(\bar{d}_{n=30})} = \frac{S_d}{\sqrt{n}} = \frac{0.2416}{\sqrt{30}} = 0.0987$$

**Step 3** – The Confidence Coefficient

For a 99% confidence interval, this number will be the 99.5th percentile of the Student’s t-Distribution that has degrees of freedom $= (n-1) = 29$. This value is 2.756.

**Step 4** – Substitute into the formula for a confidence interval

Lower limit $= \text{point estimate} - \text{(conf coeff.) (SE of point estimate)}$

$$= 0.51 - (2.756)(0.0987)$$

$$= 0.2380$$

Upper limit $= \text{point estimate} + \text{(conf coeff.) (SE of point estimate)}$

$$= 0.51 + (2.756)(0.0987)$$

$$= 0.7820$$
Percentiles of selected F Distributions are provided in Table 9, beginning page 762.

- Each row defines a different denominator df.
- Each column defines a different numerator df.
- The body of the table gives values of selected percentiles of the F-distribution.
- Only the upper-tail percentiles (.90, ..., .999) are provided to you.

**Example** - What is the 95th percentile value of an F distribution random variable with numerator degrees of freedom equal to 24 and denominator degrees of freedom equal to 3?

- Locate on page 762 the column for numerator df = 24 and row for denominator df = 3.
- Within this “block”, find the 95th percentile value = 8.64

**Example** - What is the 5th percentile value of an F distribution random variable with numerator degrees of freedom equal to 8 and denominator degrees of freedom equal to 6?

- To obtain this percentile value requires using the “wonderful result” just described.
- Locate on page 762 the column for numerator df = 6 and row for denominator df = 8.
- Within this “block”, find the 95th percentile value. This is $F_{6,8,.95} = 3.58$
- Thus, $F_{8,6,.05} = 1/F_{6,8,.95} = (1/3.58) = 0.279$
3. The Confidence Coefficient is again a Percentile from the Normal(0,1)

\[ z_{0.975} = 1.96 \]

4. Putting it all together.

Lower \( = (\text{estimate}) - (\text{multiple}) \times (\text{SE}) = 0.38 - (1.96)(0.0932) = 0.197 \)
Upper \( = (\text{estimate}) + (\text{multiple}) \times (\text{SE}) = 0.38 + (1.96)(0.0932) = 0.56 \)

<table>
<thead>
<tr>
<th>Confidence Interval for a difference between two independent proportions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left[ \hat{\pi}_1 - \hat{\pi}<em>2 \right] \pm (z</em>{1-\alpha/2}) \times \text{SE} \left( \hat{\pi}_1 - \hat{\pi}_2 \right) )</td>
</tr>
</tbody>
</table>

where the required calculations are

(1) \( \bar{X} = \frac{X}{N_1} \) and \( \bar{Y} = \frac{Y}{N_2} \)
(2) \( \hat{\pi}_1 = \bar{X} \) and \( \hat{\pi}_2 = \bar{Y} \)
(3) \( \text{SE} = \sqrt{\frac{\bar{X}(1-\bar{X})}{N_1} + \frac{\bar{Y}(1-\bar{Y})}{N_2}} \)
(4) For small number of trials \( (N \leq 30 \text{ or so}) \) in either group, use \( \text{SE} = \sqrt{\frac{0.5(0.5)}{N_1} + \frac{0.5(0.5)}{N_2}} \)