Speed-Density Relationship: From Deterministic to Stochastic

Haizhong WANG (Corresponding Author)
Department of Civil and Environmental Engineering
University of Massachusetts Amherst
142D Marston Hall
130 Natural Resources Road
Amherst, MA 01003
Email: wang@engin.umass.edu

Jia LI
Department of Civil and Environmental Engineering
University of Massachusetts Amherst
142D Marston Hall
130 Natural Resources Road
Amherst, MA 01003
Email: jli0@engin.umass.edu

Qian-Yong CHEN, Ph. D
Department of Mathematics and Statistics
Lederle Graduate Research Tower
University of Massachusetts
Amherst, MA 01003-4515
Phone: +1 413 545-9611
Fax: +1 413 545-1801
Email: qchen@math.umass.edu

Daiheng NI, Ph. D
Department of Civil and Environmental Engineering
University of Massachusetts Amherst
219 Marston Hall
130 Natural Resources Road
Amherst, MA 01003
Phone: +1 413 545-5408
Fax: +1 413 545-9569
Email: ni@ecs.umass.edu

Number of words: 4500
Number of figures and tables: 10 (10 x 250 = 2500)
Total: 7000
Abstract

Traffic flow is a many-car system with complex and stochastic movement. It is difficult to describe the system dynamics solely using deterministic models which describe average system behaviors. Therefore, a stochastic speed-density relationship is proposed as a further step forward to overcome the well-known drawbacks of deterministic models. Modeling results show that by taking care of second-order statistics (i.e., mean and variance) a stochastic speed-density model is suitable for describing the observed phenomenon as well as matching the empirical data. From here, a stochastic fundamental diagram of traffic flow can be established. The stochastic speed-density relationship model can potentially be used for real-time on-line prediction and to explain phenomena in a similar manner.
1 Introduction

The speed-density relationship serves as the basis to understand system dynamics in various disciplines. It can be used to model moving objects (or particles) in many scientific areas: pedestrians [1] [2], conveyors, network information packages [3], crowd dynamics [4], molecular motors, and biological systems [5]. In this paper, we focus on the fundamental speed-density relationship of vehicular traffic flow. Our goal in this paper is to propose a stochastic speed-density relationship based on empirical ITS data (100 stations from the entire year of 2003) from GA400 in Atlanta, Georgia.

In nature, deterministic models describe average system behaviors based on physical laws. In a live transportation system, a totally deterministic model is unlikely to include various dynamical randomness effects (or uncertainties). The stochastic behavior of real-world traffic systems is often difficult to describe or predict exactly when the influence of unknown randomness is sizable. However, it is quite possible to capture the chance that a particular value will be observed during a certain time interval in a probabilistic sense. In particular, speed-density (or concentration) models in a deterministic sense, whether single or multi-regimes, have a ‘pairwise’ relationship; that is, given a density there exists a corresponding speed from a deterministic formula. Empirical observations from GA400 ITS verified the existence of another picture. There is a distribution of traffic speeds at a certain density level due to the stochastic nature of traffic flow, this is in contrast to the ‘pairwise’ pattern from deterministic models. There is debate on whether the scattering phenomenon observed in fundamental diagrams is due to either the measurement errors, the inherent nature of traffic flow, or a combination of the two. Essentially, there are two main sources of randomness [6] [7]. The first type of randomness is derived from the irregularity in traffic observations that come intrinsically from the data collection system and the computational processing that follows. This can be observed from the scattered plots of speed-density relationships at all of the stations from GA400, refer to Fig. 2. This type of randomness has been well-understood and can be controlled statistically. The second type of randomness is inherently generated by traffic dynamics, and it relates to a general lack of knowledge about what stochastic process is involved [6]. To be specific, drivers’ behaviors vary on an individual basis; the collective behaviors of driver groups would be better described in distributional law rather than in deterministic terms [7]. This type of randomness is assumed to underly the proposed stochastic speed-density relationship referred to Fig. 1. Thus, a stochastic speed-density model is more realistic and more capable to capture the traffic dynamics and the randomness involved in a many-car system. Before marching forward, it is important to mention the fundamental principles that we abide by; Herman and his colleagues mention that, “Traffic theory was inherently an experimental science and should be pursued as such. The second one was that the mathematical model should be chosen as the one most suitable for describing a particular phenomenon, rather than trying to fit a phenomenon to a model particularly familiar or attractive to a researcher” [8]. Our main contribution is the proposed stochastic speed-density relationship model, its validity has been verified by empirical observations and performance compared with existing deterministic models.

2 Literature Review

It has been almost 75 years since Greenshields’ seminal paper Study of Traffic Capacity in 1935 [9]. Attaching empirically derived curves to a fitted linear model of the speed-density relationship started a new era of transportation science and engineering. Due to its strong
empirical nature, the efforts to find a perfect theory to explain these particular shapes mathematically never cease, but they always achieve limited success. There is a fairly large amount of effort devoted to revising or improving such an over-simplified relationship. These efforts include single-regime models: Greenberg’s Model [10], the Underwood Model [11], Northwestern [12], Drew [14], and the Pipes-Munjal Generalized Model [13]. There are also multi-regime models which include: two-regime models such as Edie Model [20], multi-regime model by cluster analysis [16], two-regime model [17], modified Greenberg, and three-regime models [17] [12]. Tables 1 and 2 list most of the well-known speed-density [12] [17]. The tables are followed with a brief discussion of each model. These models are usually called regression models or fitting models by standard statistical techniques, they can be generalized as

\[ v_i(k_i) = f(k_i|\text{params}) + \epsilon_i \]  

\[ \text{(1)} \]
TABLE 2 Deterministic Multi-regime Speed-Density Models

<table>
<thead>
<tr>
<th>Multi-regime Model</th>
<th>Free-flow Regime</th>
<th>Congested Regime</th>
</tr>
</thead>
<tbody>
<tr>
<td>Edie</td>
<td>$V = 54.9e^{-\frac{k}{163.9}} (k \leq 50)$</td>
<td>$V = 26.8In(\frac{162.5}{k}) (k \geq 50)$</td>
</tr>
<tr>
<td>Two-regime Model</td>
<td>$V = 60.9 - 0.515k (k \leq 65)$</td>
<td>$V = 40 - 0.265k (k \geq 65)$</td>
</tr>
<tr>
<td>Modified Greenberg</td>
<td>$V = 48(k \leq 35)$</td>
<td>$V = 32In(\frac{145.5}{k}) (k \geq 35)$</td>
</tr>
</tbody>
</table>

They are characterized by a deterministic part, symbolized by the function $f$, plus a measurement error $\epsilon$. Statistical theory proves these techniques will make the fitted model represent the average of the data. A generalized deterministic form of speed-density relationship can be written as

$$v_k = v(k) + \epsilon_k$$

Here, $v(k)$ is the deterministic speed-density relationship in which given density $k$ there is one fixed corresponding speed value. Deterministic models are justified on the basis that they describe the average value of the dependent variable given the independent variable inputs [15].

The advantages and drawbacks of deterministic single- and multi-regime speed-density models have been well-documented in a multitude of publications [17] [18] [19]. The advantages of deterministic models are their mathematical simplicity and analytical tractability. Deterministic speed-density models are mostly parameter based (usually requiring a combination of two parameters from free flow speed, $v_f$; jam density, $k_j$; optimal speed, $v_m$; or optimal density, $k_o$) without consideration of data correlation. Usually, $v_f$ is relatively easy to estimate from empirical data and mostly lies between speed limit and highway design speed, but $k_j$ is not easy to observe; however, an approximate value of 185-250 veh/mile is a reasonable range [17]. Since optimum density is difficult to observe and varies with highway geometry and environment, a rough estimate of optimum speed is to halve the highway design speed [17].

The fact that deterministic speed-density models match empirical observations poorly has been well-recognized in the transportation community. Consider the seminal linear speed-density by Greenshields: there are only seven data points collected from one lane in a two-way rural road in which six of the data points are below 60 mi/hr and the seventh data point was taken from a different road [9] [18]. Seven data points are not enough to generate a whole picture of a speed-density model. Later, the Greenberg model gained popularity because it bridged the gap analytically from a macroscopic stream model to a microscopic car-following model [26] [27]. However, the main criticism of this model is its inability to predict speed at lower densities, because as density approaches zero, speed tends to increase to infinity.

Later, Underwood derived an exponential model that attempted to overcome the limitation of the Greenberg model [11]. In the Underwood model, the main drawback is that speed becomes zero only when density reaches infinity. Hence, this model cannot be used for predicting speeds at high densities. The Pipes-Munjal model [13] resembles Greenshields’ model. By varying the values of $n$, a family of models can be developed.

Recent modeling efforts (usually called modified speed-density relationship) and empirical investigations are mostly based on the models in Table 1. For a generic characterization of the fundamental speed-flow relationship based on the classical deterministic speed-density models, interested readers are referred to [21]. A modified speed-density relationship is generalized in the form of

$$v = v_f(1 - \left(\frac{k}{k_j}\right)^a)^b$$

(3)
and can be seen in [22] [23]. Later, [24] [25] reviewed the historic footprints of speed-density relationships, and they completed a detailed analysis of the speed-density curve’s mathematical properties using empirical evidence. [18] investigated the properties of the fundamental relationship and briefly reviewed the modeling efforts up to the early 1990’s. In his paper, whether the scattering phenomenon is due to measurement error or the nature of traffic flow and the underlying principles under different possible shapes of fundamental plots is discussed using logical considerations but not empirically verified by field observations. For the calibration and application of dynamic speed-density relationships, interested readers are referred to [29] [28]. Despite their varying degrees of success in approximating empirical observations, the aforementioned models are driven by deterministic nature.

Modeling traffic flow in a stochastic sense is not fresh at all [40] as can be seen in a variety of papers: stochastic models for intersection control [39], stochastic traffic cellular automaton model [32], stochastic noise approach [36] [35], nonlinear stochastic dynamic model [38], stochastic real-time traffic simulation [41], and stochastic force model of traffic flow [37]. Recently, [7] proposed a stochastic free flow speed function in solving LWR type conservation equation, and [42] came up with the concept of a stochastic capacity comparing conventional capacity from speed-flow diagram.

3 Development of Stochastic Speed-Density Model and Simulation

Models are generally used to describe how theory fit empirical observations in a meaningful way. In certain situations, there are mathematical arguments showing that a deterministic model represents the mean of a similar stochastic model [15]. However, the mean alone may not be capable to describe the behavior of traffic dynamics. Using a stochastic speed-density model to replace a deterministic speed-density model is justified by this argument and investigations of empirical observations. The stochastic speed-density relationship model is largely motivated by the prevalent randomness exhibited in empirical observations that mainly comes from drivers, vehicles, roads, and environmental conditions. Randomness, here, is defined as lacking consistent pattern and regularity. Traffic observations show a scattering effect which is readily seen if speed or flow is plotted against density as shown in Fig. 2. It is such an intrinsic property of transportation system observations that deterministic models should be considered as incomplete. The underlying mechanism behind the observed scattering phenomenon is frequently the effect of a large number of factors (i.e., driver behaviors, highway geometries, and vehicle characteristics) that influence the plot but are not modeled explicitly.

3.1 Derivation of Stochastic Speed-Density Model

A conceptual form of the stochastic speed-density relationship can written as

\[ V = f(k, \omega) : (\mathbb{R}^+, \mathbb{R}, \mathbb{R}^+, \Omega) \mapsto \mathbb{R}^+ \]  

in which \( f \) is a real-valued continuous function defined on real space \( \mathbb{R} \). The basic assumption here is that \( \forall k, x, \) and \( v \) are a function of a random process consisting of the density, \( k \), and a random variable, \( \omega \):

\[ V = v(k, \omega) \]
Including randomness, a more detailed form of the stochastic $k$-$v$ relationship can be read from

$$V(k) = v(k, \omega(x,t)) : (\mathbb{R}^+, \mathbb{R}, \mathbb{R}^+, \Omega) \mapsto \mathbb{R}^+$$

(6)

Here, $\omega$ is an appropriately defined set on $\Omega$, representing the involved randomness for each fixed density $k$. $\Omega$ is the probability space equipped with measure $P_{x,t}(\cdot)$. Random parameters in a stochastic process are usually modeled by second-order statistics mostly defined as mean and covariance function. A series expansion (i.e., the Karhunen-Loève expansion) provides a second-moment characterization of random processes by means of uncorrelated random variables and deterministic orthogonal functions. It is known to converge in the mean square sense for any distribution of $v(k, \omega)$. Considering the spectral properties of covariance function, the Karhunen-Loève expansion can be used to discretize a random process by representing the process through a denumerable set of orthogonal random variables [6], usually in form of a Fourier-type series as

$$V(k, \omega) = \sum_{i=0}^{+\infty} \sqrt{\lambda_i} f_i(k) \xi_i(\omega)$$

(7)

in which $\xi_i(\omega)$ is a set of random variables to be determined, $\lambda_i$ is a constant eigenvalue, and $f_i(k)$ is an orthonormal set of deterministic functions. The distinguishing feature of this stochastic speed-density model is that at a certain density, there is a distribution of traffic speeds instead of a single speed from deterministic models. Let $A$ be a set defined on $\mathbb{R}^+$, so $\forall i, l \in A, i \neq l: k_i$ and $k_l$ are correlated ($k_i$ and $k_l$ are two distinct densities indexed by $i, l \in A$), but somehow the correlation information is lost in the original data. Therefore, we assume there exists an exponential correlation between different densities in the raw data. One physical explanation is that driver population’s collective behavior is not totally independent. A correlation function, here, is defined as the correlation between random variables at two different points in space or time. It contains information about the distribution of points or events. Since we assume $V(k)$ is a random process, for random variables $V(k_i)$ and $V(k_l)$, the correlation function is given by

$$R(k_i, k_l) = \text{corr}(V(k_i), V(k_l))$$

(8)
An exponential correlation function is assumed in this paper which is popularly used in varying disciplines in the form of

$$R(k_i, k_l) = \langle v(k_i, \omega)v(k_l, \omega) \rangle = \sigma^2 \exp\left(-\frac{|k_i - k_l|}{\alpha}\right)$$  \hspace{1cm} (9)

The constants $\sigma^2$ and $\alpha$ are, respectively, the variance and the correlation length of the process. The correlation length is a measurement of range over which fluctuations in one region of space are correlated with those in another region. $\sigma$ itself is the standard deviation. This process is known as the Ornstein-Uhlenbeck process. For more information about exponential correlation function, interested readers are referred to [34].

Let $V(k, \omega)$ be a random function of $k$ defined over domain $D$, with $\omega$ defined on the random events space $\Omega$. Let $\bar{V}(k)$ represent the expectation of $V(k, \omega)$ over all possible random process realizations. $C(k_i, k_l)$ is defined as its covariance function with bounded, symmetric, and positive definite properties [6]. It is related to the correlation function by

$$C(k_i, k_l) = R(k_i, k_l)\sigma(k_i)\sigma(k_l)$$  \hspace{1cm} (10)

Following Mercer’s Theorem, by spectral decomposition of covariance function

$$C(k_i, k_l) = \sum_{n=0}^{+\infty} \lambda_n f_n(k_i)f_n(k_l)$$  \hspace{1cm} (11)

where $\lambda_n$ and $f_n(k)$ are, respectively, the eigenvalues and the eigenvectors of the covariance kernel. They can be obtained by solving the homogeneous Fredholm integral equation given by

$$\int_D C(k_i, k_l)f_n(k)d k_l = \lambda_n f_n(k_i)$$  \hspace{1cm} (12)

Equation (12) arises due to the fact that the eigenfunction $f_i(k)$ forms a complete orthogonal set satisfying

$$\delta_{il} = \int_D f_i(k)f_l(k)d k$$  \hspace{1cm} (13)

in which $\delta_{il}$ is the Kronecker-delta function. Through some algebra, $V(k, \omega)$ can be written as

$$V(k, \omega) = \bar{V}(k) + \beta(k, \omega)$$  \hspace{1cm} (14)

where $\beta(k, \omega)$ is a process with a mean of 0 and a covariance function $C(k_i, k_l)$. In terms of the eigenfunction, $f_n(k)$, the process $\beta(k, \omega)$ can be expanded as

$$\beta(k, \omega) = \sum_{i=0}^{+\infty} \sqrt{\lambda_i} f_i(k)\xi_i(\omega)$$  \hspace{1cm} (15)

Thus, the decomposed form of the random process can be written as

$$V(k, \omega) = \bar{V}(k) + \sigma(k)\sum_{i=0}^{+\infty} f_i(k)\sqrt{\lambda_i}\xi_i(\omega)$$  \hspace{1cm} (16)

in which, $\langle \xi_i(\omega) \rangle = 0$, $\langle \xi_i(\omega)\xi_j(\omega) \rangle = \delta_{ij}$, and $\delta_{ij} = \int_D f_i(k)f_j(k)d k$. For practical implementation, the series is approximated to a finite number by truncating equation (16) at a $N^{th}$ term, a more specific form of stochastic speed-density relationship can be processed as

$$V(k, \omega) = \bar{V}(k) + \sigma(k)\sum_{i=0}^{N} f_i(k)\sqrt{\lambda_i}\xi_i(\omega)$$  \hspace{1cm} (17)
in which $\bar{V}(k)$ is the expected speed value of a deterministic speed-density relationship model, and it is open to classic or modified speed-density relationship models. The value of $N$ is governed by the accuracy of the eigen-pairs (eigenvalue and eigenfunction) in representing the covariance function rather than the number of random variables. $\xi_i(\omega)$ are pairwise uncorrelated random variables. An explicit form of $\xi_i(\omega)$ can be obtained from

$$\xi_i(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_D (V(k, \omega) - \bar{V}(k)) f_i(k) dk$$  \hfill (18)

with mean and covariance function given by

$$E[\xi_i(\omega)] = 0$$  \hfill (19)

$$E[\xi_i(\omega)\xi_l(\omega)] = \delta_{il}$$  \hfill (20)

A Gaussian or Beta distribution is assumed for the random variable $\xi_i(\omega)$ through investigation of empirical data. The Beta distribution matches the empirical observations well, but it could not guarantee the convergence of the Karhunen-Loève expansion. Due to the convergence concern, a Gaussian distribution is used as it surely guarantees the convergence of Karhunen-Loève expansion. However, the limitation of Gaussian distribution is that it generates negative speeds during simulation.

### 3.2 Algorithm

In sum, the algorithm to simulate the proposed stochastic speed-density model is devised and coded as followed,

1. Read empirical speed-density data $k$ and $v$, the data is sorted by $k$ in a non-decreasing order $k = 1, 2, \ldots, k_{max}$ (Note: $k_{max}$ is the maximum observed density at the specific station).

2. Compute the empirical mean speed $\bar{V}(k)$ and $\sigma(k)$, the computed mean and variance curve is smoothed by smoothing techniques. (Note: It seems self-evident if the simulation totally depends on the empirical data and then compares the simulated speed-density model with the empirical observations. To improve model transferability and predictability, a parameter based deterministic speed-density model is needed to approximate the mean. The approximation function of variance comes from the empirical investigation of 78 stations’ variance assuming variance is a function of density. Now, for a new station (location), the stochastic speed-density model will not require the empirical data, but rather it will need a set of location-based parameters varying with road geometry).

3. Determine a target covariance function $C(k_i, k_l)$, compute the correlation by the assumed exponential correlation function (9) and decompose the covariance function into eigenvalues and eigenfunctions using Equation (11).

4. Generate $N$ sample functions of the non-Gaussian process using the truncated K-L expansion: $V_N^{(n)}(k, \omega_i) = \bar{V}(k) + \sigma(k) \sum_{i=0}^{N} f_i(k) \sqrt{\lambda_i} \xi_i^{(n)}(\omega)$, $i = 1, 2, \ldots, N$, here $n$ is iteration number and $i$ is sample number. $\bar{V}(k)$ and $\sigma(k)$ is known from step 2, $\lambda_i$ and $f_i(k)$ is given by solving equation (12). The resulting eigenfunctions are

$$f_i(k) = \frac{\cos(\theta_i k)}{\sqrt{a + \frac{\sin(2\theta_i k)}{2\theta_i}}} \quad \text{and} \quad f^*_i(k) = \frac{\sin(\theta_i k)}{\sqrt{a - \frac{\sin(2\theta_i k)}{2\theta_i}}}$$

for even and odd $i$ respectively. The corresponding eigenvalues are $\lambda_i = \frac{2c}{\theta_i^2 + c^2}$ and $\lambda^*_i = \frac{2c}{\theta_i^2 + c^2}$. 

5. Pick $k_1$, generate $N$ samples, also generate $N$ samples of identically independent distributed Gaussian random variables $\{\xi_i(\omega)\}_{i=1}^N$. Go to step 4.

6. Pick another $k_1$, repeat step 5 until $k_{\text{max}}$, stop.

4 Study Site, Data Extraction and Aggregation

The raw GA400 ITS data at each station provides averaged values at a 20 second aggregation level, when used to generate the fundamental diagrams, the original data were aggregated every 5 minutes. The measured mean values of parameters over a certain long time interval is usually used to describe fundamental diagrams [30]. At each station, we have one-year of continuous observations in 2003. This time interval is long enough for describing equilibrium fundamental diagrams. In the computational process, since only time mean speed is available in the raw GA400 ITS data, time mean speed is used to calculate the density instead of space mean speed, which is known as the correct variable in the fundamental relationship ($q = kv_s$). A general setting of 100 ITS detectors from the study site (GA400) in Atlanta can be obtained from Fig. 3.

![Study Site: GA400 southbound and northbound with 100 Stations](image)

**FIGURE 3** Study Site: GA400 southbound and northbound with 100 Stations
5 Results of Stochastic Speed-Density Modeling

The developed stochastic speed-density model and devised algorithm was coded in Matlab. To verify its validity, we compared the performance of the proposed stochastic model with some well-known speed-density models listed in Table 1 against empirical observations from GA400, which were briefly discussed in Section 4. We tested the model using 78 stations’ data from the northbound and southbound basic freeway segments of GA400. Since there is not enough room, and no need, to include all the processed figures here, we choose two stations from each direction of GA400 to show how the performance of the stochastic speed-density model compares to other deterministic models. We also show how the simulated stochastic speed-density data compares to empirical speed-density data. The results compared to empirical observations show that:

1. The proposed stochastic speed-density model agrees fairly well with the empirical data at observed stations. It works consistently well either in the free-flow region or the congested phase.

2. The stochastic speed-density model is transferable from basic freeway segments to on-ramps and off-ramps.

3. For single-regime deterministic models, Greenshields’ and the Greenberg model tend to overestimate empirical observations. The Drew model tends underestimate speed at free-flow phase, but it overestimates the speed in the congested phase. Northwestern works in the reverse manner; it tends to overestimate the free-flow region a little bit while underestimate the congested region. This further verified the known drawbacks of deterministic models over certain density ranges.

4. The Underwood model works better than other deterministic models from our empirical results. The reason responsible for this is that for observations over an enough long time period (i.e., half a year or more), the empirical speed-density relationship tends to be more exponential than linear or logarithmic. This explains the observed performance differences among the single-regime family of models compared.

There is no personal preference for any observed station used in this paper; the selected stations include on-ramps, off-ramps, and basic freeway segments with varying number of lanes and traffic conditions. The fact that deterministic models have deficiencies over a certain portion of density ranges is well-known [17]. This can be verified from the empirical results here in Fig. 4 and 6. A. D. May pointed out in his book [17] that a disconcerting feature of deterministic models is their inability to track the empirical data in the vicinity of capacity condition. From our results Fig. 4 and 6, we find that the stochastic speed-density model tracks the empirical data faithfully and works consistently well over the whole range of densities. Though in progress, the proposed stochastic speed-density model strives to overcome some of the well-known drawbacks of deterministic models.

Fig. 5 shows that the proposed stochastic speed-density model is also capable of capturing the dynamics at on-ramps and off-ramps. 4005005 is a single lane on-ramp to GA400 northbound while 4005006 is a single lane off-ramp from GA400 northbound. From the empirical results of 22 on-ramps and off-ramps, the fundamental relationship has a similar shape but with distinguishable features from that of the basic freeway segments. One observation is that the speed-density curve on basic freeway segments is location based rather than freeway based. The speed-density curves at on-ramps and off-ramps are ramp geometry and characteristic based including, but not limited to, the ramp speed limit, number of lanes, ramp elevation,
FIGURE 4 Performance Comparison of Different Speed-Density Models at 4001118 (four lanes) and 4001119 (four lanes)
and slope.

**FIGURE 5** Comparison of Different Speed-Density Models at 4005005 (on-ramp) and 4006008 (off-ramp)

Fig. 7 compares the simulated speed-density model with empirical speed-density relationships at two stations (4001118 and 4001119) from GA400 southbound towards Atlanta, Georgia. Simultaneously, the mean and variance of the simulated speed-density models and the empirical speed-density data are plotted. The results show a fairly good match between the simulation of the stochastic speed-density models and the empirical speed-density observations. By doing this, we want to demonstrate the performance of the proposed stochastic speed-density model. Most importantly, we want to demonstrate the model’s capability to track empirical data, and its robustness to work consistently at varying traffic conditions. The point we want to conclude is that the proposed stochastic speed-density model potentially performs better than deterministic models by taking care of second-order statistics. Fig. 8 took two stations from GA400 northbound, out of Atlanta. Similarly, the results further verified the arguments and assumptions underlying the proposed stochastic speed-density model.
FIGURE 6 Comparison of Different Speed-Density Models and Corresponding Flow-Density Models at 4000058 (two lanes) and 4000059 (three lanes)
FIGURE 7 Stochastic Simulation of Speed-Density Relationship and its Corresponding Simulated and Empirical Mean, Standard Deviation at 4001118 (four lanes) and 4001119 (four lanes)
FIGURE 8 Stochastic Simulation of Speed-Density Relationship and Corresponding Simulated and Empirical Mean, Standard Deviation plotted for Comparison at 4000058 (two lanes) and 4000059 (three lanes)
6 Conclusion and Future Work

Though deterministic speed-density relationship models can explain physical phenomena underlying fundamental diagrams, the stochastic model is more accurate and more suitable to describe traffic phenomenon. From the results of stochastic model, we find out that a stochastic speed-density model matches the empirical observations better than deterministic ones. Following from this result, the \textit{LWR} type conservation equation can be revisited in a stochastic setting of speed-density relationship. The \textit{LWR} model in a stochastic setting is potentially capable to capture some interesting features (i.e., spontaneous congestion) where deterministic models fail. Another benefit from a stochastic speed-density model is its capability to perform real-time on-line prediction while deterministic models are claimed to be insufficient. Future work could be done to improve the model transferability and predictability. The accuracy and optimality of the \textit{Karhunen-Loève Expansion} algorithm could be improved and tested with other correlation functions to further fine-tune the proposed stochastic speed-density model as compared to empirical observations.

7 Acknowledgement

The authors thank Steven Andrews for his help with improving the quality and readability of this paper.
References


