Investigation of Hill’s optical turbulence model by means of direct numerical simulation

ANDREAS MUSCHINSKI1,* AND STEPHEN M. DE BRUYN KOPS2

1NorthWest Research Associates, 3380 Mitchell Lane, Boulder, Colorado 80301, USA
2Department of Mechanical and Industrial Engineering, University of Massachusetts Amherst, 160 Governors Drive, Amherst, Massachusetts 01003-2210, USA
*Corresponding author: andreas@nwra.com

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For almost four decades, Hill’s “Model 4” [J. Fluid Mech. 88, 541 (1978)] has played a central role in research and technology of optical turbulence. Based on Batchelor’s generalized Obukhov–Corrsin theory of scalar turbulence, Hill’s model predicts the dimensionless function $h(\kappa l_0, \Pr)$ that appears in Tatarskii’s well-known equation for the 3D refractive-index spectrum in the case of homogeneous and isotropic turbulence, $\Phi_s(\kappa) = 0.033C_n^2\kappa^{-11/3}h(\kappa l_0, \Pr)$. Here we investigate Hill’s model by comparing numerical solutions of Hill’s differential equation with scalar spectra estimated from direct numerical simulation (DNS) output data. Our DNS solves the Navier–Stokes equation for the 3D velocity field and the transport equation for the scalar field on a numerical grid containing $4096^3$ grid points. Two independent DNS runs are analyzed: one with the Prandtl number $Pr = 0.7$ and a second run with $Pr = 1.0$. We find very good agreement between $h(\kappa l_0, \Pr)$ estimated from the DNS output data and $h(\kappa l_0, \Pr)$ predicted by the Hill model. We find that the height of the Hill bump is $1.79 \Pr^{1/3}$, implying that there is no bump if $Pr < 0.17$. Both the DNS and the Hill model predict that the viscous-diffusive “tail” of $h(\kappa l_0, \Pr)$ is exponential, not Gaussian. © 2015 Optical Society of America

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1. INTRODUCTION

Optical propagation through the atmosphere is affected by turbulent fluctuations of the air temperature (“optical turbulence”), and the performance of optical systems that rely on light propagating through the atmosphere may be severely limited by optical turbulence. Such systems include free-space optical communication systems, optical imaging and surveillance systems, and optical directed-energy systems. Moreover, some optical remote sensing systems, such as optical scintillometers, work only if optical turbulence is present. Therefore, a quantitative understanding of optical turbulence is important for a quantitative understanding of the performance of a wide range of optical systems, regardless of whether the presence of turbulence is required or detrimental.

The modern physics of electromagnetic and acoustic wave propagation through turbulent media was pioneered by Tatarskii [1,2] in the 1950s and 1960s. The statistics of phase fluctuations, angle-of-arrival fluctuations, and irradiance fluctuations (“scintillation”) can be analytically or computationally predicted based on observations, computer simulations, or theoretical models of the 3D temperature spectrum $\Phi(\kappa)$ along the propagation path [1–11].

The classical conceptual framework for the physical understanding of $\Phi(\kappa)$ in a fully turbulent, statistically isotropic, scalar field is the Obukhov–Corrsin similarity theory, which was pioneered by Obukhov [12] in 1949 and independently studied by Corrsin [13] in 1951 and later generalized by Batchelor and coworkers [14,15] to passive scalars other than temperature and to Prandtl–Schmidt numbers that differ significantly from 1. The traditional Obukhov–Corrsin theory predicts that $\Phi(\kappa)$ follows a $\kappa^{-11/3}$ law in the inertial-convective subrange and decreases more rapidly with $\kappa$ at wavenumbers where molecular viscosity and/or molecular diffusion become important. Tatarskii modeled the viscous-diffusive subrange as a sharp cut-off [1, p. 48] and later as a Gaussian function [2, p. 67].

However, observations of atmospheric and oceanic turbulence became available which showed that the “compensated spectrum” $\Phi(\kappa)\kappa^{11/3}$ does not decrease monotonically with increasing $\kappa$ but has a bump in the transition region between the inertial-convective range and the viscous-diffusive range. Tatarskii’s models predict no such bump. Recognizing the importance of this bump for a wide range of propagation problems, Hill [16] developed refined $\Phi(\kappa)$ models that are consistent with the Obukhov–Corrsin theory and predict the
existence of a bump. Out of Hill’s four models, “Model 4” [16, p. 551] most accurately reproduced the bump that Champagne et al. [17] and Williams and Paulson [18] observed in the compensated temperature spectra obtained from fine-wire measurements of turbulence in the atmospheric surface layer over land.

In the optical propagation community, Hill’s “Model 4” has become the standard model for \( \Phi(\kappa) \), and the bump has become known as the “Hill bump.” Hill’s model, however, consists of a differential equation that can be solved only numerically, and it appears that no tabulated solution has been published in the open literature. Recently, one of us has shown [19] that the closed-form approximations of Hill’s model suggested by Churnside [20], Andrews [21], and Grayshan et al. [22] violate the temperature transport equation, a finding that motivated us to take a fresh look at Hill’s model.

The purpose of this paper is to investigate Hill’s model by comparing scalar spectra predicted by Hill’s model with scalar spectra estimated from direct numerical simulation (DNS) output data. DNS, which was pioneered more than forty years ago by Orszag and Patterson [23], has matured into a standard technique of computational fluid dynamics (CFD) [24]. DNS stands out among other CFD techniques (such as Reynolds-averaged modeling or large-eddy simulation) in that DNS relies solely on first principles of fluid mechanics, namely, the Navier–Stokes equations and the scalar transport equation. In other words, DNS does not rely on any \textit{a priori} assumptions or models about the statistical nature of turbulence. In that regard, a state-of-the-art DNS is the computational counterpart of a well-designed laboratory or field experiment. DNS is particularly well suited for the investigation of turbulence at the transition range between the inertial-convective subrange and the viscous-diffusive subrange.

The paper is organized as follows. Section 2 summarizes the relevant theoretical background including the Obukhov–Corrsin theory, Tatarskii’s models, and Hill’s model. A brief description of the DNS technique is given in Section 3. In Section 4, we present numerical solutions of Hill’s model and spectra estimated from the DNS data, and we compare both results with each other. The results are discussed in Section 5. Section 6 provides a summary and conclusions.

2. THEORETICAL BACKGROUND

A. Obukhov–Corrsin Similarity Theory

The 3D temperature spectrum, \( \Phi(\kappa) \), plays a central role in the physics of optical wave propagation through the turbulent atmosphere [1–11]. The Obukhov–Corrsin similarity theory [12–15] of the spatial statistics of temperature fluctuations in fully developed, statistically homogeneous, and isotropic turbulence predicts

\[
\Phi(\kappa) = \beta_3 \nu e^{-1/3} \kappa^{-11/3} g(\kappa \eta, \Pr),
\]

where the wavenumber \( \kappa \) is the magnitude of the wave vector \( \mathbf{k} \); \( \chi \) is the dissipation rate of temperature variance; \( \epsilon \) is the dissipation rate of turbulent kinetic energy per unit mass; \( g(\kappa \eta) \) is a universal, dimensionless function, which goes to 1 for \( \kappa \eta \ll 1 \) and to 0 for \( \kappa \eta \gg 1 \);

\[
\eta = \left( \frac{\nu^3}{\epsilon} \right)^{1/4}
\]

is the Kolmogorov length with \( \nu \) as the kinematic viscosity of the medium; the Prandtl number,

\[
\Pr = \frac{\nu}{\alpha},
\]

is the ratio of \( \nu \) and the thermal diffusivity \( \alpha \); and \( \beta_3 \) is a universal, dimensionless coefficient.

Integrating the spherically symmetric \( \Phi(\kappa) \) over all directions gives the so-called shell-averaged spectrum,

\[
E(\kappa) = 4\pi \Phi(\kappa) \kappa^2.
\]

Therefore,

\[
E(\kappa) = \beta_3 \nu e^{-1/3} \kappa^{-5/3} g(\kappa \eta, \Pr),
\]

where the dimensionless, universal coefficient

\[
\beta = 4\pi \beta_3
\]

is known as the Obukhov–Corrsin constant, the value of which is empirically known to be approximately 0.7 [16,25,26].

Tatarskii [1,2] used the alternative formulation

\[
\Phi(\kappa) = \frac{\Gamma(8/3) \sin(\pi/3)}{4\pi^2} C_2^2 \kappa^{-11/3} h(\kappa l_0),
\]

where \( \Gamma(8/3) \sin(\pi/3)/4\pi^2 = 0.033005 \),

\[
h(\kappa l_0) = g(\kappa \eta),
\]

and \( l_0 \) is the inner temperature scale, which is related to \( \eta \) through

\[
\frac{l_0}{\eta} = \left( \frac{27\Gamma(1/3) \beta_3}{5 \Pr} \right)^{3/4}.
\]

The temperature variance dissipation rate can be obtained from

\[
\chi = 2\alpha \int_0^\infty E(\kappa) \kappa^2 d\kappa.
\]

Inserting \( E(\kappa) \) in the form

\[
E(\kappa) = \beta_3 \nu e^{-1/3} \kappa^{-5/3} h(\kappa l_0, \Pr),
\]

gives, regardless of the value of \( \Pr \),

\[
\int_0^\infty h(y, \Pr) y^{1/3} dy = \frac{27\Gamma(1/3)}{10} = 7.223
\]

[19,27], where \( y = k l_0 \).

B. Tatarskii’s Models

The earliest models for \( h(y) \) were Tatarskii’s cutoff model

\[
h(y) = \begin{cases} 1 & y \leq y_c \\ 0 & y > y_c \end{cases}
\]

[1, Eq. (3.25) on p. 48] and Tatarskii’s Gaussian model

\[
h(y) = \exp[-(y/y_m)^2]
\]

[2, Eq. (47) on p. 67]. The temperature variance dissipation constraint in Eq. (12) gives \( y_c = [18\Gamma(1/3)/5]^{1/4} = 5.473 \) and \( y_m = [27\Gamma(1/3)/5\Gamma(2/3)]^{1/4} = 5.909 \) [19]. These results are close to Tatarskii’s values 5.48 and 5.92, respectively.
C. Hill’s Model

Careful observations of turbulence in the atmospheric surface layer [17,18] showed that \( b(\kappa l_0) \) does not decrease monotonically but has a pronounced maximum at \( \kappa l_0 \approx 1 \). Tatarskiǐ’s models did not account for that “bump.”

Hill [16] developed four theoretical models for \( \Phi(\kappa) \), one of which, “Model 4,” reproduces the observations most accurately. Model 4, which we refer to here simply as “Hill’s model,” consists of a homogeneous, ordinary, second-order differential equation for \( E(x\kappa^1) \) [\( \Gamma(\kappa) \) in Hill’s notation]:

\[
\frac{d}{dx} \left\{ x^{1/3}(x^{2b} + 1)^{-1/3b} \frac{d}{dx} [x^{-2} E(x\kappa^1)] \right\} = \frac{22}{3} \frac{\beta}{\Pr} \left( \kappa^3 \right)^{4/3} x^2 E(x\kappa^1). \tag{15}
\]

Here,

\[
E(x) = \beta x^{-1/3} \kappa^{-5/3} f(x), \tag{16}
\]

where

\[
f(x) = h(y), \tag{17}
\]

and where the dimensionless wavenumber

\[
x = \frac{\kappa}{\kappa}, \tag{18}
\]

is scaled by the wavenumber \( \kappa^* \), which Hill hypothesized to be a universal multiple of the Kolmogorov wavenumber \( \eta \). In other words, Hill hypothesized that

\[
a = \kappa^* \eta \tag{19}
\]

is a universal, dimensionless constant.

Rewriting Eq. (15) in terms of \( f(x) \) gives

\[
\frac{d}{dx} \left\{ (x^{2b} + 1)^{-1/3b} \left[ -\frac{11}{3} f(x) + x \frac{df}{dx} \right] \right\} = \frac{22}{3} c x^{1/3} f(x), \tag{20}
\]

where

\[
c = \frac{\beta}{\Pr} x^{4/3}. \tag{21}
\]

As in Eq. (15), Eq. (20) is a homogeneous, ordinary, second-order differential equation. The equation is homogeneous in the sense that if \( f(x) \) is a solution, then \( rf(x) \) for any real number \( r \) is also a solution. Equation (20) has two free parameters, \( b \) and \( c \). That is, the solution \( f(x) \) depends only on \( b \) and \( c \) but not on the parameters \( \beta \), \( \Pr \), and \( a \) individually.

From Eqs. (9) and (17) and

\[
x = \frac{1}{a} l_0 y, \tag{22}
\]

we obtain

\[
h(y) = f \left[ \frac{5}{a} \left( \frac{2^{2/3}}{\Gamma(1/3)} \right) \frac{\Pr}{\beta} y^{3/4} \right]. \tag{23}
\]

So we can use Eq. (23) to calculate \( h(y) \) from numerical solutions of \( f(x) \). Alternatively, we could have rewritten Hill’s differential equation, Eq. (15), in terms of \( h(y) \) instead of \( f(x) \), and then solve the differential equation for \( h(y) \). In the following, however, we solve Eq. (20) numerically to find \( f(x) \) and then obtain \( h(y) \) by means of Eq. (23).

3. DIRECT NUMERICAL SIMULATION

As stated in Section 1, DNS is a standard technique in CFD [24]. In contrast with many other CFD techniques, such as Reynolds-averaged modeling or large-eddy simulation, DNS does not rely on any \textit{a priori} assumptions about the statistical nature of turbulent fluctuations. Instead, DNS solves “directly” the fundamental equations of fluid mechanics, namely, the Navier–Stokes equations for the velocity field and the transport equation for one or more scalar fields.

Two important features of DNS suitable for research studies are that the numerical schema be highly accurate and that the spatial and temporal resolution be sufficient. For the DNS reported here, a pseudo-spectral method is used because the phase errors are very small, rates of convergence are very high, and the truncation error decreases exponentially as the number of Fourier modes becomes large [28]. The simulations are resolved so that \( \kappa_{\text{max}} \eta = 1.8 \). Here, \( \kappa_{\text{max}} \) is the maximum wave number in the simulation after removal of those potentially contaminated by aliasing errors.

The simulations are very similar to case R4 in [29], which is derived from the simulations in [30], and the reader is referred to those papers for details on the numerical method and statistics regarding the velocity fields. Briefly, the simulations are of forced, isotropic homogeneous turbulence with a mean passive-scalar gradient. While the scalar field is not periodic, the scalar fluctuations about the mean gradient are, and so periodic boundary conditions are used in all directions. This allows the equations of motion to be discretized with finite Fourier series using 2048\(^3\) Fourier modes.

Energy is added to the velocity field at each time step [31,32] in wave numbers less than \( \kappa \eta = 0.06 \) so that the shell-averaged spectrum is almost stationary and equal to Pope’s model spectrum with his \( \rho_0 = 2 \) and his \( c_t = 6.78 \) [24, Eq. (6.247)]. The fields are dealiased through two methods. First, the discretized equations of motion are written so that all aliasing errors cancel, i.e., the momentum equation is written in rotational form and the scalar transport equation in conservation form and convective form on alternating time steps to approximate the skew-symmetric form. Second, a spherical truncation filter is applied in Fourier space to remove all energy from wave numbers greater than about 15/16 of the maximum wave number, which has been shown to remove aliasing errors from all wave numbers that are not already affected by truncation error [33].

4. RESULTS

A. Numerical Solution of Hill’s Model

Figures 1–4 show numerical solutions of Eq. (20), represented in terms of the compensated spectra \( g(\kappa \eta) \) and \( b(\kappa l_0) \). We use Hill’s parameter values (\( a = 0.072 \), \( b = 0.19 \), \( \beta = 0.72 \), and \( \Pr = 0.72 \)) as the “baseline” parameter set and show how \( g(\kappa \eta) \) and \( b(\kappa l_0) \) vary as these parameters are changed. As discussed in Section 2, \( g(\kappa \eta) \) and \( b(\kappa l_0) \) are uniquely determined by only two parameters, namely, \( b \) and \( c = \beta a^{4/3} / \Pr \). We will see that \( c \) is the key parameter, while \( b \) plays only a minor role. That is, Hill’s model is essentially a one-parameter model, which greatly simplifies its validation by means of empirical data.
Figure 1 shows $g(k\eta)$ for Pr ranging from 0.5 to 1.0 in increments of 0.1 (black lines), with $a = 0.072$, $b = 0.19$, and $\beta = 0.72$ left unchanged. The height of the Hill bump increases with increasing Pr, with $g_{\text{max}} = 1.32$ for Pr = 0.5 and $g_{\text{max}} = 1.67$ for Pr = 1.0. With increasing Pr, the peak location shifts slightly to higher wavenumbers, with $\kappa_{\text{max}}\eta = 0.15$ for Pr = 0.5 to $\kappa_{\text{max}}\eta = 0.22$ for Pr = 1.0. For dry air (Pr = 0.72), the peak is at $\kappa_{\text{max}}\eta = 0.18$.

The blue and red lines in Fig. 1 are the solutions for water vapor ($Sc = 0.63$, $g_{\text{max}} = 1.43$) and dry air (Pr = 0.72, $g_{\text{max}} = 1.50$). The Schmidt number, Sc, of water vapor is the ratio between $\nu$ and the diffusivity of water-vapor concentration. In the context of the Hill model, there is no need to distinguish between Pr and Sc because thermal diffusivity and scalar-concentration diffusivity play the same role in the generalized Obukhov–Corrsin theory [14].

Figure 2 shows the same Obukhov–Corrsin-normalized temperature spectra as in Fig. 1, except that they are presented as functions of $kl_0$. With increasing Pr, the peaks of $b(kl_0)$ shift to smaller values of $kl_0$, in contrast to the peaks of $g(k\eta)$ (Fig. 1), which shift to higher values of $k\eta$ as Pr increases. This is caused by the fact that $l_0/\eta$ decreases like $Pr^{-3/4}$ with increasing Pr, which overcompensates the shift of the peak of $g(k\eta)$ toward higher $k\eta$ values as Pr increases.

With increasing Pr, the drop-off of $b(kl_0)$ in the viscous-diffusive range becomes steeper, and the solutions for $b(kl_0)$ in the range between Pr = 0.5 and Pr = 1.0 intersect each other at about $kl_0 = 0.3$, where $b(kl_0) \approx 0.9$. Note that the functions $g(k\eta)$ for different Pr do not intersect (Fig. 1).

Figure 3 shows the same compensated spectra $b(kl_0)$ as in Fig. 2, except that now $b$ is scaled logarithmically while $kl_0$ is scaled linearly. At high $kl_0$ values, $b(kl_0)$ varies like $\exp(-\delta kl_0)$, where $\delta$ increases with increasing Pr. That is, $b(kl_0)$ decreases more steeply with increasing Pr, as already shown in Fig. 2.

Figure 4 shows $b(kl_0)$ for the parameter $b$ ranging from 1.0 (uppermost black line) to 6.0 (lowermost black line) in increments of 1.0. The values of the other parameters were kept constant and set equal to those recommended by Hill: $a = 0.072$, $\beta = 0.72$, and Pr = 0.72 (dry air). The red line is $b(kl_0)$ for Hill’s value $b = 1.9$. For $b = 1.9$ and larger, the height and location of the Hill bump are practically independent...
of $b$. The only effect of an increasing $b$ is the increasing “sudden-ness” of the transition from the flat portion of $b(kl_0)$.

B. DNS Results

Figure 5 shows the normalized spectrum $E(\kappa)/[\chi e^{-1/3}\kappa^{-5/3}]$ obtained from two groups of DNS output datasets. There are six spectra estimated from the DNS run with $Pr = 0.7$ (blue) and six spectra estimated from the DNS run with $Pr = 1.0$ (red). For each of the two runs, each spectrum was estimated from an instantaneous, 3D realization of the velocity field and the scalar field. The velocity fields were analyzed to obtain $\epsilon$ and $\eta$ (the velocity spectra are not discussed in this study), which were used to scale the respective scalar spectra in order to obtain $g(\kappa \eta)$ and $b(kl_0)$ for each of the twelve realizations. For each of the two runs, the six realizations were separated in time by about 0.1 large-eddy turn-over times.

At low wavenumbers ($\kappa \eta < 0.02$), the twelve estimated spectra are highly correlated with each other. This is because the DNS is forced deterministically in wavenumber space. As mentioned in Section 3, the largest wavenumber at which the velocity field is forced is $\kappa \eta = 0.06$. (The scalar field is not forced.) At higher wavenumbers, both the velocity and scalar spectra represent increasingly random, turbulent flow. There is about half a decade of an inertial-convective range (between about $\kappa \eta = 0.01$ and $\kappa \eta = 0.03$), where $E(\kappa)/[\chi e^{-1/3}\kappa^{-5/3}]$ is approximately flat. In this region, the value of $E(\kappa)/[\chi e^{-1/3}\kappa^{-5/3}]$ is approximately 0.68, which is in good agreement with empirical values of the Obukhov–Corrsin constant reported earlier: $\beta = 0.72$ by Hill [16] in 1978, $\beta = (5/3) \times 0.4 = 0.67$ by Sreenivasan [25] in 1996 based on observations, and $\beta = 0.68$ by Watanabe and Gotoh [26] in 2004 based on DNS output data. Figure 5 shows that our two DNS runs (for $Pr = 0.7$ and $Pr = 1.0$) provide essentially the same value for $\beta$, which is consistent with the prediction of the Obukhov–Corrsin theory that $\beta$ is independent of $Pr$.

Figure 6 shows $h(kl_0)$ evaluated for $Pr = 0.7$ (blue) and $Pr = 1.0$ (red) from the DNS data shown in Fig. 5. For each Prandtl number, we show the average of the six estimates of $h(kl_0)$. The two solid, black lines are numerical solutions of the Hill model for $Pr = 0.7$ and $Pr = 1.0$, respectively, with the other parameters set to $a = 0.065$, $b = 1.9$ (Hill’s value), and $\beta = 0.68$. We chose the value of $a$ such that the maximum of $h(kl_0)$ predicted by the Hill model agreed with the maximum of $h(kl_0)$ obtained from the DNS data.

Figure 6 shows that the Hill model predictions of $h(kl_0)$ agree well with the DNS predictions for both Prandtl numbers.

Figure 7 shows the same data as Fig. 6, but now $h$ is scaled logarithmically while $kl_0$ is scaled linearly. Both the Hill model and the DNS results indicate that $h(kl_0)$ tends to decrease like $\exp(-\delta kl_0)$, with the coefficient $\delta$ increasing with $Pr$. For a given $Pr$, however, the DNS-predicted $\delta$ appears to be somewhat smaller than $\delta$ predicted by the Hill model. It is not clear why this is so.

The cusp on the spectra (at $kl_0 \approx 9$ for $Pr = 1.0$ and at $kl_0 \approx 13$ for $Pr = 0.7$) is truncation error, not dealiasing error [34, left column on p. 244]. It is obvious that the cusp must be truncation error because the spectrum must incorporate the correct dissipation rate regardless of resolution, provided that the resolution is high enough, which is the case here. The high-wavenumber end of the spectrum adjusts to make this true. It is a positive attribute of spectral DNS that the cusp taints just a few wave numbers, whereas other methods have significant numerical viscosity and so smear the error over a wider range of wavenumbers. Also, since the spectrum drops
exponentially, and the truncation error decreases exponentially in a spectral schema, the cusp always looks big on a plot where the spectrum is scaled logarithmically, no matter what the resolution; see, e.g., Fig. 3(a) in Ref. [35] and Fig. 2 in Ref. [36]. In fact, the cusp gets bigger as resolution is increased, presumably because the spectrum decreases faster than the truncation error.

Fig. 6. DNS predictions and Hill-model predictions of $b(kl_0)$ for $Pr = 0.7$ (blue: DNS; black: Hill model) and for $Pr = 1.0$ (red: DNS; black: Hill model).

Fig. 7. Same as Fig. 6, except that now $b$ is scaled logarithmically and $kl_0$ is scaled linearly.

5. DISCUSSION

A. Height of the Hill Bump

In Section 4, we have shown that the Hill model and our DNS runs for $Pr = 0.7$ and $Pr = 1.0$ predict the same behavior of the bump in $b(kl_0)$: the height of the bump increases with increasing $Pr$, and the peak moves to smaller $kl_0$ with increasing $Pr$.

Figure 8 shows the bump height, $h_{max}$, as a function of the key parameter of the Hill model, $c = \beta a^{4/3}/Pr$, in the range from $c = 10^{-4}$ to $c = 1$. For each value of $c$, we solved numerically the differential equation in Eq. (20), assuming Hill’s value $b = 1.9$, and we determined numerically the height of the bump, $f_{max} = h_{max}$. For $c$ below about 0.1, $h_{max}(c)$ is well approximated by the power law

$$h_{max}(c) = 0.469 c^{-1/3}.$$  \hfill (24)

Once $h_{max}(c)$ drops to 1, there is no bump anymore, and the maximum of $b(kl_0)$ is equal to the limit of $b(kl_0)$ for $kl_0 \ll 1$, which is 1. The condition $0.469 c_0^{1/3} = 1$ gives $c_0 = 0.10$ as the critical value of $c$ that marks the disappearance of the Hill bump.

Because $c$ is proportional to $Pr^{-1}$, the $c^{-1/3}$ law is equivalent to a $Pr^{1/3}$ law, as shown in Fig. 9. The solid line depicts the power law

$$h_{max}(Pr) = 1.79 Pr^{1/3}.$$  \hfill (25)

Here we assumed the same parameter values ($a = 0.065$, $b = 1.9$, and $\beta = 0.68$) that we used in Section 4 to reproduce the DNS results with the Hill model. From Eq. (25) we obtain $h_{max} = 1.59$ for $Pr = 0.7$ and $h_{max} = 1.79$ for $Pr = 1.0$, which is consistent with the results shown in Fig. 6. The condition $h_{max}(Pr_0) = 1$ imposed on Eq. (25) gives $Pr_0 = 0.17$ as the critical value of $Pr$ at which the Hill bump disappears.

Fig. 8. Height of the Hill bump, $h_{max}$, as a function of the dimensionless parameter $c = \beta a^{4/3}/Pr$. 
B. Asymptotic Behavior in the Viscous-Diffusive Range: Exponential or Gaussian Tail?

Both the DNS and the Hill model predict that \( b(\kappa l_0) \) decreases for \( \kappa l_0 \) exponentially rather than like a Gaussian function. An exponential tail is in agreement with Kraichnan’s theoretical result [37, sentence after Eq. (3.12) on p. 948] and with DNS results by Bogucki et al. [38].

6. SUMMARY AND CONCLUSIONS

We have analyzed DNS output data to investigate Hill’s [16] model of optical turbulence. The agreement between the DNS results and the predictions of Hill’s model is very good. These are our most important findings:

1. The height of the “Hill bump” increases with increasing Prandtl (or Schmidt) number \( Pr \) as \( h_{\text{max}}(Pr) = 1.79 \cdot Pr^{1/3} \), implying \( h_{\text{max}} = 1.60 \) for dry air (\( Pr = 0.72 \)) and implying that there is no bump if \( Pr < 0.17 \).

2. The DNS data predict \( a = 0.065 \) for Hill’s constant \( a = \kappa^3 \eta \), slightly less than Hill’s value \( a = 0.072 \) obtained from field observations.

3. The onset of the DNS-predicted bump occurs at about \( \kappa \eta = 0.03 \), in agreement with Hill’s model.

4. Both the DNS and the Hill model predict that with increasing \( Pr \), the peak of the bump shifts to higher \( \kappa \eta \) but to smaller \( \kappa l_0 \).

5. Both the DNS and the Hill model predict an exponential not a Gaussian “tail” of \( b(\kappa l_0) \) in the far viscous-diffusive range.

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