Physics 715. HW 4

1. Consider a two-dimensional square lattice with the lattice constant $a$. Nearest neighbor atoms of mass $m$ are coupled by harmonic forces with the spring constants $\kappa_\parallel$ and $\kappa_\perp$:

$$U = U_\parallel + U_\perp$$

$$U_\parallel = (\kappa_\parallel / 2) \sum_{<ij>} \left[ b_{<ij>} \cdot (u_j - u_i) \right]^2$$

$$U_\perp = (\kappa_\perp / 2) \sum_{<ij>} \left[ b_{<ij>} \times (u_j - u_i) \right]^2$$

where $b_{<ij>}$ is a unit vector in the direction of the bond connecting nearest neighbor points $i$ and $j$, i.e. it is either $\pm a_1/a$ or $\pm a_2/a$. (see Figure).

(a) Derive phonon dispersion laws for this lattice and verify that $\omega_j(k) \to 0$ when $k \to 0$.

(b) Draw the first Brillouin zone (BZ) for this lattice, and verify that the derivative $\partial \omega_j(k) / \partial k$ is zero in the direction perpendicular to the BZ boundary (by symmetry, it is sufficient to consider only one segment of the boundary).
(c) In which directions the classification of waves in terms of transverse and longitudinal is exact (Hint: do not miss directions along which the two phonon branches have the same energy—one is free to rotate the eigenvector basis at points of degeneracy).
(d) What relation between $\kappa_\parallel$ and $\kappa_\perp$ has to be satisfied for all modes to be transverse or longitudinal?

2. **Diatomic chain.** Consider a one-dimensional diatomic chain: $ABABAB\ldots$. The masses of the atoms $A$ and $B$ are $m_1$ and $m_2$. The distance between two adjacent atoms $A$ and $B$ is $a/2$. The interaction is only between the adjacent atoms $A$ and $B$; the interaction is harmonic with the spring constant $\kappa$.
(a) Solve for the phonons. (b) Study the limit $m_1 \to m_2$: Interpret the result at $m_1 = m_2$ in terms of the single-atomic chain; explain a subtle difference between the cases $m_1 = m_2$ and $|m_1 - m_2| \ll m_1$.

3. **Chain with alternating interaction.** Consider a one-dimensional chain of atoms of the same mass $m$, with the harmonic interaction between the nearest-neighbor atoms characterized by alternating spring constants: $\kappa_1, \kappa_2, \kappa_1, \kappa_2, \ldots$. The distance between the nearest-neighbor atoms is $a/2$.
(a) Solve for the phonons. (b) Study the limit $\kappa_1 \to \kappa_2$: Interpret the result at $\kappa_1 = \kappa_2$ in terms of the chain with non-alternating interaction; explain a subtle difference between the cases $\kappa_1 = \kappa_2$ and $|\kappa_1 - \kappa_2| \ll \kappa_1$. 

2