Understanding Predictors of Realized Random Effects in Mixed Models

- SRS Unit versus Random Effect
- Two Stage Sample Results
  - Comparisons with balance
  - Balance and Response error
  - Unbalance
- SRS- Adding a Covariate (Domains)
- SRS- Further Extensions
  - Godambe's Result
  - Interpreting BLUP as estimating a unit
  - Extending random variables
- Open Questions
  - Variance estimation
  - Extensions to other settings
  - General 'guidelines'

SRS: Unit versus Random Effect

$$Y_i = \sum_{s=1}^{N} U_{is} y_s \quad \text{versus} \quad \mathbf{Y}_i = \begin{pmatrix} U_{i1} y_1 \\ U_{i2} y_2 \\ \vdots \\ U_{iN} y_N \end{pmatrix}$$

Different linear combinations define:

The parameter for unit s: y_s

The random variable Y_i $\qquad Y_i = \mu + \mathrm{B}_i$

Uncertainty in interpretation- more work needed

Problem too simple - need extensions to other settings (Predictors of Y_i)

Two Stage Sample Results

(Predictors of Y_i ith PSU)

- Mixed models
- Super-population models
- Random permutation models
  - Balanced: N Clusters; M subjects/cluster
    - Sample: n clusters; m subjects/cluster
  - Balanced with response error
  - Unbalanced

Mixed Model:

$Y_{ij} = \mu + B_i + E_{ij}$ where

$$B_i \sim iid\ N\left(0\ ,\ \sigma^2\right)$$

and $E_{ij} \sim iid\ N\left(0\ ,\ \sigma_i^2\right)$

$$\hat{p} = \hat{\mu} + k_i\left(\bar{Y}_i - \hat{\mu}\right)$$

where $k_i = \dfrac{\sigma^2}{v_i}$ and $v_i = \sigma^2 + \dfrac{\sigma_i^2}{m}$

$$\hat{\mu} = \sum_{i=1}^{n} w_i \bar{Y}_i \ ;\ \ \bar{Y}_i = \frac{\sum_{j=1}^{m} Y_{ij}}{m} \ ;\ \ w_i = \frac{1/v_i}{\sum_{i^*=1}^{n} 1/v_{i^*}}$$

Scott and Smith

$$E\left(Y_{ij}\right) = \mu$$

$$\begin{aligned}\operatorname{cov}\left(Y_{ij}, Y_{kl}\right) &= \delta^2 + \sigma_i^2 \ \text{ when } i = k; j = l \\ &= \delta^2 \ \text{ when } i = k; j \neq l \\ &= 0 \ \text{ otherwise.}\end{aligned}$$

$$\hat{P} = \frac{m}{M}\bar{Y}_i + \left(\frac{M-m}{M}\right)\left[\hat{\mu}^* + k_i^*\left(\bar{Y}_i - \hat{\mu}^*\right)\right]$$

where $k_i^* = \dfrac{\delta^2}{v_i^*}$ and $v_i^* = \delta^2 + \dfrac{\sigma_i^2}{m}$

$$\hat{\mu}^* = \sum_{i=1}^{n} w_i^* \bar{Y}_i \quad ; \quad w_i^* = \frac{1/v_i^*}{\sum_{i^*=1}^{n} 1/v_{i^*}^*}$$

Summary

Mixed Model

$$E(\mathbf{Y}) = \mathbf{X}\mu$$

$$\operatorname{var}(\mathbf{Y}) = \bigoplus_{i=1}^{N}\left(\sigma_i^2\mathbf{I}_M + \sigma^2\mathbf{J}_M\right)$$

$$\hat{p} = \mathbf{e}_i'\left(\mathbf{1}_n\hat{\mu} + k_i\left(\bar{\mathbf{Y}}_I - \mathbf{1}_n\hat{\mu}\right)\right)$$

Scott and Smith Model

$$E_{\xi_1\xi_2}(\mathbf{Y}) = \mathbf{X}\mu$$

$$\operatorname{var}_{\xi_1\xi_2}(\mathbf{Y}) = \bigoplus_{i=1}^{N}\left(\sigma_i^2\mathbf{I}_M + \delta^2\mathbf{J}_M\right)$$

$$\hat{P}_i = f\mathbf{e}_i'\bar{\mathbf{Y}}_I + (1-f)\mathbf{e}_i'\left(\mathbf{1}_n\hat{\mu}^* + k_i^*\left(\bar{\mathbf{Y}}_I - \mathbf{1}_n\hat{\mu}^*\right)\right)$$

where $f = \dfrac{m}{M}$ for $i \le n$

2-Stage Random Permutation Model

$$E_{\xi_1\xi_2}(\mathbf{Y}) = \mathbf{X}\mu$$

$$\operatorname{var}_{\xi_1\xi_2}(\mathbf{Y}) = \bigoplus_{i=1}^{N}\left(\sigma_e^2\mathbf{I}_M + \sigma^{*2}\mathbf{J}_M\right) - \sigma^{*2}\frac{\mathbf{J}_{NM}}{N}$$

$$\hat{T}_i = f\mathbf{e}_i'\bar{\mathbf{Y}}_I + (1-f)\mathbf{e}_i'\left(\mathbf{1}_n\bar{Y} + k_i\left(\bar{\mathbf{Y}}_I - \mathbf{1}_n\bar{Y}\right)\right)$$

$$\bar{Y} = \frac{\sum_{i=1}^{n}\bar{Y}_i}{n}$$

2-Stage Random Permutation Model

$$E_{\xi_1\xi_2}\left(\mathbf{Y}\right)=\mathbf{X}\mu$$

$$\operatorname{var}_{\xi_1\xi_2}\left(\mathbf{Y}\right)=\bigoplus_{i=1}^{N}\left(\sigma_e^2\mathbf{I}_M+\sigma^{*2}\mathbf{J}_M\right)-\sigma^{*2}\frac{\mathbf{J}_{NM}}{N}$$

$$\hat{T}_i=f\mathbf{e}_i'\bar{\mathbf{Y}}_I+\left(1-f\right)\mathbf{e}_i'\left(\mathbf{1}_n\bar{Y}+k_i\left(\bar{\mathbf{Y}}_I-\mathbf{1}_n\bar{Y}\right)\right)$$

$$\bar{Y}=\frac{\sum_{i=1}^{n}\bar{Y}_i}{n}$$

2-Stage Random Permutation Model With Response Error

$$E_{\xi_1\xi_2}\left(\mathbf{Y}\right)=\mathbf{X}\mu$$

$$\operatorname{var}_{\xi_1\xi_2}\left(\mathbf{Y}\right)=\bigoplus_{i=1}^{N}\left(\left(\sigma_e^2+\bar{\bar{\sigma}}^2\right)\mathbf{I}_M+\sigma^{*2}\mathbf{J}_M\right)-\sigma^{*2}\frac{\mathbf{J}_{NM}}{N}$$

$$\hat{T}_i=f\mathbf{e}_i'\left(\mathbf{1}_n\bar{Y}+k_r^*\left(\bar{\mathbf{Y}}_I-\mathbf{1}_n\bar{Y}\right)\right)+\left(1-f\right)\mathbf{e}_i'\left(\mathbf{1}_n\bar{Y}+k^*\left(\bar{\mathbf{Y}}_I-\mathbf{1}_n\bar{Y}\right)\right)$$

Mixed $\hat{p} = \mathbf{e}_i'\left(\mathbf{1}_n\hat{\mu} + k_i\left(\overline{\mathbf{Y}}_I - \mathbf{1}_n\hat{\mu}\right)\right)$

S&S $\hat{P}_i = f\mathbf{e}_i'\overline{\mathbf{Y}}_I + (1-f)\mathbf{e}_i'\left(\mathbf{1}_n\hat{\mu}^* + k_i^*\left(\overline{\mathbf{Y}}_I - \mathbf{1}_n\hat{\mu}^*\right)\right)$

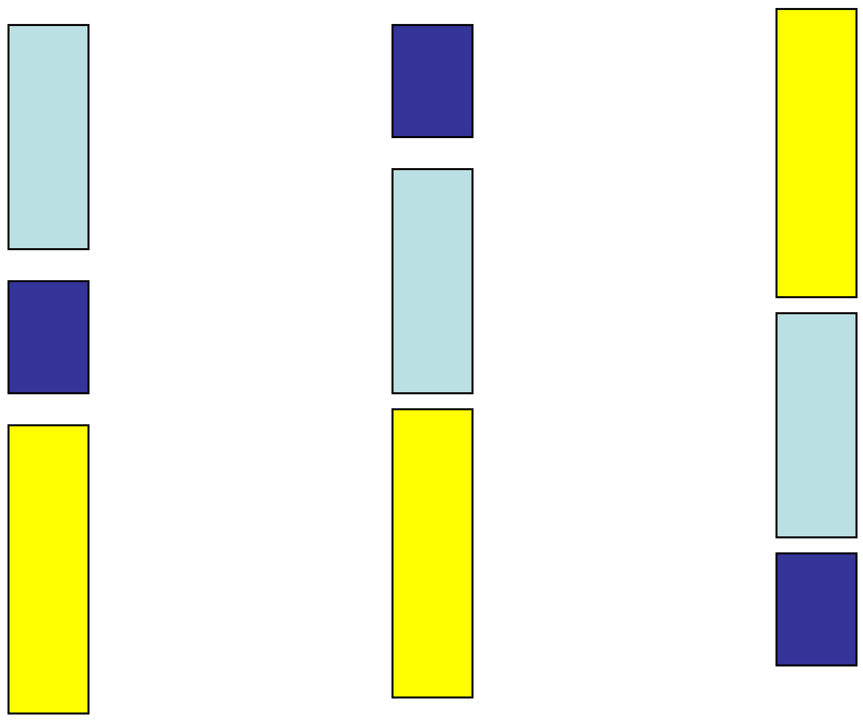
RP $\hat{T}_i = f\mathbf{e}_i'\overline{\mathbf{Y}}_I + (1-f)\mathbf{e}_i'\left(\mathbf{1}_n\overline{Y} + k\left(\overline{\mathbf{Y}}_I - \mathbf{1}_n\overline{Y}\right)\right)$

RP&E $\hat{T}_i = f\mathbf{e}_i'\left(\mathbf{1}_n\overline{Y} + k_r^*\left(\overline{\mathbf{Y}}_I - \mathbf{1}_n\overline{Y}\right)\right) + (1-f)\mathbf{e}_i'\left(\mathbf{1}_n\overline{Y} + k^*\left(\overline{\mathbf{Y}}_I - \mathbf{1}_n\overline{Y}\right)\right)$

	Between	Within			
Mixed	σ^2	σ_i^2	$k_i = \dfrac{\sigma^2}{v_i}$	$v_i = \sigma^2 + \dfrac{\sigma_i^2}{m_i}$	
S&S	δ^2	σ_i^2	$k_i^* = \dfrac{\delta^2}{v_i^*}$	$v_i^* = \delta^2 + \dfrac{\sigma_i^2}{m_i}$	
RP	σ^2	$\sigma_s^2 \ \sigma_e^2$	$k = \dfrac{\sigma^{*2}}{v}$	$v = \sigma^{*2} + \dfrac{\sigma_e^2}{m}$	$\sigma^{*2} = \sigma^2 - \dfrac{\sigma_e^2}{M}$
RP&E	σ^2	$\sigma_s^2 \ \sigma_e^2$	$k^* = \dfrac{\sigma^{*2}}{v^*}$	$v^* = \sigma^{*2} + \dfrac{\sigma_e^2 + \bar{\bar{\sigma}}^2}{m}$	
		Response			
		$\sigma_{st}^2 \ \bar{\bar{\sigma}}^2$	$k_r^* = \dfrac{v}{v_r}$	$v_r = v + \dfrac{\bar{\bar{\sigma}}^2}{m}$	

Unequal Size Clusters

Permutations



Pop. Values $y_{st} = \mu + \beta_s + \varepsilon_{st}$

Usual RVs $Y_{ij} = \sum_{s=1}^{N} \sum_{t=1}^{M_s} U_{is} U_{jt}^{(s)} y_{st}$

Expanded RVs $R_{isjt}^{*} = U_{is} U_{jt}^{(s)} y_{st}$

$$\mathbf{R}_{st}^{*} = \left(\mathbf{U}_s \otimes \mathbf{U}_t^{(s)} \right) y_{st}$$ where $$\mathbf{U}_s = \left(U_{1s} \quad U_{2s} \quad \cdots \quad U_{Ns} \right)'$$

$$\mathbf{U}_t^{(s)} = \left(U_{1t}^{(s)} \quad U_{2t}^{(s)} \quad \cdots \quad U_{M_s t}^{(s)} \right)'$$

$$\mathbf{R}^{*} = \left(\mathbf{R}_{1+}^{*}{}' \quad \mathbf{R}_{2+}^{*}{}' \quad \cdots \quad \mathbf{R}_{N+}^{*}{}' \right)'$$ where $$\mathbf{R}_{s+}^{*} = \left(\mathbf{R}_{s1}^{*}{}' \quad \mathbf{R}_{s2}^{*}{}' \quad \cdots \quad \mathbf{R}_{sM_s}^{*}{}' \right)'$$

$$\mathbf{R}^* = \begin{pmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_N \end{pmatrix} \mu + \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_N \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_N \end{pmatrix} + \begin{pmatrix} \mathbf{X}_{1+} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_{2+} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_{N+} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_N \end{pmatrix} + \mathbf{E}$$

where $\mathbf{X}_s = \dfrac{\mathbf{1}_{NM_s^2}}{NM_s}$ and $\mathbf{X}_{s+} = \mathbf{I}_{M_s} \otimes \dfrac{\mathbf{1}_{NM_s}}{NM_s}$

$$E_{\xi_1 \xi_2}\left(\mathbf{R}^*\right) = \begin{pmatrix} \mathbf{X}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_N \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_N \end{pmatrix} + \begin{pmatrix} \mathbf{X}_{1+} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_{2+} & \cdots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{X}_{N+} \end{pmatrix} \varepsilon$$

$$\operatorname{var}_{\xi_1\xi_2}\left(\mathbf{R}^*\right) = \bigoplus_{s=1}^{N}\left[\left(\frac{1}{M_s - 1}\right)\left(\bigoplus_{t=1}^{M_1} y_{st}^2 - \frac{\mathbf{y}_s\mathbf{y}_s{}'}{M_s}\right)\otimes\frac{\mathbf{I}_N}{N}\otimes\mathbf{P}_{M_s} + \left(\frac{1}{N-1}\right)\mathbf{y}_s\mathbf{y}_s{}'\otimes\mathbf{P}_N\otimes\frac{\mathbf{J}_{M_s}}{M_s^2}\right]$$

$$-\frac{1}{N\left(N-1\right)}\begin{pmatrix}\mathbf{y}_1\otimes\mathbf{P}_N\otimes\frac{\mathbf{1}_{M_1}}{M_1}\\ \mathbf{y}_2\otimes\mathbf{P}_N\otimes\frac{\mathbf{1}_{M_2}}{M_2}\\ \vdots\\ \mathbf{y}_N\otimes\mathbf{P}_N\otimes\frac{\mathbf{1}_{M_N}}{M_N}\end{pmatrix}\begin{pmatrix}\mathbf{y}_1\otimes\mathbf{P}_N\otimes\frac{\mathbf{1}_{M_1}}{M_1}\\ \mathbf{y}_2\otimes\mathbf{P}_N\otimes\frac{\mathbf{1}_{M_2}}{M_2}\\ \vdots\\ \mathbf{y}_N\otimes\mathbf{P}_N\otimes\frac{\mathbf{1}_{M_N}}{M_N}\end{pmatrix}'$$

Parameters of Interest

Cluster Total $P_i^{\circ} = \sum_{s=1}^{N} U_{is} M_s \mu_s$ or $P_i^{\circ} = \mathbf{g}_i'^{\circ} \mathbf{R}^*$

where $\mathbf{g}_i'^{\circ} = \mathbf{1}_N' \left(\bigoplus_{s=1}^{N} \left[\mathbf{1}_{M_s}' \otimes \left(\mathbf{e}_i' \otimes \mathbf{1}_{M_s}' \right) \right] \right)$

Cluster Mean $P_i^{*} = \sum_{s=1}^{N} U_{is} \mu_s$ or $P_i^{*} = \mathbf{g}_i'^{*} \mathbf{R}^*$

where $\mathbf{g}_i'^{*} = \mathbf{1}_N' \left(\bigoplus_{s=1}^{N} \left[\mathbf{1}_{M_s}' \otimes \left(\mathbf{e}_i' \otimes \frac{\mathbf{1}_{M_s}'}{M_s} \right) \right] \right)$

Collapsing Random Variables

$$\mathbf{R}^* = \mathbf{A}'\mathbf{Y} + \mathbf{B}'\mathbf{R}^*$$

where $P_i = \mathbf{g}_i'\ \mathbf{A}'\mathbf{Y}$ and $\mathbf{B}'\mathbf{R}^* = \mathbf{0}$

For unbiasedness, it is necessary to have PPS sampling.

Unequal Sampling Fractions, change target parameter.

Predict

$P_i^{\bullet} = \mathbf{g}_i'^{\bullet}\mathbf{Y}^{\bullet}$ where $P_i^* = E_{\xi_2}\left(P_i^{\bullet}\right)$

How to Do Collapsing?

Collapse to sample/remainder Cluster totals

$$\left(\frac{\sum_{s=1}^{N} U_{is} f_s M_s \overline{Y}_{sI}}{\sum_{s=1}^{N} U_{is} \left(1 - f_s\right) M_s \overline{Y}_{sII}} \right)$$

Collapse to scaled sample/remainder Cluster totals

$$\left(\frac{\sum_{s=1}^{N} U_{is} f_s \overline{Y}_{sI}}{\sum_{s=1}^{N} U_{is} \left(1 - f_s\right) \overline{Y}_{sII}} \right)$$

Collapse to sample mean/remainder mean for Clusters

$$\left(\frac{\sum_{s=1}^{N} U_{is} \overline{Y}_{sI}}{\sum_{s=1}^{N} U_{is} \overline{Y}_{sII}} \right)$$

RP-Balance $$\hat{T}_i = f\mathbf{e}_i'\overline{\mathbf{Y}}_I + (1-f)\mathbf{e}_i'\left(\mathbf{1}_n\overline{Y} + k\left(\overline{\mathbf{Y}}_I - \mathbf{1}_n\overline{Y}\right)\right)$$

Predict Totals (use totals)

$$\hat{P}_i^{\circ} = \mathbf{e}_{iI}'\mathbf{Y}_I^{\circ} + \left(\frac{1-f}{f}\right)\mathbf{e}_{iI}'\left(\mathbf{1}_n\left(\frac{\mathbf{1}_n'\mathbf{Y}_I^{\circ}}{n}\right) + fk^{\circ}\left(\mathbf{Y}_I^{\circ} - \mathbf{1}_n\left(\frac{\mathbf{1}_n'\mathbf{Y}_I^{\circ}}{n}\right)\right)\right)$$

Predict Means (use weighted totals)(with PPS)

$$\hat{P}_i^{*} = \mathbf{e}_{iI}'\mathbf{Y}_I^{*} + \left(\frac{1-f}{f}\right)\mathbf{e}_{iI}'\left(\mathbf{1}_n\left(\frac{\mathbf{1}_n'\mathbf{Y}_I^{*}}{n}\right) + fk^{*}\left(\mathbf{Y}_I^{*} - \mathbf{1}_n\left(\frac{\mathbf{1}_n'\mathbf{Y}_I^{*}}{n}\right)\right)\right)$$

Predict Means (use means)(for non-PPS sampling)

$$\hat{P}^{\bullet} = c\mathbf{e}_{iI}'\mathbf{Y}_I^{\bullet} + \mathbf{e}_{iI}'\left[(1-c)\mathbf{1}_n\left(\frac{\mathbf{1}_n'\mathbf{Y}_I^{\bullet}}{n}\right) + ck^{\bullet}\left(\mathbf{Y}_I^{\bullet} - \mathbf{1}_n\left(\frac{\mathbf{1}_n'\mathbf{Y}_I^{*}}{n}\right)\right)\right]$$

RP σ^2 σ_s^2 σ_e^2 $k = \dfrac{\sigma^{*2}}{v}$ $v = \sigma^{*2} + \dfrac{\sigma_e^2}{m}$ $\sigma^{*2} = \sigma^2 - \dfrac{\sigma_e^2}{M}$

Totals $k^\circ = \dfrac{\sigma_{\tau+}^2}{v^\circ}$ $v^\circ = f\sigma_{\tau+}^2 + \sigma_{e+}^2$ $\sigma_{\tau+}^2 = \sigma_\tau^2 - \sigma_{e+}^2$

$\tau_s = M_s \mu_s$ $\sigma_\tau^2 = \dfrac{\sum_{s=1}^{N} (\tau_s - \bar{\tau})^2}{N-1}$ $\sigma_{e+}^2 = \dfrac{\sum_{s=1}^{N} M_s \sigma_s^2}{N}$

Means (PPS) $k^* = \dfrac{\sigma^{*2}}{v^*}$ $v^* = f\sigma^{*2} + \sigma_e^{*2}$ $\sigma^{*2} = \sigma^2 - \sigma_e^{*2}$

$\sigma_e^{*2} = \dfrac{1}{N} \sum_{s=1}^{N} \dfrac{\sigma_s^2}{M_s}$

Means (non-PPS) $k^\bullet = \dfrac{\sigma^{*2}}{v^\bullet}$ $v^\bullet = \sigma_{eI}^{\bullet 2} + \sigma^{*2}$ $\sigma_{eI}^{\bullet 2} = \dfrac{1}{N} \sum_{s=1}^{N} \dfrac{\sigma_s^2}{m_s}$

Accounting for a Covariate

Basic Idea: (Wenjun Li's Dissertation)

Use Seemingly Unrelated Regression Models to represent the ''Usual' N Random Variables for y and x

Simplest Example:

SRS w/o rep.

Focus on estimating population total

x= 0 or 1

Pop total for x could be known, or unknown

With Known X Totals

Model $$\begin{pmatrix} \mathbf{Y} \\ \mathbf{X} \end{pmatrix} = \left(\mathbf{I}_2 \otimes \mathbf{1}_N \right) \mu + \mathbf{E}$$

Transform X $$\mathbf{X} = \mathbf{P}_N \mathbf{X} + \mathbf{1}_N \mu_x$$

$$\mathbf{P}_N \mathbf{X} = \left(\mathbf{X} - \mathbf{1}_N \mu_x \right)$$
$$= \mathbf{X}^*$$

$$\begin{pmatrix} \mathbf{Y} \\ \mathbf{X}^* \end{pmatrix} = \left(\mathbf{I}_2 \otimes \mathbf{1}_N \right) \begin{pmatrix} \mu_y \\ 0 \end{pmatrix} + \mathbf{E}^*$$

Predictor of Total $$\hat{P} = n\bar{Y} + (N-n)\left[\bar{Y} - \left(\frac{N}{N-n} \right) k \left(\bar{X} - \mu_x \right) \right]$$

where $$k = \frac{\sigma_{xy}}{\sigma_x^2}$$

SRS- Further Extensions

Question: Can we use the predictor of a random variable at a position as an estimator of a unit?

Godambe's Result: $(N-1)^n$ random variables

Royall (1969): N random variables

Expanded RVs: $(N-1)^2$ random variables

Example: N=3, n=2. Use a predictor of a position to predict unit=1

			Permutation					
Position (i)	**Unit (j)**	**Response**	**abc**	**acb**	**bac**	**bca**	**cab**	**cba**
1	1	Y11	1	1	0	0	0	0
2	1	Y21	0	0	1	0	1	0
3	1	Y31	0	0	0	0	0	0
1	2	Y12	0	0	0	0.5	0	0
2	2	Y22	0	0	0	0	0	0.5
3	2	Y32	0	0	0	0	0	0
1	3	Y13	0	0	0	0	0	0.5
2	3	Y23	0	0	0	0.5	0	0
3	3	Y33	0	0	0	0	0	0

Source: c02ed53.xls

Example: N=3, n=2. Use a predictor of a position to predict unit=1

			Permutation					
Position (i)	Unit (j)	Response	abc	acb	bac	bca	cab	cba
1	1	Y11	1	1	0	0	0	0
2	1	Y21	0	0	1	0	1	0
3	1	Y31	0	0	0	0	0	0
1	2	Y12	0	0	0	0.5	0	0
2	2	Y22	0	0	0	0	0	0.5
3	2	Y32	0	0	0	0	0	0
1	3	Y13	0	0	0	0	0	0.5
2	3	Y23	0	0	0	0.5	0	0
3	3	Y33	0	0	0	0	0	0

Source: c02ed53.xls

Interpreted as the predictor of a unit, this isn't a linear combination of the expanded random variables.

We need to allow for the coefficient for other units to change depending upon whether or not the unit of interest is in the sample.

g	Unit (j) not in Sample	Unit (j) in Sample	**Position (i)**	**Unit (j)**	**Response (Z)**	**abc**	**acb**	**bac**	**bca**	**cab**	**cba**
1	0	1	1	1	Vs(1)U(11)y1	1	1	0	0	0	0
1	0	1	2	1	Vs(1)U(21)y1	0	0	1	0	1	0
0	0	1	3	1	Vs(1)U(31)y1	0	0	0	0	0	0
0	0	1	1	2	Vs(1)U(12)y2	0	0	0	0	0	0
0	0	1	2	2	Vs(1)U(22)y2	0	0	0	0	0	0
0	0	1	3	2	Vs(1)U(32)y2	0	0	0	0	0	0
0	0	1	1	3	Vs(1)U(13)y3	0	0	0	0	0	0
0	0	1	2	3	Vs(1)U(23)y3	0	0	0	0	0	0
0	0	1	3	3	Vs(1)U(33)y3	0	0	0	0	0	0
0	1	0	1	1	Vr(1)U(11)y1	0	0	0	0	0	0
0	1	0	2	1	Vr(1)U(21)y1	0	0	0	0	0	0
1	1	0	3	1	Vr(1)U(31)y1	0	0	0	0	0	0
0	1	0	1	2	Vr(1)U(12)y2	0	0	0	0.5	0	0
0	1	0	2	2	Vr(1)U(22)y2	0	0	0	0	0	0.5
0	1	0	3	2	Vr(1)U(32)y2	0	0	0	0	0	0
0	1	0	1	3	Vr(1)U(13)y3	0	0	0	0	0	0.5
0	1	0	2	3	Vr(1)U(23)y3	0	0	0	0.5	0	0
0	1	0	3	3	Vr(1)U(33)y3	0	0	0	0	0	0

Partition into a sample and remainder

Collapse random variables similarly to unequal cluster size problems

Results with N=3 and n=2 lead to a non-unique solution. ???

Maybe larger N and n are needed?

Maybe solutions will always be non-unique?

Summary of Ideas

- If first stage is exchangeable, second stages can not easily retain nesting of units in the first stage unit. Identifiability of 1st stage unit matters.
- By expanding the random variables, units can be tracked. High dimension singular variance structures result, and the problem needs to be 'reduced'.
- Projections to lower dimensional spaces appear 'arbitrary', but seem to 'work'.
- Extending the expanded random variables may enable a unit to be identified, and estimated. Our hope is that we obtain an estimator that equals the predictor of a position.
- A variety of variance components need to be estimated to implement the results.