Index to Homework 4.2 from Lecture 4.2

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I. The homework questions.

Suggested priorities: Do at least one “Claim” and one “Example” from question 1, and all of questions 2 and 3, and 5. If you have time, do 4 or part of it. Number 8 (in PtMW) would be good if you’d like more examples of posets and practice seeing which posets are lattices. If you have more time, try one or two of the ones listed as “6”. Number 7 is completely optional. We certainly don’t expect you to do all, or even most, of these.

1) Show that the claims given in connection with Definitions 1 and 2 of *lattices* in section 1.2.1 above hold. [For each of these, you need its context in the text of 1.2.1]

Claim 1: The operations $\wedge$ and $\vee$ defined by (Op) have the properties (L1)-(L5).

Claim 2: If we consider the order $\leq_1$ defined by these operations by the condition

(Ord) $a \leq_1 b$ iff $a \wedge b = a$

we will get the same order that in we have in the lattice *A*.

Claim 3: On the other hand, if we begin from definition 2 and define on the carrier *A* of the algebra A the order corresponding to the condition (Ord), the conditions of the definition 1 will be satisfied.

Claim 4: The operations defined by this order by the conditions (Op) will coincide with the operations of algebra A.

Example 1: Show that the Poset in Figure 11-1 is a lattice.

Example 2: Verify that the power set algebra given in Example 2 is always a lattice. [Optional addition: Can’t the claim in Example 2 be strengthened by dropping the restriction to starting with a nonempty set *A*? Doesn’t the power set of the empty set also produce a lattice?]

2) Define algebras corresponding to join and meet semilattices. [answer in text p.282]

1) Define the notion of sublattice, using the algebraic definition of lattice (not the poset definition) and give examples.

2) [optional] a. The notion of “subposet” is not defined in the text or in these notes, although it is used in the text on p. 281 and in Fig. 11-4 on p. 282. Let us define it as follows: $\langle B, \leq_B \rangle$ is a subposet of a poset $\langle A, \leq_A \rangle$ iff (i) $B \subseteq A$ and (ii) $\leq_B = \leq_A|B$. Given that definition, let’s look at the question about whether the interdefinability of lattices as a kind of poset and lattices as a kind of algebra extends also to the “sub” concepts. I.e. are subposets as defined above equivalent to sublattices on the algebraic definition? The answer turns out to be “no”.
To show it, can you find an example of a subposet of a lattice such that the subposet fits the poset definition of a lattice, but which is not a sublattice of the original lattice according to the algebraic definition? (There is an example given in the text of PtMW, p. 281, diagram p. 282. If you look it up, then find a different one.) NB These are good examples of something that came up in class on Thursday Sept 20: namely that the third condition in the definition of subalgebra is not redundant!

Note: this result does not threaten the equivalence of the poset definition and the algebraic definition of lattices, but it does underscore the fact that posets are not algebras (why?), so the "sub-X" definitions that apply uniformly to algebras don't apply in the same way to posets.

b. Under Figure 11-4 on p.282 it says "The subposet L' = <{a,c,d,e}, □} is a lattice and a subalgebra, but not a sublattice." I (BHP) think that it's false that it's a subalgebra. Yes, VB confirms it. Show that the given subposet, which is indeed a lattice, is neither a subalgebra nor a sublattice. (In fact I suspect, without having tried to prove it, that it's impossible for a subset of a lattice together with the original lattice operation to be a lattice and a subalgebra without being a sublattice. Is that right?)

5) Prove that $\wp(A)$ and T are Boolean algebras (verify that properties (B1) – (B5) hold). What subalgebras do these algebras have?

6) and more: See homework problems for the xeroxed handout IIID of Partee (1979), and try some of them. Postpone the ones about the propositional logic model (1,2, and 7; in fact don't read section 5 about the propositional logic model itself – we'll return to it later. I didn't know then that what I presented there has a name, the "Lindenbaum algebra" corresponding to propositional logic). But you can try any of 1 (any one or more of 1a-d; these are good to do), 4 (highly recommended), and the 'starred problem' (= more challenging) 5 (similar to 4 but harder). You can do 6 but it's less interesting. Number 7 is interesting but should be postponed.

7) We defined Boolean algebras "from scratch". How might the definition be shortened if we defined Boolean algebras in terms of lattices plus additional properties? (The answer is given in a possibly cryptic form at the beginning of 12.1 in PtMW; cryptic because we haven't asked you to read 11.5, where complemented and distributive lattices are defined. But you can get an idea from that paragraph what axioms you would have to write down in addition to saying that <B, , □> forms a lattice.)

8) Try part or all of exercise 2 in PtMW, p. 293. Those are good examples for playing with posets and seeing which posets are and aren't lattices.

II. Index to answers on this website.

Student Solution 1 to Exercise1. One of several correct proofs of Claim 1. Some were longer, some were shorter with fewer steps explicit, this one was a nice length with clear explanations.

Student Solution 2 to Exercise 1. An almost correct proof for Claim 2. All the ingredients are there and all the parts are done correctly, just stopped short of proving the actual Claim 2.

Student Solution 3 to Exercise 1. A good proof for Claim 3. The proof could have been shortened by half by invoking "duality", which we didn't explicitly define or discuss. Duality is a kind of symmetry. The operations and □ are *duals*, together with sup and inf. So if you prove something about , you can then repeat the same claim (with replacement of all notions by their duals) for □. (Read the beginning of PtMW Chapter 11 for some discussion of duality.) When invoking such a claim, we usually say “Dually, ...”, or “By duality, ...”.

Student Solution 4 to Exercise 1. A correct answer for “Example 1” of Exercise 1. Many did it just like this: simple, straightforward, ‘brute force’ because it’s just a small concrete example.

Student Solution 1 to Exercise 2. A perfect answer to question 2.

Student Solution 1 to Exercise 3. A perfect answer to question 3, with examples.

Student Solution 1 to Exercise 5. A very good answer to the problem of proving that $\boldsymbol{\wp(A)}$ and **T** are Boolean algebras. Could be shortened at some points – there are some annotations about how. Did not go on to say what subalgebras these algebras have. (Did anybody do that part and get it right? Maybe not.)

Student Solution 1 to Exercise 6. Good answers to most of the suggested problems from the supplementary Boolean algebra handout from Partee (1979).