Validity

We have seen the syntax and semantics of Statement Logic. We can now use Statement Logic to analyze when an argument is valid.

Arguments An argument consists of

1. a set of statements “hypothesized” to be true (its premises)
2. a single statement, its conclusion, whose truth allegedly follows from the truth of the premises.

We will not be worrying with whether, in fact, the premises are or aren’t true. We will be just considering whether if we assume that the premises are true, then we are bound to assume that the conclusion is also true.

Valid Arguments Intuitively speaking, a valid argument is one in which the conclusion follows from its premises. A valid argument is one in which, if we assume the truth of its premises, we have to admit the truth of its conclusion. In more precise terms, given the semantics of Statement Logic, we can say that an argument is valid iff all assignments of truth values to its atomic statements that make its premises true, make the conclusion true.

Equivalently, if $P_1, \ldots, P_n$ are the premises of argument $A$ and $Q$ its conclusion, if $A$ is valid, then the statement $((P_1 \land \ldots \land P_n) \rightarrow Q)$ must be a tautology. Can you see why?

Invalid Arguments An argument is invalid iff there is at least one way to assign truth values to its atomic statements that make its premises true and its conclusion false.
2 Rules of Inference

Consider the following argument in English. (Read “∴” as “therefore”)

(1) 1. Premise 1: If John loves Mary, then Mary is happy
    2. Premise 2: John loves Mary
    3. ∴ Mary is happy.

The argument is intuitively valid. To show that it is we have to show that, if we assume the truth of its premises, we are bound to assume the truth of its conclusion. Let’s translate the argument into Statement Logic:

(2) John loves Mary ⊨ p
(3) Mary is happy ⊨ q
(4) If John loves Mary, then Mary is happy ⊨ (p → q)
(5) 1. Premise 1: p → q
    2. Premise 2: p
    3. ∴ q

Can we show that if we assume the truth of its premises, we have to assume the truth of the conclusion? Let’s construct a truth table:

<table>
<thead>
<tr>
<th>p</th>
<th>q</th>
<th>(p → q)</th>
<th>((p → q) ∧ p)</th>
<th>(((p → q) ∧ p) → q)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

We can now see that if both (p → q) and p are true, then q must be true as well. In other words, q is a logical consequence of ((p → q) ∧ p). The statement (((p → q) ∧ p) → q) is then a tautology.

Recall that if a statement is a tautology, it is true no matter the truth values of its constituents. That means that any statement of the form (((P → Q) ∧ P) → Q) will be a tautology. And any argument of the form

(7) 1. P → Q
    2. P
    3. ∴ Q
will be truth-preserving, i.e. valid!

If an argument is of the form of (7), we know it is valid. We now have a very handy tool. Can you tell whether the following argument is valid?

(8) 1. \((p \land ((\neg q \rightarrow r) \lor (r \land \neg p))) \rightarrow (r \rightarrow (\neg q \lor p))\)
2. \((p \land ((\neg q \rightarrow r) \lor (r \land \neg p)))\)
3. \(\therefore (r \rightarrow (\neg q \lor p))\)

There is a traditional name for the argument form in (7): Modus Ponens (Latin for “mode that affirms”)

There is a similar argument form that is nonetheless invalid:

(9) 1. \(P \rightarrow Q\)
2. \(Q\)
3. \(\therefore P\)

To show that this argument form is invalid, we consider any instance, for example

(10) 1. \((p \rightarrow q)\)
2. \(q\)
3. \(\therefore p\)

Or, to make things clearer, the following version in English:

(11) 1. *If John loves Mary, then she is happy*
2. *Mary is happy*
3. \(\therefore John loves Mary\)

Can we show that \(((p \rightarrow q) \land q) \rightarrow p\) is *not* a tautology? Hint: let’s look at its truth table:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>(p → q)</th>
<th>((p → q) ∧ q)</th>
<th>(((p → q) ∧ q) → p)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Can you see why it is not a tautology? Which is the crucial row? What’s the truth value of the premises in that row? What’s the truth value of the conclusion?
Fallacies  A fallacy is an invalid argument form that intuitively seems valid. The argument form in (9) is known as the fallacy of affirming the consequent

2.1 Proofs

Up until now we have determined the validity of an argument by showing that its conclusion is a logical consequence of the premises. That method can be quite inconvenient: an argument containing n atomic statements requires a truth table of 2^n lines!

Fortunately, there is a more convenient alternative: we can analyze the argument into a sequence of simpler arguments. If all the simpler arguments have been shown to be truth-preserving, then the whole argument must be truth preserving too.

Here's an example:

(13) 1. **Premise:** \((p \rightarrow (q \rightarrow (r \rightarrow (s \rightarrow \neg t))))\)
  2. **Premise:** \(p\)
  3. **Premise:** \(q\)
  4. **Premise:** \(r\)
  5. **Premise:** \(s\)
  6. \(\therefore \neg t\)

We can show that the conclusion follows from the premises by a sequence of four instances of Modus Ponens:

(14) 1. **Premise:** \((p \rightarrow (q \rightarrow (r \rightarrow (s \rightarrow \neg t))))\)
  2. **Premise:** \(p\)
  3. **Premise:** \(q\)
  4. **Premise:** \(r\)
  5. **Premise:** \(s\)
  6. \((q \rightarrow (r \rightarrow (s \rightarrow \neg t)))\) (1, 2, MP)
  7. \((r \rightarrow (s \rightarrow \neg t))\) (3, 6, MP)
  8. \((s \rightarrow \neg t)\) (4, 7, MP)
  9. \(\therefore \neg t\) (5, 8, MP)

Rules of Inference  are simple valid argument forms that can be used to prove that a certain argument is valid. The book lists some on page 117.
Proofs  A derivation of a conclusion $P$ from a set of premises by means of the application of a series of rules of inferences is called a proof of $P$.

We have seen some examples before, here’s another one: a proof for the argument in (15):

(15)  1. $Premise: p \land q$
      2. $Premise: (p \lor r) \rightarrow s$
      3. $Premise: (t \rightarrow \neg c)$
      4. $\therefore \neg t$

(16)  1. $Premise: p \land q$
      2. $Premise: (p \lor r) \rightarrow s$
      3. $Premise: (t \rightarrow \neg s)$
      4. $p$ (1, Simp)
      5. $(p \lor r)$ (4, Add.)
      6. $s$ (4,2, MP)
      7. $\therefore \neg t$ (3, 6, MT)

Once we are given a proof, it is easy to check out that it is indeed a proof by verifying that each step is an instance of a rule of inference.

Substitutions  Sometimes the premises are not of the right form to apply any rules of inference that we are given directly. In those cases we have to change the form of the premises by finding a logically equivalent formulation that can match the rules of inference. Recall that the laws of Statement Logic give us a recipe to find logically equivalent expressions.

Here’s an example, from the book (page 118):

(17)  1. show: $(p \rightarrow q)$
      2. premises:
         (a) $(p \rightarrow (q \lor r))$
         (b) $\neg r$
      3. $(\neg p \lor (q \lor r))$ (1, Comm)
      4. $(\neg p \lor q) \lor r$ (3, Ass)
      5. $r \lor (\neg p \lor q)$ (4, Comm)
      6. $\neg p \lor q$ (2, 5, DS)
      7. $\therefore p \rightarrow q$ (6, Cond)


2.2 Conditional Proof

In order to prove a conclusion whose main connective is “→” it is convenient to use a particular strategy (a conditional proof) that consists of assuming the antecedent of the conditional as an extra premise and proving its consequent.

The validity of a conditional proof is based on the fact that \(((P_1 \land \ldots \land P_n) \rightarrow (Q \rightarrow R))\) is logically equivalent to \(((P_1 \land \ldots \land P_n \land Q) \rightarrow R)\). [Useful exercise: prove it.]

When doing a conditional proof, we are working under a special assumption. In order to remind ourselves of that, we indicate with a bar each line which is based on the additional premise:

(18) 1. SHOW: \(p \rightarrow q\)
   2. PREMISES:
      (a) \(p \rightarrow (q \lor r)\)
      (b) \(\neg r\)
   3. |: Auxiliary Premise: \(p\)
   4. \(q \lor r\) (2,3, MP)
   5. \(r \lor q\) (4, Comm.)
   6. \(| q\) (2,5, D.S.)
   7. \(p \rightarrow q\) (3-6, Conditional Proof)

You must cancel the auxiliary premise by the rule of Conditional Proof before the whole proof is finished. Note that, since you are working under an assumption, it is forbidden to use any lines of the conditional proof as part of the proof after its cancellation.

Auxiliary premises can be any statement whatsoever that could be useful to reach our goal.

**Strategy:** If the desired conclusion is of the form \(P \rightarrow Q\), it’s often easiest to prove it with a conditional proof, using \(P\) as the extra premise.

2.3 Reductio

All the proofs we have seen so far are direct proofs: there is a direct derivation from the premises to the conclusion. Here’s another strategy: in order to prove \(Q\), assume, as an auxiliary premise \(\neg Q\) and find a contradiction. Since we are assuming that all the other premises are true, the auxiliary premise must be false. If we know that \(\neg Q\) is false, we have already proved that \(Q!\) We have found an indirect proof for \(Q\).

Here’s an example of an indirect proof, this time, carried out in plain English:

(19) 1. Is there only one empty set? To prove that there is only one empty set
2. Assume, as an auxiliary premises, that there are two different empty sets: call them $\text{Emp}_1$ and $\text{Emp}_2$.
3. Since two sets are different only when they have different members, $\text{Emp}_1$ and $\text{Emp}_2$ must have different members.
4. But we have reached a contradiction: $\text{Emp}_1$ and $\text{Emp}_2$ have exactly the same number of members: none!
5. From that we can conclude that our assumption that there are two empty sets is false. Therefore, there is only one.

If you are curious . . .

If you are curious about natural deduction systems for Statement Logic, you can check Gary Hard-degree’s introductory book (available on-line at http://www-unix.oit.umass.edu/~gmhwww/110/text.htm) There you can find lots of derivations and a very elegant natural deduction system.

You should also check Jan Jaspar’s excellent java applets, which are real fun: