Stock Return Characteristics, Skew Laws, and the Differential Pricing of Individual Equity Options

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November 28, 2000

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Abstract

How do risk-neutral return skews evolve over time and in the cross-section of individual stocks? We document the differential pricing of individual equity options versus the market index, and relate it to variations in the skew. The change-of-measure induced by marginal-utility tilting of the physical density can introduce skews in the risk-neutral return density. We derive the skew laws that decompose individual return skewness into a systematic skewness component and an idiosyncratic skewness component. Our empirical analysis of OEX options and 30 of its individual components demonstrates that individual risk-neutral distributions differ from that of the market index by being far less negatively skewed, and substantially more volatile.

Skewness continues to occupy a prominent role in equity markets. In the traditional asset pricing literature, Harvey and Siddique (2000) show that variations in conditional skewness can explain the cross-section of stock returns, even in the presence of book-to-market and size. In particular, stocks with negative co-skews command a higher equilibrium risk compensation (see also Kraus and Litzenberger (1976)). Realizing the inherent importance of skewness, Ait-Sahalia and Lo (1998), Bakshi, Cao, and Chen (1997), Bates (2000), Pan (1999), Rubinstein (1994), and Toft and Prucyk (1997) have devised option models to characterize asymmetries in the underlying risk-neutral pricing distributions. Despite tremendous advances in empirical and theoretical modeling of skews, extant work has not yet formalized restrictions on the physical return density and the pricing kernel process that could shift the risk-neutral distributions to the left. How do risk-neutral skews arise? What are its economic implications for individual equity options? Our present thrust is to fill specific gaps from theoretical and empirical standpoints. First, we provide a theoretical characterization that links risk-neutral index skews to risk aversion, and to the higher-order moments of the physical distribution (for a wide class of marginal utilities). Second, we develop the skew laws that relate individual skews to market index skews and idiosyncratic skews. Third, we establish the differential pricing of individual equity options versus the market index. Critical to this thrust is the link, to first-order, between skew laws and the differential pricing of individual equity options that makes our empirical study more tractable.

To make it easy to draw comparisons across option strikes and in the cross-section of equity options, the structure of option prices - how option prices differ across strikes - is often represented through the slope of the implied volatility curve (Rubinstein (1985, 1994)). Given their equivalence, we will use the slope of the implied volatility curve (or, the smile) and the structure of option prices to exemplify the same primitive object throughout. Granted, a one-to-one correspondence also exists between the smile and the risk-neutral density, modeling the smile as a stochastic process is now a central feature of some option models. While it is widely acknowledged that the smile is somehow due to the existence of negatively-skewed and heavy-tailed risk-neutral return distributions, a formal test of this simple idea has proven hard to implement. For example, is it skewness or kurtosis that is of first-order importance for explaining the observed variation in the structure of option prices? When the return distribution is skewed to the left, will a higher level of kurtosis induce a flatter smile?

The hurdles in quantifying the basic link across a wide spectrum of options stem from three sources. First, to infer the smile from the initial higher moments requires the identification of the underlying risk-neutral return density, and there is no natural way to reconstruct the density from just its higher moments. Second, although parametric option pricing models can be fitted and used to estimate the risk-neutral moments, in practice, all models are likely to be mis-specified.

Even when option models are well-enough specified across the strike price range, it is not clear that any derived relation between option prices and risk-neutral moments is a generic property, as opposed to being a reflection of the particular modeling choice (i.e., parameterization can force an artificial interdependence between skewness and kurtosis). Thus, it appears important that skews be recovered in a model-free fashion. Third, most stock options are American and therefore their risk-neutral densities cannot be so easily characterized using existing methods (Rubinstein (1994) and Ait-Sahalia and Lo (1998)). Consequently, much research in the estimation of risk-neutral distributions, and its moments, has concentrated on index, as opposed to individual equity, options. From a general asset pricing perspective, it is unsettling that we do not yet understand the properties of individual equity risk-neutral return distributions, or the structure of their option prices.

To build the connection between the differential pricing of individual stock options and the moments of the risk-neutral distribution in a model-free manner, we rely on the basic result from Bakshi and Madan (2000): any payoff can be spanned and priced using an explicit positioning across option strikes. Their results can be applied here as the moment, viewed as a payoff, is within the class of functions that can be spanned via options. Define, as usual, the skewness (kurtosis) as the third (fourth) central moment of the risk-neutral return distribution normalized by the cube (fourth power) of the return standard deviation, and the option positions that represent the third and fourth non-central moment as the cubic and quartic contracts, respectively. We show that the cubic contract can quantify return asymmetry by a specific position that simultaneously involves a long position in *out of money* calls and a short position in *out of money* puts. When the risk-neutral distribution is left-skewed, the combined cost of the positioning in puts is larger than that of the combined positioning in calls. We refer to the cost of reproducing the risk-neutral skewness and kurtosis as the price of skewness and kurtosis even though the respective payoffs are not actually discounted. The contingent claims theory that we use here is applicable to both European and American options, and the derived measures of tail asymmetry and tail size are readily comparable across equities and over time.

In order to analyze what may create a wedge between individual and index risk-neutral return distributions, we posit a market model in which individual stock return can be decomposed into a systematic component and an idiosyncratic component. Provided the idiosyncratic risk component is symmetric (or positively skewed) and the index distribution is negatively skewed, we can restrict the risk-neutral individual skew to be less negative than the market. Our economic analysis reveals that negatively-skewed risk-neutral index distributions are viable even when the physical return distribution is symmetric. Curiously, this outcome is achieved when the return process is in the family of fat-tailed physical distributions and the representative agent is risk-averse. We also demonstrate that variations in short-term and medium-term skews are related, and risk-neutral skews do not aggregate linearly over time. Finally, by appealing to a perturbation of the underlying density, we verify that, at any level of moneyness, the slope of the smile is, in general, affine in skewness and kurtosis. We use our skew paradigms to test the following hypotheses: (1) The index volatility smiles are more negatively-sloped than individual smiles. (2) In a cross-section of stocks, or in the time-series, the more negatively skewed the risk-neutral return distribution, the steeper the volatility smiles. On the other hand, in the presence of risk-neutral skews, a higher risk-neutral kurtosis flattens the implied volatility curve. (3) Individual risk-neutral distributions are less skewed to the left, and possibly more volatile (and fat-tailed) than the index distributions.[1]

Our empirical study is based on nearly 350,000 option quotes written on the S&P 100 index (hereby, OEX) and its 30 largest individual equity components. Covering January 1991 through December 1995, our principal conclusions are as follows. First, the slope of the individual equity smiles are persistently negative, but are much *less* negative than the index. To be concrete, when we regress log individual implied volatility onto log moneyness, the magnitude of the individual equity slope is only about one-fourth of the index. The documented differences in the slope of index and individual smiles produces a substantial difference in the relative price of options: for the OEX (a representative stock), the implied volatility of a deep out of the money put is about 22% (29%), as compared to at the money implied of 14% (26%). Clearly, the pricing structure of individual equity options is relatively flatter compared with that of the market index. Overall, these empirical findings support the view that the pricing kernel only prices systematic factors.

Second, variations in the risk-neutral skew are instrumental in explaining the differential pricing of individual equity options. In our dynamic regressions involving the contemporaneous slope of the smile and the contemporaneous risk-neutral moments, we find that the more negatively skewed the return distribution, the steeper is its volatility smile. Yet, when risk-neutral distributions evolve to be more fat-tailed, the smile gets less downward-sloping. Specifically, a higher risk-neutral kurtosis flattens the smile in the presence of left-tails. The cross-sectional regressions confirm that less negatively skewed stocks will have flatter smiles, on average.

Third, our inquiry consolidates a number of core properties mirrored by all (in our sample) individual risk-neutral (pricing) distributions:

[1]Amongst academics, Mark Rubinstein (we are told) has often lectured on (i) the potential sources of negative risk-neutral index skews, and (ii) on the relative flatness of individual equity option smiles. For example, this has been noted in the introduction of Jackwerth and Rubinstein (1996), and possibly elsewhere. Although Toft and Prucyk (1997) and Dennis and Mayhew (1999) construct various measures of the skew, the lack of theoretical foundation makes it difficult to interpret these measures consistently in either the cross-section or in the time-series. This point will become evident from our Theorems 2 and 3. In one particular example, it is shown that the leverage argument (i.e., equity returns and volatility are inversely related) for skews produces the counterfactual implication that some individual equity returns are risk-neutrally more left-skewed than the index.

- Individual stocks are mildly left-skewed (or even positively skewed), while index return distributions are heavily left-skewed. By way of contrast, the price of volatility is far more expensive for individual stocks. On balance, there is no consistent pattern for the price of the fourth moment in the cross-section;
- Although individual skews are negative much of the time, their magnitudes are seldom more negative relative to the index. The index skews are never positive, even periodically.

Since individual skew is a positively weighted combination of the market and idiosyncratic skew, the less negative individual skew (relative to the market index) can only transform into the statement that the idiosyncratic return component cannot be strongly skewed to the left.

Finally, using generalized method of moments, we empirically relate the risk-neutral index skews to the higher-moments of the physical distribution. Our results indicate that the substantial differences in the magnitudes of risk-neutral and physical skews are primarily a consequence of risk aversion and long-tailed physical distributions. The resulting estimates of risk aversion parameter appear reasonable. A variety of extended diagnostics support our main empirical findings.

This article is divided into several parts. Section 1 is devoted to formulating the key elements of the problem. In Section 2, we relate the structure of option prices to higher-order risk-neutral return moments. Section 3 reviews the equity options data. The differential pricing of individual equity options versus the market index is demonstrated in Section 4. We empirically examine the role of skews. Conclusions are offered in Section 5. All proofs are collected in an Appendix.

1 Understanding and Recovering Risk-Neutral Skews

This section accomplishes three tasks. At the outset, we propose a methodology to span and price skewness and kurtosis. This step is rendered feasible using only out of money calls and puts, and without imposing any structure on the underlying forcing process. Next, we establish when risk aversion causes the aggregate index to have negative-skews under the risk-neutral measure. We then decompose the price of individual return skewness into market-induced skewness and idiosyncratic skewness. Each conceptualization is critical for the later empirical exercises.

1.1 Generic Spanning and Pricing Characterizations in Bakshi-Madan (2000)

Since our intent is to frugally represent the risk-neutral distribution (or some feature thereof) in terms of traded option prices, it is only convenient to adopt the setting outlined in Bakshi and Madan (2000). That is, to fix notation, denote the time t price of the stock n by $S_n(t)$ (for

$n = 1, \cdots, N$), and the market index by $S_{\mathrm{m}}(t)$. Without any loss of generality, let the interest rate be a constant r, and $S(t) > 0$ with probability 1 for all t (suppressing the subscript n).

To ease equation presentation, write the $t + \tau$ period price of the stock, $S(t + \tau)$, as S and define the set $\Omega \equiv \{S > 0\}$. Let the risk-neutral pricing density $q[t, \tau; S]$, or simply $q[S]$, embody all remaining uncertainty about S. The physical density, $p[S]$, and the associated Radon-Nikodym derivative that delivers $q[S]$, for a given pricing kernel, will be formalized in Section 1.3. For any universal claim payoff $H[S] \in \mathcal{L}^1(q)$ (i.e., $\int_\Omega \mid H[S] \mid q[S]\, dS < \infty$), the symbol $\mathcal{E}_t^*\{.\}$ will mean to represent the expectation operator under risk-neutral density. That is, in what follows,

$$\mathcal{E}_t^*\{H[S]\} = \int_\Omega H[S]\, q[S]\, dS. \tag{1}$$

With this understanding, we can express the price of the European call and put written on the stock with strike price K and expiring in τ-periods from time t as: $C(t, \tau; K) = \int_\Omega e^{-r\tau}\, (S - K)^+\, q[S]\, dS$, and $P(t, \tau; K) = \int_\Omega e^{-r\tau}\, (K - S)^+\, q[S]\, dS$, where $(S - K)^+ \equiv \max(0, S - K)$ represents the maximum operator.

As articulated in Bakshi and Madan (2000), any payoff function with bounded expectation can be spanned by a continuum of out of money European calls and puts. In particular, a special case of their Theorem 1 is that the entire collection of twice-continuously differentiable payoff functions, $H[S] \in \mathcal{C}^2$, can be spanned algebraically (see also Carr and Madan (1996)), as in:

$$H[S] \;=\; H[\bar{S}] + (S - \bar{S}) H_S[\bar{S}] + \int_{\bar{S}}^{\infty} H_{SS}[K](S - K)^+ dK + \int_0^{\bar{S}} H_{SS}[K](K - S)^+ dK \tag{2}$$

where $H_S[\bar{S}]$ ($H_{SS}[K]$) represents the first-order (second-order) derivative of the payoff with respect to S evaluated at some $\bar{S}$ (the strike price). Intuitively, the position in options enables one to buy the curvature of the payoff function.

Applying the martingale pricing operator to both sides of (2), we have the arbitrage-free price of the hypothetical claim

$$\begin{aligned}\mathcal{E}_t^*\{e^{-r\tau}\, H[S]\} \;=\; & (H[\bar{S}] - \bar{S}\, H_S[\bar{S}])\, e^{-r\tau} + H_S[\bar{S}]\, S(t) \\ & + \int_{\bar{S}}^{\infty} H_{SS}[K]\, C(t, \tau; K)\, dK + \int_0^{\bar{S}} H_{SS}[K]\, P(t, \tau; K)\, dK \end{aligned} \tag{3}$$

which merely formalizes how $H[S]$ can be synthesized from (i) a zero-coupon bond with positioning: $H[\bar{S}] - \bar{S}\, H_S[\bar{S}]$, (ii) the stock with positioning: $H_S[\bar{S}]$, and (iii) a linear combination of calls and puts (indexed by K) with positioning: $H_{SS}[K]$. By observing the relevant market prices and appealing to (3), we can statically construct the intrinsic values of most contingent claims.

1.2 Mimicking Risk-Neutral Skewness and Kurtosis

To streamline the discussion of stock return characteristics and the structure of option prices, let the τ-period return be given by the log price relative: $R(t,\tau) \equiv \ln[S(t+\tau)] - \ln[S(t)]$. Define the volatility contract, the cubic contract, and the quartic contracts to have the payoffs:

$$H[S] = \begin{cases} R(t,\tau)^2 & \textbf{Volatility Contract} \\ R(t,\tau)^3 & \textbf{Cubic Contract} \\ R(t,\tau)^4 & \textbf{Quartic Contract.} \end{cases} \tag{4}$$

Let $V(t,\tau) \equiv \mathcal{E}_t^* \{e^{-r\tau} R(t,\tau)^2\}$, $W(t,\tau) \equiv \mathcal{E}_t^* \{e^{-r\tau} R(t,\tau)^3\}$, and $X(t,\tau) \equiv \mathcal{E}_t^* \{e^{-r\tau} R(t,\tau)^4\}$ represent the fair value of the respective payoff. The following theorem is a consequence of (2)-(3).

Theorem 1 *Under all martingale pricing measures, the following contract prices can be recovered from the market prices of out of money European calls and puts:*

1. *The τ-period risk-neutral return skewness, $SKEW(t,\tau)$, is given by*

$$\begin{aligned} SKEW(t,\tau) &\equiv \frac{\mathcal{E}_t^* \left\{ (R(t,\tau) - \mathcal{E}_t^*[R(t,\tau)])^3 \right\}}{\left\{ \mathcal{E}_t^* \left(R(t,\tau) - \mathcal{E}_t^*[R(t,\tau)]\right)^2 \right\}^{3/2}} \\ &= \frac{e^{r\tau} W(t,\tau) - 3\mu(t,\tau) e^{r\tau} V(t,\tau) + 2\mu(t,\tau)^3}{[e^{r\tau} V(t,\tau) - \mu(t,\tau)^2]^{3/2}}. \end{aligned} \tag{5}$$

2. *The risk-neutral kurtosis, denoted $KURT(t,\tau)$, is*

$$\begin{aligned} KURT(t,\tau) &\equiv \frac{\mathcal{E}_t^* \left\{ (R(t,\tau) - \mathcal{E}_t^*[R(t,\tau)])^4 \right\}}{\left\{ \mathcal{E}_t^* \left(R(t,\tau) - \mathcal{E}_t^*[R(t,\tau)]\right)^2 \right\}^{2}} \\ &= \frac{e^{r\tau} X(t,\tau) - 4\mu(t,\tau) e^{r\tau} W(t,\tau) + 6 e^{r\tau} \mu(t,\tau)^2 V(t,\tau) - 3\mu(t,\tau)^4}{[e^{r\tau} V(t,\tau) - \mu(t,\tau)^2]^2}, \end{aligned} \tag{6}$$

with $\mu(t,\tau)$ displayed in (40) of the Appendix. The price of the volatility contract

$$V(t,\tau) = \int_{S(t)}^{\infty} \frac{2\left(1 - \ln\left[\frac{K}{S(t)}\right]\right)}{K^2} C(t,\tau;K)\, dK + \int_0^{S(t)} \frac{2\left(1 + \ln\left[\frac{S(t)}{K}\right]\right)}{K^2} P(t,\tau;K)\, dK \tag{7}$$

and the price of the cubic, and the quartic, contract

$$W(t,\tau) = \int_{S(t)}^{\infty} \frac{6 \ln\left[\frac{K}{S(t)}\right] - 3\left(\ln\left[\frac{K}{S(t)}\right]\right)^2}{K^2} C(t,\tau;K)\, dK$$

$$
- \int_0^{S(t)} \frac{6 \ln \left[\frac{S(t)}{K}\right] + 3(\ln \left[\frac{S(t)}{K}\right])^2}{K^2} P(t, \tau; K)\, dK, \tag{8}
$$

$$
\begin{aligned}
X(t,\tau) \;=\; & \int_{S(t)}^{\infty} \frac{12(\ln \left[\frac{K}{S(t)}\right])^2 - 4(\ln \left[\frac{K}{S(t)}\right])^3}{K^2} C(t, \tau; K)\, dK \\
& + \int_0^{S(t)} \frac{12(\ln \left[\frac{S(t)}{K}\right])^2 + 4(\ln \left[\frac{S(t)}{K}\right])^3}{K^2} P(t, \tau; K)\, dK,
\end{aligned} \tag{9}
$$

can each be formulated through a portfolio of options indexed by their strikes.

The theorem formalizes a mechanism to extract the volatility, the skewness, and the kurtosis, of the risk-neutral return density from a collection of out of money (hereby OTM) calls and puts. Notably, one must always pay to go long the volatility and the quartic/kurtosis contracts. Specifically, to unwind the price of volatility, all OTM calls and puts are to be weighted by the strike price dependent amount: $\frac{2-2\ln[K/S(t)]}{K^2}$. In the quartic contract, the positioning is cubic in moneyness, however. Heuristically, a more pronounced fourth moment can only give rise to heavy-tailed distributions, a feature that will bid up the prices of both deep out of money and in the money calls and puts. When fitting implied volatility curves, this effect sometimes surfaces as a parabola in the space of moneyness and implied volatility. Therefore, the weighting structure assigning far higher weight to OTM (versus near the money) options does have intuitive justification.

The cubic contract displayed in (8) permits a play on the skew. With return distributions that are left-shifted, all OTM put options will be priced at a premium relative to OTM calls. In this environment, the cost of the short position in the linear combination of OTM puts will generally exceed the call option counterpart. Equation (5) thus blends qualitative as well as quantitative dimensions of asymmetry. More exactly, when the cubic contract is normalized by $V(t, \tau)$, it quantifies asymmetry both across time, and in the stock cross-section. In this sense, the price of skewness complements the skewness premium measure of Bates (1991), which is the ratio of a single call to put price. As we shall see, the option portfolio (5) is instrumental in quantifying fluctuations in the smile and in reconciling the relative structure of individual option prices.

Although it is possible to parameterize skews via a specific jump model (Bates (2000) and Pan (1999)), for reasons already discussed, the model-free determination of skews is desirable on theoretical and empirical grounds. In our context, moment discovery can be contemplated as summing a coarsely available grid of OTM calls and puts; it also generalizes to American options. The latter assertion can be supported in two ways. First, OTM options have negligible early exercise premiums. Second, even when early exercise premiums are not modest (i.e., OTM options in the neighborhood of at the money), the portfolio weighting in these options is small by construction. In the converse, larger weighting applies to deep OTM options but their market prices are declining rapidly with strikes. In reality, a finite positioning in options should effectively span the payoffs of interest. We address issues of accuracy in our implementations.

Equation (5) may be useful to researchers interested in measuring risk compensation for individual/index skews (see Harvey and Siddique (2000) for an innovative approach). Suppose an individual holds the claim: $\frac{(R_n(t,\tau)-\mu_n)^3}{[e^{r\tau}V_n(t,\tau)-\mu_n(t,\tau)^2]^{3/2}}$, with no idiosyncratic exposure. The market price of this exposure is precisely given by equation (5). For any admissible stochastic discount factor, ξ, and covariance operator, $\text{Cov}_t(.,.)$, the reward for bearing skewness risk, μ_S, is then:

$$\mu_S - r = -\text{Cov}_t\left(\frac{\xi(t+\Delta t)}{\xi(t)}, \frac{\text{SKEW}(t+\Delta t,\tau)}{\text{SKEW}(t,\tau)}\right) \tag{10}$$

which is, in principle, computable once the stochastic discount factor has been identified. The identification of ξ can be rather involved, and requires the joint estimation and formulation of the physical and risk-neutral processes. For details on this procedure, we refer the reader to Chernov and Ghysels (2000), Pan (1999) and Harvey and Siddique (2000).

1.3 Sources of Risk-Neutral Index Skews

For our synthesis involving the relationship between risk-neutral and physical densities, let $p[R_{\text{m}}]$ denote the physical density of the τ-period index return, R_{m}. Similarly denote the joint physical density of the stock collection by $p[R_1,\cdots,R_N,R_{\text{m}}]$. Under certain conditions, we must have, by the Radon-Nikodym theorem, the identities:[2]

$$q[R_{\text{m}}] \quad = \quad \frac{e^{-\gamma R_{\text{m}}}\times p[R_{\text{m}}]}{\int e^{-\gamma R_{\text{m}}}\times p[R_{\text{m}}]\,dR_{\text{m}}}, \quad \text{and} \tag{11}$$

[2]Strictly, the Radon-Nikodym theorem is a statement about two equivalent probability measures, Q and $\overline{P}$ on some measurable space (recall we have reserved P for the put price). In general, we have measures on a sigma-field of subsets of Ω and the Radon-Nikodym theorem allows us to assert: $Q[d\omega]=\xi[\omega]\,\overline{P}[d\omega]$, where $\xi[\omega]$ is an $\mathcal{L}^1$ measurable function with respect to the underlying sigma-field (Halmos (1974)). For any (Borel measurable) test function $f[S]$, the density of the stock price (if it exists) is defined by the condition: $\int f[S]\,p[S]\,dS=\int f[S]\,\overline{P}[d\omega]$. Analogously, the risk-neutral density satisfies: $\int f[S]\,q[S]\,dS=\int f[S]\,\xi[\omega]\,\overline{P}[d\omega]$. Armed with this result, define the conditional expectation of ξ, given the filtration generated by the stock price as: $E[\xi\mid S]$ by the condition that (for all test functions $f[S_m]$): $\int f[S_{\text{m}}]\,\xi[\omega]\,\overline{P}[d\omega]=\int f[S_{\text{m}}]\,E[\xi\mid S_{\text{m}}]\,\overline{P}[d\omega]=\int f[S_{\text{m}}]\,E\{\xi\mid S_{\text{m}}\}\,p[S_{\text{m}}]\,dS_{\text{m}}$. Applying this property of conditional expectations to the above equation, we get $\int f[S_{\text{m}}]\,q[S_{\text{m}}]\,dS_{\text{m}}=\int f[S_{\text{m}}]\,E\{\xi\mid S_{\text{m}}\}\,p[S_{\text{m}}]\,dS_{\text{m}}$. Thus, we may deduce $q[S_{\text{m}}]=E\{\xi\mid S_{\text{m}}\}\times p[S_{\text{m}}]$. As is traditional, one conjectures a form for the un-normalized Radon-Nikodym derivative, and in this case: $q[S_{\text{m}}]=\frac{E\{\xi\mid S_{\text{m}}\}\times p[S_{\text{m}}]}{\int E\{\xi\mid S_{\text{m}}\}\times p[S_{\text{m}}]\,dS_{\text{m}}}$, where ξ can be interpreted as a general un-normalized change-of-measure pricing kernel. Under the maintained hypothesis of a power utility function in wealth, we may specialize the stochastic discount factor to $E\{\xi\mid S_{\text{m}}\}=S_{\text{m}}^{-\gamma}=e^{-\gamma\ln(S_{\text{m}})}$. Then dividing the denominator and numerator by $S_{\text{m}}^{-\gamma}(t)$ and making a change of variable, we derive (11). For an extended treatment of admissible stochastic discount factors (and their uniqueness/non-uniqueness), see, for example, Hansen and Jagannathan (1997) and Harrison and Kreps (1979). For recent applications of (11), check Amin and Ng (1993), Chernov and Ghysels (2000), Stutzer (1996), and Jackwerth (2000).

$$q[R_1, \cdots, R_N, R_{\rm m}] \;\; = \;\; \frac{e^{-\gamma R_{\rm m}} \times p[R_1, \cdots, R_N, R_{\rm m}]}{\int e^{-\gamma R_{\rm m}} \times p[R_1, \cdots, R_N, R_{\rm m}]\, dR_1 \cdots dR_N\, dR_{\rm m}}, \qquad (12)$$

where $e^{-\gamma R_{\rm m}}$ is the pricing kernel in power utility economies, with coefficient of relative risk aversion γ. Here, the risk-neutral index density is obtained by exponentially tilting the physical density. Note that the normalization factor in the denominator of (11) ensures $q[R_{\rm m}]$ is a proper density function that integrates to unity. We now prove the main result of this subsection.

Theorem 2 *Up to a first-order of γ, the risk-neutral skewness of index returns is analytically attached to its physical counterparts via*

$$SKEW_m(t,\tau) \approx \overline{SKEW}_m(t,\tau) - \gamma \left(\overline{KURT}_m(t,\tau) - 3 \right) \overline{STD}_m(t,\tau), \qquad (13)$$

where $\overline{STD}(t,\tau)$, $\overline{SKEW}(t,\tau)$ and $\overline{KURT}(t,\tau)$ are a stand-in for return standard deviation, skewness, and kurtosis, under the physical probability measure, respectively. Thus, exponential tilting of the physical density will produce negative skew in the risk-neutral index distribution provided the physical distribution is fat-tailed (with non-zero γ).

Because our characterization of individual equity skews hinge on negative skewness in the risk-neutral index distribution, the result (13) is of special relevance. At a theoretical level, Theorem 2 provides sound economic reasons for the presence of risk-neutral skews even when the physical process is symmetric. Essentially, it states that there are three sources of negative skew in the risk-neutral index distribution. First, a negative skew in the physical distribution causes the risk-neutral index distribution to be left-skewed even under $\gamma = 0$ restriction. Second, risk-neutral index skews and the kurtosis of the physical measure appear to be inversely related: for a given volatility level and risk aversion, raising the level of kurtosis beyond 3 generates a more pronounced left-tail. In a likewise manner, higher stock market volatility will not guarantee left-skew unless the parent distribution is fat-tailed.

At the least, these features match the observations in Bates (1991) and Rubinstein (1994) that the index distributions have become (risk-neutrally) more negatively skewed after the crash of 1987. Finally, risk aversion makes the risk-neutral density inherit negative skew, provided the kurtosis of the physical distribution is in excess of 3. Since physical distributions estimated in practice are often symmetric (see the evidence in Rubinstein (1994) and Jackwerth (1999) on realized S&P 500 index returns), according to (13), heavy-tailed index distributions and risk aversion are the most likely root of the risk-neutral index skew.

To see the working behind this counterintuitive finding that exponential tilting of the physical index distribution produces no skew when kurtosis is equal to 3, let us take a parametric example in which index returns are distributed normal with mean $\overline{\mu}_{\mathrm{m}}$, and variance $\overline{\sigma}^2_{\mathrm{m}}$. With the aid of (11) and Gaussian $p[R_{\mathrm{m}}]$, we have (for some constants $A_0 > 0$ and $A_1 > 0$)

$$q[R_{\mathrm{m}}] = A_0 \exp(-\gamma R_{\mathrm{m}}) \times \exp\left(-\frac{(R_{\mathrm{m}} - \overline{\mu}_{\mathrm{m}})^2}{2\,\overline{\sigma}^2_{\mathrm{m}}}\right)$$

$$= A_1 \exp\left(-\frac{[R_{\mathrm{m}} - (\overline{\mu}_{\mathrm{m}} - \gamma\,\overline{\sigma}^2_{\mathrm{m}})]^2}{2\,\overline{\sigma}^2_{\mathrm{m}}}\right) \qquad (14)$$

which is again a mean-shifted (i.e., $\overline{\mu}_{\mathrm{m}} - \gamma\,\overline{\sigma}^2_{\mathrm{m}}$) Gaussian variate with zero skewness. This is consistent with our first-order analysis that indicates a need for excess kurtosis to generate a change in skew. The excess kurtosis is well-known to be prevalent statistically in index returns. So long as the physical distribution is fat-tailed, the end-result is similar in stochastic volatility and pure-jump models as well. In (47)-(49) of the Appendix, it is explicitly shown how exponential tilting of the physical density alters the (first) three moments of the risk-neutral distribution (whether the physical density is generated via a partial equilibrium, or a general equilibrium Lucas, economy).

Can Theorem 2 be generalized to a broader family of utility functions? Is the power utility assumption crucial for generating the negative skew phenomena? To resolve this issue, consider the wider class of marginal utility functions, $U'[R_{\mathrm{m}}]$, given by

$$U'[R_{\mathrm{m}}] = \int_0^\infty e^{-z\,R_{\mathrm{m}}}\,\nu(dz), \qquad (15)$$

for a measure ν on $\Re^+$. This includes as candidates for marginal utility, all bounded Borel functions vanishing at infinity (Revuz and Yor (1991)). For example, the choice of the gamma density for the measure $\nu(.)$ results in HARA marginal utility. In particular, we can also accommodate, as a special case, the bounded versions of the loss aversion utility functions considered by Kahneman and Tversky (1979). With positive $\nu(.)$, all completely monotone utility functions (i.e., $U'[R_{\mathrm{m}}] > 0$, $U''[R_{\mathrm{m}}] < 0$, $U'''[R_{\mathrm{m}}] > 0$, and so on) are also nested within equation (15). Clearly, the coefficient of relative risk aversion $\gamma[R_{\mathrm{m}}] \equiv -\frac{R_{\mathrm{m}}\,U''[R_{\mathrm{m}}]}{U'[R_{\mathrm{m}}]}$, can vary stochastically with R_{m}.

For all such stochastic discount factors, define a ϕ-approximation by: $U'[R_{\mathrm{m}};\phi] = \int_0^\infty e^{-\phi\,z\,R_{\mathrm{m}}}\,\nu(dz)$, which is just a functional arc approximation in the space of marginal utilities. Hence

$$q[R_{\mathrm{m}}] = \frac{p[R_{\mathrm{m}}] \times \int_0^\infty e^{-\phi\,z\,R_{\mathrm{m}}}\,\nu(dz)}{\int \int_0^\infty e^{-\phi\,z\,R_{\mathrm{m}}}\,\nu(dz)\,p[R_{\mathrm{m}}]\,dR_{\mathrm{m}}}. \qquad (16)$$

It then follows that: $\mathrm{SKEW}_{\mathrm{m}} \approx \overline{\mathrm{SKEW}}_{\mathrm{m}} - \{\phi \int_0^\infty z\,\nu(dz)\}\left(\overline{\mathrm{KURT}}_{\mathrm{m}} - 3\right)\overline{\mathrm{STD}}_{\mathrm{m}}$ (see the Appendix for intermediate steps). Even though risk aversion may no longer be time-invariant, the skew dynamics are still being determined by higher-order moments of the physical distribution. In particular, as we have shown, the even moments are being weighted by a constant proportional to risk aversion. This is the outcome, as the risk aversion dependence on the market is getting integrated out. In the more general case of state-dependent preferences, the skew dynamics can depend on conditional moves in risk aversion.

Depending on the nature of return autocorrelation, the risk-neutral skews may not aggregate linearly across the time spectrum. To develop this argument in some detail, suppose the one-period (say, weekly) rate of return follows an AR-1 process under the physical measure: $R_{\mathrm{m}}(t) = \rho R_{\mathrm{m}}(t-1) + u(t)$. As usual, let the white noise, $u(t)$, have zero mean with $|\rho| < 1$. Keep the higher moments, $\overline{\mathrm{STD}}_u(t)$, $\overline{\mathrm{SKEW}}_u(t)$ and $\overline{\mathrm{KURT}}_u(t)$, unspecified for now. Define the term structure of risk-neutral skews, $\mathrm{SKEW}_{\mathrm{m}}(t,\tau)$, as a function of τ. Using a standard logic, we determine (for $\tau = 1, 2, 3, \cdots, \infty$)

$$\overline{\mathrm{STD}}_{\mathrm{m}}(t,\tau) = \overline{\mathrm{STD}}_u(t) \times \sqrt{\frac{\tau - L_a[\rho,\tau]}{(1-\rho^2)^2}}, \tag{17}$$

$$\overline{\mathrm{SKEW}}_{\mathrm{m}}(t,\tau) = \overline{\mathrm{SKEW}}_u(t) \times \frac{\tau - L_b[\rho,\tau]}{(\tau - L_a[\rho,\tau])^{3/2}}, \quad \text{and} \tag{18}$$

$$\overline{\mathrm{KURT}}_{\mathrm{m}}(t,\tau) - 3 = \left(\overline{\mathrm{KURT}}_u(t) - 3\right) \times \frac{\tau - L_c[\rho,\tau]}{(\tau - L_a[\rho,\tau])^2}, \tag{19}$$

where $L_a[\rho,\tau] \equiv \frac{2\rho(1-\rho^\tau)}{1-\rho} - \frac{\rho^2(1-\rho^{2\tau})}{1-\rho^2}$, $L_b[\rho,\tau] \equiv \frac{3\rho(1-\rho^\tau)}{1-\rho} - \frac{3\rho^2(1-\rho^{2\tau})}{1-\rho^2} + \frac{\rho^3(1-\rho^{3\tau})}{1-\rho^3}$ and $L_c[\rho,\tau] \equiv \frac{4\rho(1-\rho^\tau)}{1-\rho} - \frac{6\rho^2(1-\rho^{2\tau})}{1-\rho^2} + \frac{4\rho^3(1-\rho^{3\tau})}{1-\rho^3} - \frac{\rho^4(1-\rho^{4\tau})}{1-\rho^4}$. By combining (17)-(19) with Theorem 2, several observations are apparent:

- When $\rho = 0$, $L_a = L_b = L_c = 0$. Therefore, $\mathrm{SKEW}_{\mathrm{m}}(t,\tau) = \frac{1}{\sqrt{\tau}}\overline{\mathrm{SKEW}}_u(t) - \frac{\gamma}{\sqrt{\tau}}(\overline{\mathrm{KURT}}_u(t) - 3)\,\overline{\mathrm{STD}}_u(t)$. As a result, absolute skews are declining like the square-root of maturity (irrespective of the distributional characteristics of u).

- With moderate levels of positive autocorrelation, the skew term structures display a U-shaped tendency: getting more negative with τ initially, and then gradually shrinking to zero with large τ. With $\rho < 0$, the term structure of skews bears the trait that short skews are always more negative than long skews. In either case, the presence of autocorrelation slows down the rate at which the central limit theorem is holding.

- If u is symmetric with kurtosis 3, the term structure of risk-neutral index skews is flat regardless of the nature of return dependency and risk taking behavior.

In summary, the preceding analysis integrates two insights about the term structure of skews. First, the part of skew that relies on risk aversion and fat-tailed distribution is more consistent with the daily/weekly frequency. Second, as the observation frequency is altered from weekly to monthly, the term structure of absolute skews can get upward-sloping even when $\overline{\text{KURT}}_{\text{m}}(t,\tau)$ approaches 3. Although not pursued here, higher-order autoregressive processes would lead to more flexible forms for absolute skew term structures.

One can take advantage of equation (13) to reverse-engineer an estimate of the risk aversion coefficient, and requires only simple inputs. To make the point precise, the risk-neutral index skew is recoverable from option positioning (5), and higher-moments of the physical distribution can be computed, with some sacrifice of quality, from the time-series of index returns. Informal as it may be, the reasonableness of the estimates can serve as an additional metric to assess conformance with theory. One such estimation strategy is discussed in the empirical section.

1.4 Skew Laws for Individual Stocks

To formalize the next aspect of the problem, assume that the individual stock return, $R_n(t,\tau)$, conforms with a generating process of the single-index type

$$R_n(t,\tau) = a_n(t,\tau) + b_n(t,\tau)\, R_{\text{m}}(t,\tau) + \varepsilon_n(t,\tau) \qquad n = 1,\cdots,N, \tag{20}$$

where $a_n(t,\tau)$ and $b_n(t,\tau)$ are scalers. Provided drift induced restrictions are placed on the parameters $a_n(t,\tau)$ and $b_n(t,\tau)$, the return process (20) is also well-defined under the risk-neutral measure. Presume that the unsystematic risk component $\varepsilon_n(t,\tau)$ has zero mean (whether risk-neutral or physical) and is independent of $R_{\text{m}}(t,\tau)$ for all t. Due to this property, the co-skews, $E\{\varepsilon_n(t,\tau)\,(R_{\text{m}}(t,\tau) - \mu_{\text{m}}(t,\tau))^2\}$ and $E\{\varepsilon_n^2(t,\tau)\,(R_{\text{m}}(t,\tau) - \mu_{\text{m}}(t,\tau))\}$, are zero. Impose the square-integrability conditions: $V_\varepsilon(t,\tau) \equiv e^{-r\tau}\int \varepsilon^2 q[\varepsilon]d\varepsilon < \infty$ and $V_{\text{m}}(t,\tau) \equiv e^{-r\tau}\int R_{\text{m}}^2 q[R_{\text{m}}]dR_{\text{m}} < \infty$, which bound the price of idiosyncratic volatility and index volatility. We can now state.

Theorem 3 *If stock returns follow the one-factor linear model displayed in (20), then*

(a) The price of individual skewness, denoted $SKEW_n(t,\tau)$, is linked to the price of market skewness, $SKEW_m(t,\tau)$, as stated below (for $n = 1,\cdots,N$):

$$SKEW_n(t,\tau) \;=\; \Psi_n(t,\tau)\, SKEW_m(t,\tau) + \Upsilon_n(t,\tau)\, SKEW_\varepsilon(t,\tau), \tag{21}$$

where $SKEW_\varepsilon(t,\tau)$ represents the skewness of ε; and

$$\Psi_n(t,\tau) \;\equiv\; \left(1 + \frac{V_\varepsilon(t,\tau)}{b_n^2(t,\tau)[V_m(t,\tau) - e^{-r\tau}\mu_m^2(t,\tau)]}\right)^{-3/2} \tag{22}$$

$$\Upsilon_n(t,\tau) \equiv \left(1 + \frac{b_n^2(t,\tau)[V_m(t,\tau) - e^{-r\tau}\mu_m^2(t,\tau)]}{V_\varepsilon(t,\tau)}\right)^{-3/2} \tag{23}$$

with $0 \leq \Psi_n(t,\tau) \leq 1$ *and* $0 \leq \Upsilon_n(t,\tau) \leq 1$.

(b) The individual skew will be less negative than the skew of the market

$$SKEW_n(t,\tau) > SKEW_m(t,\tau) \qquad n = 1, \cdots, N, \tag{24}$$

under the following conditions: (i) $\varepsilon_n(t,\tau)$ *belongs to a member of distributions that are symmetric around zero (i.e.,* $\mathcal{E}_t^*\{\varepsilon_n^3(t,\tau)\} = 0$*). In this case, the variation in the price of individual skewness can be bounded to be no more than that of the stock market index:* $0 \leq \frac{SKEW_{n(t,\tau)}}{SKEW_{m(t,\tau)}} \leq 1$*; or, (ii) the distribution of* $\varepsilon_n(t,\tau)$ *is positively-skewed.*

In (24), the risk-neutral index distribution is regarded as being left-skewed. The risk-neutral individual and index skews can be recovered from the option tracking portfolio (5).

Since the idiosyncratic return component requires no measure-change conversions, the skewness laws postulated in (21) will be obeyed under both the physical and risk neutral measures (with appropriate adjustments to $\Psi(t,\tau)$ and $\Upsilon(t,\tau)$). Either way, this statement of the theorem should not be interpreted to mean that individual return skewness will move in lockstep with market skewness. From equation (12), one can understand why total volatility matters for pricing derivatives even though the stochastic discount factor only prices systematic risk. This feature is reflected in the price of skewness, as the latter is merely a portfolio of options.

Two polar cases can shed light on the precise role of idiosyncratic skewness. **Case A**: $R_n(t,\tau) = a_n(t,\tau) + b_n(t,\tau)\, R_{\mathrm{m}}(t,\tau)$, accommodates a generating structure in which individual return is perfectly correlated with the stock market. When this is so, the risk-neutral skewness of the individual stock coincides with that of the market. **Case B**: $R_n(t,\tau) = a_n(t,\tau) + \varepsilon_n(t,\tau)$. In this setting, the stock contains no systematic component, and the sole source of individual skewness is the idiosyncratic skewness. In reality, the individual skewness will be partly influenced by market skewness, and partly by idiosyncratic skewness. In a later empirical exercise, we study the skew law implication that: $0 \leq \Psi_n(t,\tau) \leq 1$. We can also note that if restrictions (21)-(24) hold simultaneously and the market is heavily skewed to the left, then the idiosyncratic skews are bounded below and cannot be highly negative.

Even though not stressed in the theorem, more can be said about the character of individual risk-neutral distributions. Relying on the properties of variance operators and (20), first observe that: $V_n(t,\tau) = b_n^2(t,\tau)\, V_{\mathrm{m}}(t,\tau) + V_\varepsilon(t,\tau)$. Thus, provided the variance of the unsystematic factor is sufficiently well-behaved, the individual risk-neutral distributions will be inherently more volatile than the index. Next, as even moments are correlated in general, we can also expect individual stocks to display more leptokurtosis than the market (but this feature is not clear-cut analytically).

Due to the aforementioned, the differential pricing of index and individual equity options is likely. First, as expected, the less negative individual equity skew tempers the way all individual OTM puts are priced vis-a-vis all OTM calls. In particular, the skewness premium should get alleviated for individual stock options. Second, as individual stocks are more inclined to extreme moves than the market, the valuation of deep OTM calls/puts versus near the money calls/puts can be expected to diverge as well. These departures between the index and individual risk-neutral distributions will modify the structure of option prices (i.e, the smiles).

To get a flavor of the skew laws outside of the single-factor model, consider the return generating process: $R_n(t,\tau) = a_n(t,\tau) + b_n(t,\tau)\, R_{\mathrm{m}}(t,\tau) + c_n(t,\tau)\, F(t,\tau) + \varepsilon_n(t,\tau)$, which incorporates a systematic factor, F, besides the market index. Assume the independence of $R_{\mathrm{m}}(t,\tau)$, $F(t,\tau)$ and $\varepsilon_n(t,\tau)$. It can be shown that

$$\mathrm{SKEW}_n(t,\tau) = \Psi_n(t,\tau)\,\mathrm{SKEW}_{\mathrm{m}}(t,\tau) + \bar{\Psi}_n(t,\tau)\,\mathrm{SKEW}_F(t,\tau) + \Upsilon_n(t,\tau)\,\mathrm{SKEW}_{\varepsilon}(t,\tau) \qquad (25)$$

where $\Psi_n > 0$, $\bar{\Psi}_n > 0$ and $\Upsilon_n > 0$ are given in (54)-(56) of the Appendix. Two parametric cases are of special appeal. Suppose $\mathrm{SKEW}_F(t,\tau) < 0$. Then, as in the single-factor characterization, ε_n cannot be relatively far left-skewed with negative index skews. Next, when $\mathrm{SKEW}_F(t,\tau) > 0$, then $\mathrm{SKEW}_n(t,\tau) > \mathrm{SKEW}_{\mathrm{m}}(t,\tau)$ with symmetry of ε_n. Under the auxiliary assumption that each systematic factor contributes equally to the variance of R_n, we can see that $\mathrm{SKEW}_n(t,\tau) > (\mathrm{SKEW}_{\mathrm{m}}(t,\tau) + \mathrm{SKEW}_F(t,\tau))/2$, for all n and all τ.

Our framework is sufficiently versatile to recover co-skews between individual stocks and the market. To see how this can be accomplished using individual equity option prices, define, from Harvey and Siddique (2000), the risk-neutral co-skew as:

$$\mathrm{COSKEW}_n(t,\tau) \equiv \frac{\mathcal{E}_t^*\left\{(R_n(t,\tau) - \mathcal{E}_t^*[R_n(t,\tau)]) \times (R_{\mathrm{m}}(t,\tau) - \mathcal{E}_t^*[R_{\mathrm{m}}(t,\tau)])^2\right\}}{\left\{\mathcal{E}_t^*\,(R_n(t,\tau) - \mathcal{E}_t^*[R_n(t,\tau)])^2 \times \mathcal{E}_t^*\,(R_{\mathrm{m}}(t,\tau) - \mathcal{E}_t^*[R_{\mathrm{m}}(t,\tau)])^2\right\}^{1/2}} \qquad (26)$$

$$= b_n\,\mathrm{SKEW}_{\mathrm{m}}(t,\tau)\,\frac{e^{r\tau}V_{\mathrm{m}}(t,\tau) - \mu_{\mathrm{m}}^2(t,\tau)}{\sqrt{e^{r\tau}V_n(t,\tau) - \mu_n^2(t,\tau)}} \qquad n = 1, \cdots, N, \qquad (27)$$

from the single-factor assumption (20). As before, $V(t,\tau)$ and $\mu(t,\tau)$ are known from option positioning (7) and (40). However, recognize that $b_n(t,\tau)$ is a risk-neutralized parameter and can be estimated from individual equity option prices. The exact procedure is as follows. First, under the single-factor assumption, the call option price, $C_n(t,\tau)$, is

$$\begin{aligned} C_n(t,\tau) &= e^{-r\tau}\int_{S_n>K_n}(S_n(t,\tau)-K_n)\,q[S_n]\,dS_n \qquad\qquad n=1,\cdots,N, \\ &= e^{-r\tau}\,v_n(t)\,f_n(t,\tau;-i)\,\Pi_{1,n}(t,\tau)-K_n\,e^{-r\tau}\,\Pi_{2,n}(t,\tau) \end{aligned} \tag{28}$$

where $v(t)$, the characteristic function, $f_n(t,\tau;u)$, and the risk-neutral probabilities, $\Pi_{1,n}(t,\tau)$ and $\Pi_{2,n}(t,\tau)$ are presented in the Appendix. Second, if the process for (i) market index is consistent with, say, a general jump-diffusion with stochastic volatility (i.e., Bakshi, Cao, and Chen (1997), Bates (2000) and Pan (1999)), and (ii) the characteristic function for $\varepsilon_n(t,\tau)$ is exogenously specified, the individual option price can be computed in closed-form. As one can anticipate, this particularly parameterized option model is a function of a_n, b_n and the parameters governing the idiosyncratic component and the market index. Thus, each parameter can be estimated by minimizing the distance between the actual and the model determined option prices (according to some metric). We leave this application on co-skews to a future empirical examination.

Before closing this subsection, we need to bridge one remaining gap: Can the *leverage effect* reproduce risk-neutral skewness patterns, where the aggregate index is more negatively skewed than any individual stock. For this purpose, we parameterize, in the Appendix, a model in which stock returns and volatility correlate negatively at the individual stock level. In this setting, we demonstrate that the leverage effect does impart a negative skew to the individual stock and to the aggregate index. But its predictions for the skew magnitudes are sharply at odds with those asserted in Theorem 3. Specifically, leverage suggests that index skews will be *less* negative than some individual stocks. The model's implications for the joint behavior of risk-neutral and physical distributions are unknown, and outside our scope. When we comment on the empirical properties of individual/index skews, we will provide further elaboration. For a different strand of the leverage argument, the readers are referred to Toft and Pruyck (1997).

2 The Structure of Option Prices and Skewness/Kurtosis

We can now merge theoretical elements of the risk-neutral distributions of the market and the individual stocks on one hand, and the mapping that exists between the structure of option prices and the risk-neutral moments, on the other. As such, this formalizes the empirical framework for exploring the observed structure of option prices–individual equities or the stock market index.

To fix ideas, define the *implied volatility* as the volatility that equates the market price of the option to the Black-Scholes value. Accordingly, for risk-neutral density, $q[S]$, the implied volatility,

σ, is obtained by inverting the Black-Scholes formula

$$\int_{\Omega} e^{-r\tau}\,(K-S)^{+}\,q[S]\,dS \;=\; K e^{-r\tau}\,[1-\mathcal{N}(d_2)] - S(t)\,[1-\mathcal{N}(d_1)] \qquad (29)$$

where $d_1[y] = -\frac{\ln(y\,e^{-r\tau})}{\sigma\sqrt{\tau}} + \frac{1}{2}\sigma\sqrt{\tau}$, $d_2[y] = d_1[y] - \sigma\sqrt{\tau}$ and moneyness $y \equiv \frac{K}{S}$. Clearly, to know the implied volatility, one must know the form of the risk-neutral density $q[S]$ or the structure of option prices.

We will refer to the *implied volatility curves* as measuring the relation among put implied volatilities that differ only by their moneyness, going from deep out of money puts to deep in the money puts. For a fixed τ, write $\sigma[y;t,\tau]$ to reflect its dependence on y, and define the slope of the implied volatility curve as some notion of change in put implied volatility with change in moneyness. Intuitively, a flatter implied volatility curve implies that option prices of adjacent strikes are spaced closer, than far apart. The market perception of the price of jump risk is embedded in the evolution of the implied volatility curve (Rubinstein (1994) and Pan (1999)).

The following result–that relates the implied volatility function to the risk-neutral moments–is borrowed with some modification from Backus, Foresi, Lai, and Wu (1997). As in Longstaff (1995), it hinges on an approximate representation of any risk-neutral density in terms of the Gaussian.

Theorem 4 *Let $\sigma[y;t,\tau]$ denote Black-Scholes implied volatility (as recovered by solving (29)). Then, for a given moneyness, the implied volatility is affine in the risk-neutral moments that surrogate tail asymmetry and tail-size:*

$$\sigma_n[y;t,\tau] \;\approx\; \alpha_n[y] + \beta_n[y]\,SKEW_n(t,\tau) + \theta_n[y]\,KURT_n(t,\tau), \qquad n = 1,\cdots,N, \quad (30)$$

with the precise form of $\alpha[y]$, $\beta[y]$, and $\theta[y]$ analytically determinate. For a given (average) moneyness, the slope of the smile is affine in the same determinants.

The virtue of Theorem 4 is that it justifies the use of simple econometric specifications to analyze the relationship between the risk-neutral moments and the structure of option prices.[3] Theorem 4 is essentially a first-order approximation of individual implied volatility, at a given

[3] There are cases where one cannot uniquely identify the density from the knowledge of all the moments, including those for all powers above 4 (i.e., lognormal). Hence, (30) may not be true in general. We can, at best, deduce that the correct option price equals the Black-Scholes price plus other terms surrogating the price of higher risk-neutral moments. To get implied volatility, one has to pass through the inverse of the Black-Scholes formula, which does not apply additively. In fact, we will get an abstract mapping of the type: $\sigma[y;t,\tau] = \Lambda[y;V,\mathrm{SKEW},\mathrm{KURT}]$. We may then take a first-order approximation and attain (30). To emphasize reliance on one higher odd-moment and one higher even-moment, we have suppressed the dependence of α, β and θ on return volatility. As an empirical matter, we did not find smiles (its slope) to be strongly influenced by risk-neutral volatility. Its effect was already impounded in the denominators of skewness and kurtosis.

moneyness and maturity, in terms of higher-order risk-neutral moments of the individual risk-neutral density. As such, equation (30) is robust to a wide variety of specifications for the physical process of equity returns and the market price of risk. Hence, there is little economic content in the validity of equation (30); it just relates different statistics of the underlying risk-neutral density. Unlike equation (13) and (21), equation (30) is not a model of risk-neutral skews.

The basic intuition for the coefficients $\beta[y]$ and $\theta[y]$ is that firms with higher negative skew have greater implied volatility at low levels of moneyness, while firms with greater kurtosis have higher implied volatilities for both out of the money and in the money puts. With regard to the effect of higher-order moments on the shape of the implied volatility curve (at a fixed maturity), we note that skewness is a first-order effect relative to kurtosis, and a higher negative skew steepens the implied volatility curve. In contrast, kurtosis is a second-order effect that symmetrically affects out of money call and put option prices, and this should flatten the slope of the implied volatility curve controlling for skewness. If the skew variable is omitted, one would expect kurtosis to proxy for the first-order effect and therefore steepen the implied volatility curve.

The discussion of the previous section along with Theorem 4 suggest the following conjectures that can be empirically investigated:

Conjecture 1 The implied volatility curves are less negatively sloped for individual stock options than for stock index options.

Conjecture 2 The more negative the risk-neutral skewness, the steeper are the implied volatility slopes. The more fat-tailed the risk-neutral distribution, flatter are the smiles in the presence of skews.

Conjecture 3 Individual stock return (risk-neutral) distributions are, on average, less negatively skewed than that of the market. Granted, the physical distribution of the index is fat-tailed, the risk-neutral distribution of the index is generally left-skewed.

Conjecture 1 lays the foundation of the investigation - is it true, as commonly asserted, that the structure of individual option prices is flatter? Conjecture 2 associates the slope of the smile to the moments, dynamically in the time-series, as well as in the cross-section. Finally, Conjecture 3 directly follows from Theorem 3. The restriction it imposes on the price of individual skew relative to the price of market skew warrants an idiosyncratic return component that is not heavily left-skewed. These conjectures are interrelated. For instance, individual slopes are flatter than the market because individual stocks are less negatively-skewed. This implicitly requires index risk-neutral distributions to be left-displaced versions of the physical counterparts. Having consolidated the big picture in theory, we now pursue our empirical objectives in sufficient detail.

3 Description of Stock Options and Choices

The primary data used in this study is a triple panel (in the three dimensions of strike, maturity and underlying ticker) of bid-ask option quotes written on different 31 stocks/index, obtained from the Berkeley Options Database. The sample contains options on the S&P 100 index (the ticker OEX) and options on thirty largest stocks in the S&P 100 index. Each of these options are traded on the Chicago Board Options Exchange, and are American in style. For each day in the sample period of 1/1/91 through 12/31/95 (i.e., 1258 business days), only the last quote prior to 3:00 PM (CST) is retained.

For three reasons, we employ daily data to construct weekly estimates of our variables. First, the use of daily data minimizes the impact of outliers by allowing moments to be computed daily, and then averaged over the calendar week. Second, the estimation of the slope of the weekly smile for individual equity options requires daily data over the week so that there are sufficient observations to estimate the smile. Third, the daily risk-neutral index skews exhibit a Monday seasonality. This evaluation exercise is motivated by the analysis of Harvey and Siddique (1999), who have shown substantial day of the week effects in the estimates of physical skews. The exact procedure to build the time-series of the smile and its slope will be outlined shortly.

The requirement to sample options daily virtually limits the analysis to the largest 30 stocks by market capitalization. Even with the existing choice, the raw data contains over 1.4 million price quotes, and additional stocks would have made the empirical examination less manageable. We decided to include the largest stocks, as their stock options are likely to be more liquid than those in the middle or lower capitalization. The tickers and names of the individual stock options are displayed in the first two columns of Table 1. The set includes, among others, such actively traded and familiar stock options such as IBM, General Electric, Ford, General Motors, and Hewlett-Packard.

To be consistent with the existing literature, the data was screened to eliminate (i) bid-ask option pairs with missing quotes, or zero bids, and (ii) option prices violating arbitrage restrictions that $C(t,\tau;K) < S(t)$ or $C(t,\tau;K) > S(t)$ - PVD[D]- PVD[K], for present value function PVD[.] and dividends D. As longer (and very short) maturity stock option quotes may not be active, options with remaining days to expiration less than 9 days and greater than 120 days were also discarded. Finally, as in the money options are unnecessary to the construction of skewness/kurtosis tracking option portfolios, only OTM calls and puts are kept. As a result, for each date, t, the puts in the sample always have moneyness corresponding to: $\{\frac{K}{S(t)} \mid \frac{K}{S(t)} < 1\}$, and calls have moneyness corresponding to: $\{\frac{K}{S(t)} \mid \frac{K}{S(t)} > 1\}$.

Although each series for skewness and kurtosis (and implied volatility slopes) pertain to a constant τ, in practice, it is not possible to strictly observe these, as options are seldom issued daily with a constant maturity. Therefore, in our empirical exercises, if an OTM option has remaining days to expiration between 9 and 60 days, it is grouped in the *short*-term option category; if the remaining days to expiration falls between 61 and 120 days, the option is grouped in the *medium*-term category. Thus, only two classifications of smiles and option portfolios will be investigated: one short, and the other medium-term.

Table 1 reports the option price as the average of the bid and ask quotes, and the number of quotes, for both short-term and medium-term OTM calls and puts, respectively. The table also reports the weight of each stock in the OEX. As would be expected, the index has considerably more strikes quoted than individual stock options, with puts more active than calls. The number of OEX OTM puts exceed the OTM calls by a substantial margin, possibly reflecting strong demand for downside insurance. In the combined option sample, there are 358,851 OTM calls and puts.

Because each option under scrutiny has the potential for early exercise, the treatment of the smile is arguably controversial. To probe this issue, we also calculate the volatility that equates the observed option price to the American price. For estimating the price of the American option, we follow the procedure recommended by Broadie and Detemple (1996). We construct a binomial tree where the Black-Scholes price is substituted in the penultimate step. The American option price is estimated by extrapolating off the prices estimated from a 50 and 100 step trees, using Richardson extrapolation and accounts for lumpy dividends.[4] We then estimate two separate implied volatilities, as the volatility that equates the option price to the American and the Black-Scholes price, respectively. In the latter calculations, discounted dividends are subtracted from the spot stock price.

Table 2 compares the European and American implied volatilities. While presenting this comparison, three decisions are made for conciseness. First, options are divided into two moneyness intervals: [-10%,-5%) and [-5%,0), for calls and puts. Next, to conserve space, only the implied volatilities for a (randomly chosen) sample of ten stocks and the OEX are shown. Finally, we focus on the 1995 sample period, as averaging over the full five year sample narrows the differences even further. For the most part, the implied volatility curves tend to taper downward from deep out of money put options to at the money, and then moves slightly upwards as the call gets progressively out of money. Although the American option implied volatility (denoted AM) is smaller than the Black-Scholes (denoted BS) counterpart, this difference is negligible and within the bid ask

[4] In all computations, discrete dividends for each stock are collected from CRSP and are assumed known over the life of the option. For the S&P 100, the daily dividends are drawn from Standard & Poor's and converted to a dividend yield for each date maturity combination. Following a common practice, when the matching Eurodollar interest rate (datasource: Datastream) is unavailable, it is linearly interpolated.

spread. For example, in the -10% to -5% moneyness category for short-term puts, the maximum discrepancy is of the order of 0.10%. With the assurance that the bias from adopting BS implied volatility is small enough to be ignored, we adhere to convention and use only Black-Scholes smiles to surrogate the pricing structure of options.

4 Skewness and the Structure of Option Prices: Empirical Tests

This section establishes the differential pricing of individual stock options versus the market index, and empirically relates it to the asymmetry and the heaviness of the risk-neutral distributions. We also present a framework to study the empirical determinants of risk-neutral index skews.

4.1 Quantifying the Structure of Option Prices

To quantify the structure of option prices, we use options of maturity τ to estimate the model,

$$\ln(\sigma[y_j]) = \Pi_0 + \Pi \ln(y_j) + \epsilon_j, \qquad j = 1, \cdots, J, \tag{31}$$

across our sample of 30 stocks and the OEX, where, recall, $y = K/S$ is option moneyness (and deterministic). An advantage of the specification in equation (31) is its potential consistency with empirical implied volatility curves that are both decreasing and convex in moneyness (see, for example, Figure 1 in Dumas, Fleming, and Whaley (1998) and Figure 2 in Rubinstein (1994)). This suggests a Π less than 0. We interpret Π as a measure of the flatness of the implied volatility curve, and designate it as the sensitivity of the implied volatility curve to moneyness. In economic terms, a flatter implied volatility curve simply states that prices of put options of nearby strikes are closer, while those options that constitute a steeper curve have prices farther apart.

The model of equation (31) is estimated weekly, and then the estimated coefficients are pooled over all the weeks in the sample. Briefly, the procedure is as follows. Over each of the calendar days in the week, we index the available options by j, and estimate the said model by a least squares regression. Thus, for each stock, we estimate equation (31) for each of the 260 weeks for which sufficient data exists. Next, as in Fama-McBeth (1973), we time-average the regression coefficients (say, $\frac{1}{T}\sum_{t=1}^{T}\Pi(t)$), and compute the t-statistic by dividing the average slope coefficient by its standard error. The model is estimated using only OTM puts and calls. As ITM puts ($K/S > 1$) can be proxied by OTM calls, this is tantamount to using all the strikes in the cross-section of puts. The estimation is done for both short-term and medium-term options.

Table 3 reports the average slope of the implied volatility curve for each of the 30 stocks and the OEX. We also report the estimated at the money implied volatility as $\exp(\Pi_0)$. Consider, first, the results for short-term smiles. The average ATM implied volatility for the OEX is 14%, while the average ATM implied volatility over the 30 stocks is about 26%. With reference to the estimate of Π, we can make three observations. First, on average, Π is negative for all the individual stocks and the OEX. The slopes are all statistically significant, and the R^2 of the regression range from 26% (for MCQ) to 86% (for the OEX). Second, the slope for the OEX is much steeper than that for individual stocks. Compared to the short-term OEX slope of -4.42, the average slope over the 30 stocks is -1.02 (the difference between OEX and a representative individual implied volatility slope is almost seven standard deviations away).

The difference between the slopes translates into a substantial difference in pricing. For example, for the OEX, the slope of -4.42 indicates that the implied volatility of a 10% out of the money put ($y = 0.9$) will be 22% as compared with the ATM implied of 14%. In contrast, for the individual equity, the 10% OTM put will be priced at 29% as compared with the ATM implied of 26%. Finally, the table reports the statistic $\Pi < 0$, which is just a counting indicator for the number of weeks in which the slope of the implied volatility curve is negative. This statistic ranges from 71% in the case of IBM to 100% for the case of the OEX. Although over this sample period, the slope is always downward sloping for the index, it is not always so for individual equities. We also examined the slope of the smile in the yearly sub-samples, and still found index smiles to be much steeper than any individual equity smile. The regression findings from medium-term smiles are comparable and not reported.

As may be observed from Table 1, OTM puts are far more active than OTM calls for the OEX. To investigate the pricing differential between individual stocks and the OEX for OTM puts alone, we also estimated equation (31) by trimming the option data to include only OTM puts. The average (short-term) slope coefficient for the OEX is -5.00, as compared to an average of -2.04 over the 30 stocks. The conclusion from the one-sided smile is essentially the same - that the OTM puts are relatively more expensive than the ATM puts for the OEX than in the individual equity option markets.

In summary, two conclusions emerge. First, the slope of the OEX smile is persistently more negative than individual equity slopes. Second, unlike the OEX, the slopes of individual smiles are not always negative. Thus, OTM puts are consistently and substantially more expensive than OTM calls for the index. In contrast, the difference between OTM puts and OTM calls is smaller for the individual equity, and may, in fact, change signs. But, why are index smiles always downward sloping? What causes the slope of the individual smiles to reverse its sign? The differential pricing in the cross-section of strikes, and in the cross-section of stocks is puzzling.[5]

[5]To verify the results, we also model the implied volatility curve as quadratic in moneyness: $\sigma_j = \vartheta_0 + \vartheta_1(K_j/S -$

4.2 Explaining the Behavior of Options in the Stock Cross-section

Although, as in the previous subsection, it is possible to establish that the implied volatility curve is flatter for the individual equity than for the OEX, it is difficult to provide an economic rationale for the differential pricing of individual equity options. In this sub-section and the next, we investigate whether we can parsimoniously relate the structure of option prices to the respective risk-neutral moments, and, if so, what judgements can be drawn from the analysis.

Unlike the implied volatility curve, the risk-neutral moments are intrinsically unobservable. Here, we make use of our model-free characterizations in Theorem 1 to estimate each moment. Consider, as an example, the estimation of the skew. This requires first replicating the cubic contract in equation (8), and we do this by constructing positions in both OTM calls and puts that approximate the corresponding integral. The long position in the calls is discretized as,

$$\lim_{\overline{K}\to\infty} \sum_{j=1}^{\frac{\overline{K}-S(t)}{\Delta K}} w[S(t)+j\Delta K]\, C(t,\tau;S(t)+j\Delta K)\,\Delta K \qquad (32)$$

where $w[K] \equiv \frac{6\ln\left[\frac{K}{S(t)}\right]-3(\ln\left[\frac{K}{S(t)}\right])^2}{K^2}$, and the short position in the puts as

$$\sum_{j=1}^{\frac{S(t)-\Delta K}{\Delta K}} w[j\Delta K]\, P(t,\tau;j\Delta K)\,\Delta K \qquad (33)$$

where now $w[K] \equiv \frac{6\ln\left[\frac{S(t)}{K}\right]+3(\ln\left[\frac{S(t)}{K}\right])^2}{K^2}$. We similarly discretize and estimate the volatility contract and the quartic contract, and next, using the formulas in (5) and (6), we estimate the risk-neutral skewness and kurtosis. The moments are estimated daily, separately for both short-term and medium-term options.

Motivated by Theorem 4, we investigate whether a stock with a greater absolute skew has a steeper smile. This implication arises from the fact that the risk-neutral density can be approxi-

$1) + \vartheta_2\,(K_j/S-1)^2 + \epsilon_j$, $j = 1,\cdots,J$ (see Heynen (1994) and Dumas, Fleming, and Whaley (1998)). In this model, ϑ_1 can be viewed as the slope of the smile evaluated at $\frac{K}{S}=1$ (i.e., at the money slope), and ϑ_2 measures the curvature of the smile. In our implementations, we found that the ϑ_1 of the index is consistently more negative than the individual counterparts. In addition, the convexity parameter, ϑ_2, is persistently positive in the cross-section of stocks. Consequently, the quadratic specification has in common with its log predecessor the feature that the first-order (second-order) derivative of the implied volatility function with respect to moneyness is negative (positive). Overall, this segment of our investigation indicates that the relative flatness of individual smiles may not be vulnerable to smile measurement rules. Toft and Prucyk (1997) and Dennis and Mayhew (1999) adopt an alternative measure where the slope is standardized to impute the distance between the implied volatilities of a 10% in and out of money options, respectively. This measure of the implied volatility slope is particular to just two option strikes that are themselves almost two standard deviations out of the money for short-term options, and hence constitutes a crude measure of volatility skews.

mated from its higher moments. To this end, we estimate an ordinary least squares regression

$$\mathrm{SLOPE}_n(t,\tau) = \alpha + \beta\,\mathrm{SKEW}_n(t,\tau) + \theta\,\mathrm{KURT}_n(t,\tau) + \epsilon_n, \qquad n = 1, \cdots, N, \tag{34}$$

where the series for the slope of the smile are the weekly estimates of the coefficient Π obtained from regressing log implied volatility on log moneyness (the first-pass estimations are summarized in Table 3). As we have compiled weekly estimates of slopes and the corresponding moments for each of the 30 stocks, we estimate the cross-sectional regression weekly, for each of the 260 weeks in the period January, 1991 to December 1995. In so doing, we follow the standard procedure of averaging the estimated regression coefficients and their R^2. We report, in Table 4, results for both the multivariate regression, as well as the univariate regressions (with skewness/kurtosis as separate explanatory variables).

Irrespective of the sample period, and regardless of the maturity structure of options, the coefficient for skewness, β, is positive and statistically significant. Thus, as premised, each week, a more negatively skewed stock displays a steeper smile. Over the entire sample, the average coefficient for skewness is 1.45 (t-statistic of 55.88), while that of kurtosis is 0.46 (t-statistic of 16.54), for short-term smiles. Sub-period results for each year are consistent with those of the overall sample period: the estimate of β is in the range of 1.29 to 1.62, and θ in the range of 0.21 to 0.63. The results are stable across both option maturities, and the fit of the regression has an average R^2 of 51.37% for short-term options, and 56.29% for medium-term options.

To determine the individual explanatory powers of skewness and kurtosis, we performed two separate univariate regressions where we constrain $\theta \equiv 0$ and $\beta \equiv 0$, respectively. These restricted regressions support two additional findings. First, the cross-sectional behavior of equity options is primarily driven by the degree of asymmetry in the risk-neutral distributions; the average R^2 in the short-term univariate regression is 46.54% with skewness alone, as compared to 5.6% with kurtosis alone. Therefore, the model performance worsens substantially when a role for skews is omitted. We infer from this reduction in performance that the first-order effect on the implied volatility slopes is driven by risk-neutral skews. The second point to note is that although the sign on β remains unaltered between the restricted and the unrestricted regressions, the coefficient on kurtosis reverses sign and turns negative. Thus, consistent with our conjectures, in the presence (absence) of negative skewness, the kurtosis makes the smile flatter (steeper).

A possible explanation of the estimation results for θ is that a fatter tail is accompanied by a greater negative skew and a steeper smile, but that the marginal effect of kurtosis is to flatten the smile. Indeed, for our sample of individual stocks and the index, the average time-series correlation between (risk-neutral) skew and kurtosis is -0.48. Thus, the negative covariation between skew and kurtosis will downward bias θ when skewness is left uncontrolled in the estimation of equation (34). To examine the role of kurtosis separate from its covariation with the skew, we linearly project kurtosis onto skewness: $\text{KURT}(t,\tau) = a_0 + a_1\,\text{SKEW}(t,\tau) + \widehat{\text{KURT}}(t,\tau)$, and extracted the orthogonalized component of kurtosis, $\widehat{\text{KURT}}(t,\tau)$. Repeating the cross-sectional regression (34) for the entire 260 weeks, we get the following results for short-term options (all coefficients are significant):

$$\begin{aligned}\text{SLOPE}_n &= -0.81 + 1.29\,\text{SKEW}_n + 0.10\,\widehat{\text{KURT}}_n + \epsilon_n, \quad R^2 = 49.98\%, \text{ and}\\ \text{SLOPE}_n &= -1.11 + 0.12\,\widehat{\text{KURT}}_n + \epsilon_n, \quad R^2 = 5.91\%.\end{aligned}$$

As our evidence verifies, the orthogonal component of kurtosis also flattens the smile. This is also true across each of the annual sub-samples. Moreover, the average β of 1.29 is smaller than its Table 4 counterpart. To sum up, skewness does not completely subsume the effect of kurtosis (or its orthogonal component), and individual skew variation is responsible for explaining the bulk of the variation in the cross-section of individual equity option prices. We will provide an economic explanation for these results shortly.

4.3 Explaining Dynamic Variations in Individual Option Prices

We next research the link between the risk-neutral moments and the individual option prices in the time-series (suppressing dependence on τ in each entity):

$$\text{SLOPE}(t) = \alpha + \beta\,\text{SKEW}(t) + \theta\,\text{KURT}(t) + \delta\,\text{SLOPE}(t-1) + \epsilon(t), \qquad (35)$$

which involves a time-series regression of the slope of the smile on individual name risk-neutral skew and kurtosis. The inclusion of lagged $\text{SLOPE}(t-1)$ is necessary to correct for the serial correlation in the dependent variable, SLOPE. We also verified the coefficient estimates using the Cochrane-Orcutt methodology, which is essentially the same as putting in a lagged dependent variable (Hamilton (1994)). Therefore, all reported results are based on the OLS estimation of equation (35). To ensure robust inference, the standard errors are computed using the Newey-West heteroskedasticity and serial correlation consistent covariance matrix. The reported t-statistics are based on a lag length of 8. In our pre-trial estimations, we experimented with MA corrections up to 20 and found virtually similar results.

Panel A of Table 5 presents the unrestricted regression results for short-term options. For all stocks and the OEX, β is positive and statistically significant. Thus, as anticipated, the smile steepens when the risk-neutral skew becomes more negative from one week to another. The sensitivity of the slope to risk-neutral skewness is by far the highest for the OEX which has a β of 2.44, in contrast to a range of 0.35 to 1.42 for the individual stocks. The kurtosis coefficient, θ, is typically small and positive, with 21 significant t-statistics. As in the case of the cross-sectional regressions, an increase in risk-neutral kurtosis flattens the smile in the time-series as well. Again, the magnitude of 0.27 for the OEX is among the highest. Overall, all regressions appear to have a reasonable fit. For the OEX, the regression R^2 is 74.82%, and is as high as 74.85% for IBM. The serial correlation coefficient, δ, is positive and statistically significant (all names and the OEX).

To better appreciate the role of risk-neutral skew and kurtosis, two additional empirical tests are performed. First, we perform the restricted time-series regressions and examine the fit of each model (see Panel B of Table 5). For the vast majority of the stocks, risk-neutral skewness tracks the dynamic movements in the slope of the smile fairly well (on average, the R^2 is 55.40%). When kurtosis is included by itself in equation (35), there is some deterioration in model fit (on average, the R^2 is 36.57%). While not shown in a table, the key conclusions are unchanged when medium-term options are used instead. Therefore, as hypothesized, the tail-asymmetry and the tail-size of the risk-neutral distribution reflects itself in the asymmetry of the implied volatility curves.

Second, returning to Panel A of Table 5, we also present the likelihood ratio test statistic for the exclusion restriction that $\theta = 0$. As is standard (Hamilton (1994)), this statistic is distributed $\chi^2(1)$. A high value of the statistic indicates that the null hypothesis $\theta = 0$ is rejected. From the last two columns of Panel A of Table 5, we can observe that most of the χ^2 statistics are large in magnitude. In fact, 23 of the p-values are lower than 0.05 and only 6 p-values exceed 0.10. Based on this test, we can conclude that, even in the presence of negative skew, risk-neutral kurtosis is important in explaining dynamic movements in the slope of the smile. The marginal effect of omitting risk-neutral kurtosis is strongest for the market index.

One concern with the regression results that we just presented is that the slope of the smile as well as the risk-neutral moments are based on the same set of options (over a week). To verify that the results are not due to some spurious feedback, we perform integrity checks from two distinct angles. First, consistent with the term structure of risk-neutral skews, we regress the medium-term slope of the smile on short-term skewness: $\text{SLOPE}_{med}(t) = \hat{\alpha} + \hat{\beta}\,\text{SKEW}_{sh}(t) + \hat{\delta}\,\text{SLOPE}_{med}(t-1) + \epsilon(t)$, and, second, we regress the medium-term slope of the smile on lagged medium-term skewness: $\text{SLOPE}_{med}(t) = \tilde{\alpha} + \tilde{\beta}\,\text{SKEW}_{med}(t-1) + \tilde{\delta}\,\text{SLOPE}_{med}(t-1) + \epsilon(t)$. For each of these regressions, the slope and the risk-neutral skew are now estimated from a collection of option prices with no overlap. If the results of Table 5 are not spurious, then using either the lagged medium-term skew, or the short-term skew, as an instrumental variable for the medium-term skew should give qualitatively similar (albeit weaker) results. In the first candidate specification, the index and 22 of the 30 stocks show significant positive coefficients. For the second specification, all 30 stocks and the index show significant positive coefficients, with comparable goodness-of-fit R^2 statistics. Both set of regressions indicate that increasing the absolute magnitude of risk-neutral skewness makes puts more expensive relative to calls.

Equation (35) and equation (13), that relates risk-neutral index skews to physical index higher-moments via risk aversion, are part of the same underlying economic equilibrium, and may be combined. In the one-factor generating structure, one may view the implied volatility slope as reflecting risk-neutral index skews, with the idiosyncratic component providing a perturbation. On the other hand, one can potentially view the risk-neutral skew and kurtosis at the individual equity level as noisy proxies for the respective risk-neutral index moments. Following the derivation of Theorem 2, one may relate the risk-neutral index kurtosis to risk aversion and the physical moments by:

$$\mathrm{KURT_m} \approx \overline{\mathrm{KURT}}_\mathrm{m} - \gamma \left[2\,(\overline{\mathrm{KURT}}_\mathrm{m} + 2)\,\overline{\mathrm{SKEW}}_\mathrm{m} + \overline{\mathrm{PKEW}}_\mathrm{m}\right] \overline{\mathrm{STD}}_\mathrm{m}, \tag{36}$$

where $\overline{\mathrm{PKEW}}$ is the fifth (physical) moment normalized by the variance raised to the power 5/2; other physical moments are as previously defined. We thereby observe that individual name implied volatility curves are related, in a one-factor setting, to normalized physical moments up to order five. As already stated, if risk aversion is strong and there is considerable excess kurtosis, it leads to strong negative risk-neutral index skew. Consistent with this notion, as corroborated in Tables 4 and 5, risk-neutral skews accounts, to a first-order, for the observed steepness of the implied volatility curves. Conditional on negative risk-neutral skewness, the effect of risk-neutral kurtosis is of second-order, as reflected in the relative small magnitudes of θ.

Having captured the primary effect of risk aversion and fat-tailed physical distributions via risk-neutral skews in equation (35), the effect of kurtosis can be understood as addressing the symmetric move in the tails. If risk-neutral skewness is not controlled, then risk-neutral kurtosis proxies for the fundamental effect of risk-neutral skewness (especially noting the dominance of out of money puts relative to out of money calls in our sample). The findings in Tables 4 and Table 5 remain broadly consistent with the viewpoint that the primary action on the structure of option prices is aversion to market risk and the existence of fat-tailed physical distributions. It follows then, that to understand the relative structure of individual equity options and the market index, one must equivalently characterize their relative risk-neutral skews.

4.4 Skewness Patterns for Individual Stocks

Our goal here is to describe the empirical properties of the risk-neutral moments, and present the relationship that exists between the skew of the individual equities and the stock market index. Let us start with the average short-term skew for individual stocks and the OEX (shown in Table 6). In comparison with its 30 stock components, the OEX is substantially more negatively skewed with an average skewness of -1.09 (over the entire 1991-1995 sample). In contrast, the skewness of GE, HWP, and XRX are -0.53, -0.17, and -0.33, respectively. For each of the stocks, the difference between individual and OEX skews is statistically significant with a minimum t-statistic of 5.72 (not reported). We also incorporate estimates for (i) the fraction of weeks in which the individual skew is higher than the index skew (i.e., the occurrence frequency for the event $\text{SKEW}_n > \text{SKEW}_\text{m}$), and (ii) the fraction of weeks in which the individual/index skews are negative (that is, $\text{SKEW}_n < 0$). Together, these statistics again highlight the dichotomy between the market index and the individual stocks. Unlike any individual stock return distribution, the OEX risk-neutral distribution is persistently skewed to the left in each of the 260 weeks in the sample. Finally, on average across the 30 stocks, the individual skew is *less* negative than the market 89% of the times. Only occasionally do individual stocks have skews that are more negative than the OEX (respectively 13% and 2% for GE and IBM).

How do we interpret the fact that individual skews are almost always less negative than that of the market index? In light of the underlying theory, there are at least three compelling explanations. First, if there is indeed a market component in the individual return, then our characterizations indicate that the idiosyncratic return component is, most likely, *not* heavily negatively skewed. Second, if a market component is non-existent, then idiosyncratic skewness decides the skewness of the individual stock. In this hypothetical case, the small negative skew of the individual stock may simply reflect that of the idiosyncratic return component. However, amongst our sample, all stocks have a sizeable market component to its return - in a (weekly) regression of stock return on the market return (as in (20)), each stock has a significant $b(t, \tau)$. Thus, the less negative skew of the individual stock appears to be a symptom of an unsystematic return component that is either positive, symmetric, or mildly negatively skewed. Third, the leverage explanation implies that at least some stocks are more negatively skewed than the market index, which we do not empirically detect. While the feedback between return and volatility is sufficient to produce negative individual skews, it is inadequate for creating an index distribution that is overly left-skewed.

To isolate the contribution of systematic and idiosyncratic component skews to the individual return skew, consider the regression

$$\text{SKEW}_n(t) = \Psi_0 + \Psi_1 \, \text{SKEW}_{\text{m}}(t) + \epsilon(t). \tag{37}$$

In essence, this regression follows from the skew laws in equation (21) of Theorem 3, and assesses time-variations in the individual risk-neutral skew via time-variations in the risk-neutral market skew (the idiosyncratic skew is unidentified). The regression will be well-specified, for instance, if the relation between $V_m(t,\tau)$ and $V_n(t,\tau)$ (or equivalently $V_\varepsilon(t,\tau)$) is stable, so that the coefficient Ψ_1 can be assumed constant over this sample period. Again exploiting short-term options, Table 6 reports the results of this regression. The following observations can be made. First, each Ψ_1 that is significantly greater than zero at the 5% level is also significantly less than 1. This is broadly in line with our theory that the individual risk-neutral skew is a weighted combination of the risk-neutral market skew and the idiosyncratic skew, with weights that are bounded between 0 and 1. However, the coefficient Ψ_1 should not be interpreted as the coefficient of co-skewness (as defined in Harvey and Siddique (2000)). The latter captures the co-variation between the first moment in the individual names and the second moment of the market, as per equation (26). Equation (37), on the other hand, assesses the co-variation between third moments. For three stocks, the estimate of Ψ_1 is negative; however, these are not statistically significant.

Second, about a third of the stocks do not show a significant dynamic relation between the market and the individual skew. Even for the stocks that have a meaningful relation, the R^2 of the regression is small, with only three stocks having R^2 greater than 10% (GE, BMY and MMM). One possible interpretation of these results is that the time-variation in the idiosyncratic skew is more important than that of the market skew in determining the risk-neutral individual skew. Alternatively, the idiosyncratic skew may be directionally offsetting the negative market skew. Finally, the results for medium-term options are comparable (both quantitatively and quantitatively), with 21 of the 22 significant (at the 5% level) coefficients being positive and less than 1 (not reported here).[6]

[6]So far,we have not discussed the preciseness of our weekly estimates for risk-neutral return skew and kurtosis. How much of the cubic and the quartic contract price comes from outside of the available strike price range (say, $\pm 20\%$ range)? To see whether this area is negligible in general, let us compute the fourth moment in a (risk-neutral) Gaussian setting with standard deviation h (keeping r=0). The reader can verify that the area in the tail: $\frac{1}{h\sqrt{2\pi}} \int_{0.20}^{\infty} R^4 \exp[-R^2/2h^2] \, dR$, is relatively small (as a fraction of the total) for plausible values of h (over the short-term and medium-term horizons). Thus, despite the absence of a continuum of strikes (and our discretizations in (32)-(33)), the results with finite strikes appear reliable on a theoretical basis. One can, nonetheless, evaluate the maximum absolute deviation between the target cash flow and any proposed finite strike hedge. That is, taking a predetermined range for the possible values of the underlying stock returns, one can ascertain the quality of the hedge and therefore its adequacy. Since there are only a few three standard deviation events in our five-year sample, the available strikes deliver residuals that are not too large for the cubic and quartic contracts. In any case, the

The results of this section point to substantial differences in the risk-neutral distributions of individual stocks and the stock market index. While the volatility (see the price of volatility contracts in Table 6) of individual return distributions is greater than that of the index, the individual stock risk-neutral skew is less negative than the market skew. The price of individual kurtosis can be higher or lower than the market (the t-statistics are omitted, as all moments except one are statistically significant).

That the first two higher moments of the risk-neutral distribution of individual stocks can be so radically different from the index distribution has important implications. In particular, it indicates that we can make limited inference about the risk-neutral distribution of the individual stock by tracking only the risk-neutral distribution of the market. Although the single-factor model postulated in (20) is consistent with our findings, the nature of individual multivariate risk-neutral return distributions remains unresolved. Specifically, under what economic conditions can each marginal return distribution possess a low negative skew, and yet a portfolio represented by the market index be heavily left-skewed?

4.5 Determinants of Risk-Neutral Index Skews

In this final subsection, we test the market skew equation (13) using Hansen's (1982) generalized method of moments (GMM). Fix the horizon τ and define the disturbance, $\widehat{\epsilon}$, from Theorem 2, as:

$$\widehat{\epsilon}(t+1) \equiv \mathrm{SKEW}_{\mathrm{m}}(t+1) - \overline{\mathrm{SKEW}}_{\mathrm{m}}(t+1) + \gamma \left(\overline{\mathrm{KURT}}_{\mathrm{m}}(t+1) - 3 \right) \overline{\mathrm{STD}}_{\mathrm{m}}(t+1) \qquad (38)$$

where γ is the risk aversion parameter and $\overline{\mathrm{STD}}_{\mathrm{m}}(t+1)$, $\overline{\mathrm{SKEW}}_{\mathrm{m}}(t+1)$, and $\overline{\mathrm{KURT}}_{\mathrm{m}}(t+1)$ are the higher-order t+1-conditional moments of the physical index distribution. Equation (38) can be potentially viewed as a model for risk-neutral skews when $\widehat{\epsilon}(t+1)$ is independent of the physical moments. Allowing for possible dependences, we rely merely on the orthogonality of $\widehat{\epsilon}(t+1)$ with time-t determined instrumental variables, $\mathcal{Z}(t)$. Under the null hypothesis of a power utility stochastic discount factor (and those in the class of (15)) and identifying orthogonality conditions, we must have $E\{\widehat{\epsilon}(t+1) \otimes \mathcal{Z}(t)\} = 0$. As formalized in Hansen, the GMM estimator is based on minimizing the quadratic form: $\mathcal{G}_T' \mathcal{W}_T \mathcal{G}_T$, where $\mathcal{G}_T = \frac{1}{T} \sum_{t=1}^{T} \widehat{\epsilon}(t+1) \otimes \mathcal{Z}(t)$, and $\mathcal{W}_T$ represents a (positive-definite) weighting matrix.

As our intent is to estimate a single coefficient, γ, and test the restrictions embedded within (38), the GMM appears to be an attractive estimation method for several reasons. First, unlike return volatility, the estimates of physical skews and kurtosis requires a fairly long time-series,

commonality of our findings across the OEX (for which we have abundant strikes) and the individual stocks suggests that even a few strikes are reliable for mimicking skew and kurtosis. Our conclusions are, mostly, robust.

and will be measured with error (Merton (1980) and Harvey and Siddique (2000)). Therefore, the market skew formulation (38) is susceptible to an errors in variables problem. Second, compounding the situation, the return standard deviation and the excess kurtosis enter non-linearly in (38) and may be correlated with $\hat{\epsilon}$. Finally, the minimized GMM criterion function (multiplied by T), $\mathcal{J}_T$, offers a convenient approach to assess mis-specifications in (38). As is now well-recognized, the $\mathcal{J}_T$ statistic is chi-squared distributed with L-1 degrees of freedom (given L instruments).

Before turning to a discussion of GMM estimation results reported in Panels A and B of Table 7, some clarifications are in order. First, Theorem 2 applies for a particular τ. We therefore generate a series of risk-neutral index skews from options with maturities of 58 and 86 days. Second, estimates of physical skews and kurtosis are sensitive to the choice of histories. To provide a frame of reference, we experiment with moments estimated from OEX returns lagged by 350 days, 400 days, and 450 days (denoted as LAGS). All inputs into (38) are annualized for consistency. Over the 1988-1995 sample period (we have added three more years), there are, thus, 48 (32) matched observations for 58 day (86 day) index skews. Moreover, as theory offers little direction on the choice of instrumental variables to be used in the GMM estimation, three different sets were tried. Our SET 1 contains a constant and SKEW_{m} lagged once; SET 2 (SET 3) contains a constant and two (three) lags of SKEW_{m}. Each information set is picked to keep the number of orthogonality conditions manageable relative to the sample size.

Proceed now to the estimation results for 86 day skews (in Panel A). Supportive of Theorem 2 predictions, the estimate of γ are reasonable and in the range 1.76 and 2.26 for SET 1; in the range 1.99 to 2.29 for SET 2; and in the range 1.76 to 11.39 for SET 3. Each estimate of γ is statistically significant. One can also make the observation that as LAGS increase, the goodness-of-fit measure $\mathcal{J}_T$ generally falls. In fact, with LAGS set to 450, the overidentifying restrictions imposed by the model are not rejected (as reflected in the p-values higher than 5%). Otherwise, the model may be incomplete in that it has omitted higher-order terms in the first-order approximation. To appreciate the point that employing longer LAGS will possibly improve the quality of the estimation, notice that with LAGS set to 450 days, the *daily* (sample average) $\overline{\text{STD}}_{\text{m}} = 16.32\%$, $\overline{\text{SKEW}}_{\text{m}} = -1.26$, and $\overline{\text{KURT}}_{\text{m}} = 19.12$. In contrast, for LAGS set to 300 days, $\overline{\text{STD}}_{\text{m}} = 17.76\%$, $\overline{\text{SKEW}}_{\text{m}} = -0.96$, and $\overline{\text{KURT}}_{\text{m}} = 14.08$. With shorter LAGS, the skew and kurtosis may be underestimated.

If we choose $\gamma = 0$ in (38), it trivially imposes the constraint $\gamma_0 = 1$ in $\tilde{\epsilon}(t+1) \equiv \text{SKEW}_{\text{m}}(t+1) - \gamma_0\, \overline{\text{SKEW}}_{\text{m}}(t+1)$. Although ad-hoc, this alternative specification is empirically inspired: It can help evaluate the relation between the physical and the risk-neutral skews. As our GMM results demonstrate, the estimate of γ_0 is always more than 9.82 and significant. In other words, the statistical skews are too small and must be multiplied by a factor of least 10 to be consistent with risk-neutral index skews. This confirms our earlier claim that the risk-neutral skew magnitudes are not sustainable without risk aversion and fat-tailed physical index distributions.

The inferences that we have drawn are not too different with 58 day risk-neutral skews. Future work should extend the estimation methodology to include state-dependent stochastic discount factors. As risk aversion may be stochastically time-varying in that context, it may impose more stringent testable restrictions on the dynamics of risk-neutral index skews.

5 Concluding Remarks and Possible Extensions

It has been noted that higher-order risk-neutral moments like skewness and kurtosis influence the relative pricing of an option of a particular strike to that of another strike. But, basic questions like how to empirically quantify the relationship between the risk-neutral density and the moments of the physical equity return distribution have not been fully addressed. The central contributions of this article are summarized below:

- Theoretically, we reconcile when negative risk-neutral index skews are feasible from symmetric physical index distributions. For a large class of utility functions, our theory shows that risk-neutral index skews are a consequence of risk aversion and fat-tailed physical index distributions;

- We formalize the skew laws of individual equities, and propose a framework to recover risk-neutral moments from option prices. It is shown that the individual risk-neutral stock distributions are qualitatively distinct from the index counterpart. More typically, they possess a less pronounced negative skew and a more pronounced second moment;

- Empirically, we measure the structure of option prices through implied volatility slopes, and show systematic dynamic and cross-sectional variations in the volatility skews. In particular, we demonstrate the differential pricing of individual equity options: the volatility skews of individual stocks are much flatter than that of the market index. This finding is consistent with the idiosyncratic component of the return being less negatively skewed risk-neutrally than that market, with a possible implication being that only the systematic skew is priced;

- In large part, the empirical analysis suggests that when negative risk-neutral skew is internalized, a higher risk-neutral kurtosis produces a flatter volatility smile. A more negative risk-neutral skew is related to a steeper negative slope of the implied volatility curve.

Our framework allows us to understand and reconcile two stylized facts of economic significance: that the index option smile is highly skewed, and the differential pricing of individual equity options versus the market index. Overall, our findings remain consistent with the belief that the primary action on the structure of equity options is fat-tailed physical distributions and risk aversion. The econometric tests provide support for this economic argument.

The verdict is still out on a number of related research questions. First, future research should examine the nature of risk-neutral skews from other models. The class of models proposed by Grossman and Zhou (1996) with risk-averse portfolio insurers and price feedback might be a promising alternative, suggestive of generalizations to the marginal-utility tilting of the physical density studied in this paper. Another possibility is to study the interaction of biased beliefs and the pricing of puts and calls (David and Veronesi (1999)). Second, spanning the characteristic function with the option basis and then inferring the risk-neutral density is a natural extension to our work on moments. The resulting density has the convenience of using only out of money calls and puts in its construction, and can be used to integrate the estimation of objective and risk-neutral densities (Chernov and Ghysels (2000), Ferson, Heuson, and Su (1999) and Harvey and Siddique (2000)). At an abstract level, our approach of directly pricing risk-neutral moments from option portfolios can serve as a useful check in evaluating parametric methods for jointly estimating the physical and the risk-neutral densities. Third, a large body of literature (e.g., Canina and Figleswki (1993), Lamoureux and Lastrapes (1993) and Christensen and Prabhala (1998)) has attempted to determine whether at the money implied volatilities are unbiased predictors of future return volatility. Since we have designed option positioning to infer volatility, their forecasting exercises can be performed without taking any stand on the parametric option model, or on the form of the volatility risk premium. Finally, modeling the subtleties of individual stock options in various ways and testing their empirical performance might be a worthy objective. This study has provided the incentive to expand research on individual stock options.

Appendix

Proof of Theorem 1

Setting $\bar{S} \equiv S(t)$ in (2) and performing standard differentiation steps, we can observe that

$$H_{SS}[K] \;=\; \begin{cases} \frac{2(1-\ln[K/S(t)])}{K^2} & \textbf{Volatility Contract} \\ \frac{6\ln[K/S(t)]-3(\ln[K/S(t)])^2}{K^2} & \textbf{Cubic Contract} \\ \frac{12\,(\ln[S(t)/K])^2+4(\ln[S(t)/K])^3}{K^2} & \textbf{Quartic Contract.} \end{cases} \tag{39}$$

Equations (7)-(9) of Theorem 1 follow from substituting (39) into (2). For the mean stock return, we note that $\int_\Omega e^{-r\tau}\, S\, q[S]\, dS = S(t)$ (by the martingale property). Therefore,

$$\begin{aligned} e^{r\tau} &= \mathcal{E}_t^* \left\{ \frac{S(t+\tau)}{S(t)} \right\} = \mathcal{E}_t^* \left\{ \exp[R(t,\tau)] \right\} \\ &= 1 + \mathcal{E}_t^*[R(t,\tau)] + \frac{1}{2}\,\mathcal{E}_t^*[R(t,\tau)^2] + \frac{1}{6}\,\mathcal{E}_t^*[R(t,\tau)^3] + \frac{1}{24}\,\mathcal{E}_t^*[R(t,\tau)^4] \end{aligned}$$

since $\exp[R] = 1 + R + R^2/2 + R^3/6 + R^4/24 + o(R^4)$. Reorganizing,

$$\mu(t,\tau) \;\equiv\; \mathcal{E}_t^* \ln\left[\frac{S(t+\tau)}{S(t)}\right] = e^{r\tau} - 1 - \frac{e^{r\tau}}{2} V(t,\tau) - \frac{e^{r\tau}}{6} W(t,\tau) - \frac{e^{r\tau}}{24} X(t,\tau). \tag{40}$$

The final pricing formulas for risk-neutral skewness and kurtosis in equations (5) and (6) now follow by using (40), and expanding on their definitions. □

Proof of Theorem 2: Exponential Tilting of the Physical Measure can Introduce Skew in the Risk-Neutral Measure

We wish to relate the skewness of $q[R]$ to that of $p[R]$ (suppressing the subscript on R_{m}). Without loss of generality, we may suppose that the parent density, $p[R]$, has been mean-shifted and has zero mean (i.e., suppose $\overline{\kappa}_1 = 0$). Let the first three successive higher moments of $p[R]$ be

$$\overline{\kappa}_2 \;\equiv\; \int_{-\infty}^{\infty} R^2\, p[R]\, dR \tag{41}$$

$$\overline{\kappa}_3 \;\equiv\; \int_{-\infty}^{\infty} R^3\, p[R]\, dR \tag{42}$$

$$\overline{\kappa}_4 \;\equiv\; \int_{-\infty}^{\infty} R^4\, p[R]\, dR. \tag{43}$$

As is standard, define the moment generating function, $\overline{\mathcal{M}}[\lambda]$, of $p[R]$, for any real number λ, by

$$\overline{\mathcal{M}}[\lambda] \;\equiv\; \int_{-\infty}^{\infty} e^{\lambda R}\, p[R]\, dR$$

$$= \quad 1 + \frac{\lambda^2}{2}\,\overline{\kappa}_2 + \frac{\lambda^3}{6}\,\overline{\kappa}_3 + \frac{\lambda^4}{24}\,\overline{\kappa}_4 + o(\lambda^4), \tag{44}$$

and can, thus, be expressed in terms of its uncentered moments.

Now consider the moment generating function, $\mathcal{M}[\lambda]$, of $q[R]$. From the relation $q[R] = \frac{e^{-\gamma\,R} \times p[R]}{\int e^{-\gamma\,R} \times p[R]\,dR}$, it holds that

$$\begin{aligned} \mathcal{M}[\lambda] &\equiv \int_{-\infty}^{\infty} e^{\lambda R}\, q[R]\, dR \\ &= \frac{\int_{-\infty}^{\infty} e^{\lambda R}\, e^{-\gamma\,R}\, p[R]\, dR}{\int_{-\infty}^{\infty} e^{-\gamma\,R}\, p[R]\, dR} && (45) \\ &= \frac{\overline{\mathcal{M}}[\lambda-\gamma]}{\overline{\mathcal{M}}[-\gamma]}. && (46) \end{aligned}$$

Hence, $\mathcal{M}[\lambda]$ can be recovered from the (parent) moment generating function of $p[R]$.

Using the properties of moment generating functions, up to a first-order effect of γ, we see that the moments of $q[R]$ satisfy a recursive relationship. Whence, we have,

$$\kappa_1 \equiv \int_{-\infty}^{\infty} R\, q[R]\, dR \approx \overline{\kappa}_1 - \gamma\,\overline{\kappa}_2 \tag{47}$$

$$\kappa_2 \equiv \int_{-\infty}^{\infty} R^2\, q[R]\, dR \approx \overline{\kappa}_2 - \gamma\,\overline{\kappa}_3 \tag{48}$$

$$\kappa_3 \equiv \int_{-\infty}^{\infty} R^3\, q[R]\, dR \approx \overline{\kappa}_3 - \gamma\,\overline{\kappa}_4 \tag{49}$$

and $\overline{\mathcal{M}}[-\gamma] = 1 + \mathrm{o}[\gamma]$. Now we are ready to compute the risk-neutral index skew, which is,

$$\begin{aligned} \mathrm{SKEW}_{\mathrm{m}}(t,\tau) &\equiv \frac{\int_{-\infty}^{\infty} (R-\kappa_1)^3\, q[R]\, dR}{\left(\int_{-\infty}^{\infty} (R-\kappa_1)^2\, q[R]\, dR\right)^{3/2}}, \\ &= \frac{\overline{\kappa}_3 - \gamma\,(\overline{\kappa}_4 - 3\,\overline{\kappa}_2^2)}{\overline{\kappa}_2^{3/2}} + \mathrm{o}[\gamma]. && (50) \end{aligned}$$

Simplifying the resulting expression, and noting $\overline{\mathrm{KURT}} \times \overline{\kappa}_2^2 = \overline{\kappa}_4$, the theorem is proved.

For our generalization to marginal utilities in the class of $U'[R_{\mathrm{m}};\phi] = \int_0^\infty e^{-\phi\,z\,R_{\mathrm{m}}}\,\nu(dz)$, we can note that up to a first-order in ϕ that $\kappa_1 \approx \overline{\kappa}_1 - \{\phi \int_0^\infty z\,\nu(dz)\}\,\overline{\kappa}_2$, $\kappa_2 \approx \overline{\kappa}_2 - \{\phi \int_0^\infty z\,\nu(dz)\}\,\overline{\kappa}_3$, and $\kappa_3 \approx \overline{\kappa}_3 - \{\phi \int_0^\infty z\,\nu(dz)\}\,\overline{\kappa}_4$. From the same argument as in the derivation of (50), we have $\mathrm{SKEW}_{\mathrm{m}} \approx \overline{\mathrm{SKEW}}_{\mathrm{m}} - \{\phi \int_0^\infty z\,\nu(dz)\}\left(\overline{\mathrm{KURT}}_{\mathrm{m}} - 3\right)\overline{\mathrm{STD}}_{\mathrm{m}}$. $\square$

Proof of Part (a) and (b) of Theorem 3

Recall that the stock return follows a single-index return generating process. Suppressing time-arguments, write $R(t,\tau)$ as R and the risk-neutral density of stock return (idiosyncratic risk) as: $q[R]$ ($q[\varepsilon]$). The risk-neutral skewness of the index must be:

$$\mathrm{SKEW}_{\mathrm{m}}(t,\tau) \equiv \frac{\int_{-\infty}^{\infty}(R_{\mathrm{m}}-\mu_{\mathrm{m}})^3\, q[R_{\mathrm{m}}]\, dR_{\mathrm{m}}}{\left\{\int_{-\infty}^{\infty}(R_{\mathrm{m}}-\mu_{\mathrm{m}})^2\, q[R_{\mathrm{m}}]\, dR_{\mathrm{m}}\right\}^{3/2}}. \tag{51}$$

Exploiting the return generating process in (20), and using the independence of ε and R_{m}

$$\mathrm{SKEW}_n(t,\tau) = \frac{b_n^3 \int_{-\infty}^{\infty}(R_{\mathrm{m}}-\mu_{\mathrm{m}})^3\, q[R_{\mathrm{m}}]\, dR_{\mathrm{m}} + \int_{-\infty}^{\infty}\varepsilon_n^3\, q[\varepsilon]\, d\varepsilon}{\left\{b_n^2(t,\tau)\int_{-\infty}^{\infty}(R_{\mathrm{m}}-\mu_{\mathrm{m}})^2 q[R_{\mathrm{m}}]\, dR_{\mathrm{m}} + \int_{-\infty}^{\infty}\varepsilon_n^2\, q[\varepsilon]\, d\varepsilon\right\}^{3/2}} \tag{52}$$

since the co-skews, $E\{\varepsilon_n(R_{\mathrm{m}}-\mu_{\mathrm{m}})^2\}$ and $E\{\varepsilon_n^2(R_{\mathrm{m}}-\mu_{\mathrm{m}})\}$, vanish. Rearranging (52), we obtain (for $n=1,\cdots,N$)

$$\mathrm{SKEW}_n(t,\tau) = \Psi_n(t,\tau)\,\mathrm{SKEW}_{\mathrm{m}}(t,\tau) + \Upsilon_n(t,\tau)\,\frac{\mathcal{E}_t^*[\varepsilon_n(t,\tau)^3]}{\{\mathcal{E}_t^*[\varepsilon_n(t,\tau)^2]\}^{3/2}} \tag{53}$$

with $\Psi_n(t,\tau)$ and $\Upsilon_n(t,\tau)$ as displayed in (22)-(23) of the text. If the density $q[\varepsilon]$ is symmetric around origin, $\mathcal{E}_t^*[\varepsilon(t,\tau)^3]=0$. Inserting this restriction into (53) proves this element of the theorem.

This procedure can be extended to the two factor context: $R_n(t,\tau)=a_n(t,\tau)+b_n(t,\tau)\,R_{\mathrm{m}}(t,\tau)+c_n(t,\tau)\,F(t,\tau)+\varepsilon_n(t,\tau)$, which decomposes the systematic part of the individual return into two forces. Repeating the above steps, we derive (25) with

$$\Psi_n(t,\tau) \equiv \left(1+\frac{c_n^2(t,\tau)[V_F(t,\tau)-e^{-r\tau}\mu_F^2(t,\tau)]+V_\varepsilon(t,\tau)}{b_n^2(t,\tau)[V_{\mathrm{m}}(t,\tau)-e^{-r\tau}\mu_{\mathrm{m}}^2(t,\tau)]}\right)^{-3/2}, \tag{54}$$

$$\bar{\Psi}_n(t,\tau) \equiv \left(1+\frac{b_n^2(t,\tau)[V_{\mathrm{m}}(t,\tau)-e^{-r\tau}\mu_{\mathrm{m}}^2(t,\tau)]+V_\varepsilon(t,\tau)}{c_n^2(t,\tau)[V_F(t,\tau)-e^{-r\tau}\mu_F^2(t,\tau)]}\right)^{-3/2}, \quad \text{and} \tag{55}$$

$$\Upsilon_n(t,\tau) \equiv \left(1+\frac{b_n^2(t,\tau)[V_{\mathrm{m}}(t,\tau)-e^{-r\tau}\mu_{\mathrm{m}}^2(t,\tau)]+c_n^2(t,\tau)[V_F(t,\tau)-e^{-r\tau}\mu_F^2(t,\tau)]}{V_\varepsilon(t,\tau)}\right)^{-3/2} \tag{56}$$

which is the final step of the proof. □

Proof of the Individual Call Option Formula (28) under the Single-Factor Model Assumption

Define the characteristic function

$$f_n(t,\tau;u) \;=\; \mathcal{E}_t^*\left\{\exp\left[i\,u\,b_n\,\ln(S_{\mathrm{m}}(t+\tau))\right]\right\} \times \mathcal{E}_t^*\left\{\exp\left[i\,u\,\varepsilon_n(t+\tau)\right]\right\} \qquad n=1,\cdots,N \quad (57)$$

and the modified exercise region $\bar{K}_n \equiv \ln(K_n) - \ln(S_n(t)) - a_n + b_n\,\ln(S_{\mathrm{m}}(t))$. Decomposing the call option price into its constituent securities (from Bakshi and Madan (2000)), we have (28), where

$$v_n(t) \;\equiv\; e^{\ln(S_n(t))+a_n-b_n\,\ln(S_{\mathrm{m}}(t))}, \quad (58)$$

$$\Pi_{1,n}(t,\tau) \;=\; \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\,\bar{K}_n}\times f_n(t,\tau;u-i)}{i\,u\,f_n(t,\tau;-i)}\right]du \qquad \text{and,} \quad (59)$$

$$\Pi_{2,n}(t,\tau) \;=\; \frac{1}{2}+\frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{e^{-iu\,\bar{K}_n}\times f_n(t,\tau;u)}{i\,u}\right]du. \quad (60)$$

If the characteristic function for the market index and the idiosyncratic component is in closed-form, the individual call option price can be computed analytically. □

Proof that Leverage Implies Index Skew is Less Negative than Some Individual Skews

Before presenting the proof, we need a result on the moment generating function of vector standard normal variates, and its derivatives. That is: $\mathcal{E}^*\{\exp[\ell_1\,\zeta_1+\ell_2\,\zeta_2]\} = \exp[0.5\ell_1^2+0.5\ell_2^2+\eta\ell_1\ell_2]$, which is exponential affine in the variance-covariance matrix.

To stay focussed on this counterexample, we adopt a two-period and two-stock setting. Fix N=2, and hypothesize the two-period return evolution (with $\psi_n>0$)

$$R_n(1) \;=\; r+\zeta_n(1) \qquad \zeta_n \sim \mathcal{N}(0,1) \quad (61)$$

$$R_n(2) \;=\; r+\zeta_n(2)+\sigma[\zeta_n(1)]\,\varphi_n(2) \qquad \varphi_n \sim \mathcal{N}(0,1) \quad (62)$$

$$\sigma[\zeta_n(1)] \;=\; \psi_n\,\exp\left[-\zeta_n(1)\right] \quad (63)$$

for n=1,2. Equation (63) goes to the heart of the leverage argument: the volatility of the second period return increases (decreases) as lagged return innovations goes down (up) (see, Becker (1980) and Cox and Ross (1976)). Let $\zeta_n(t)$ be independent of $\varphi_n(2)$, $\eta \equiv \mathrm{Cov}_t\left(\zeta_1(t),\zeta_2(t)\right)$, and $\varrho \equiv \mathrm{Cov}_t\left(\varphi_1(t),\varphi_2(t)\right)$. Note that second-period volatilities are correlated across stocks and the individual return process is auto-correlated. This model yields

$$\text{SKEW}_n(2) = -\frac{6\,\psi_n^2\,\exp(2)}{(1+\psi_n^2\,\exp(2))^{3/2}} \qquad n = 1,2. \tag{64}$$

Therefore, leverage does produce negative skewness in individual names. Now cross-sectionally aggregate the second period return equally to get the return on the market (basket): $R_{\text{m}}(2) = \frac{R_1(2)+R_2(2)}{2}$. With some algebraic manipulation, we arrive at the *leverage implied* index skew:

$$\text{SKEW}_{\text{m}}(2) = \varsigma_0 + \sum_{n=1}^{2} \varsigma_n \times \text{SKEW}_n(2) \tag{65}$$

where

$$\varsigma_0 \equiv -\frac{6\,\varrho\,(1+\eta)\,\psi_1\,\psi_2\,\exp(1+\eta)}{2(1+\eta)+\psi_1^2\,\exp(2)+\psi_2^2\,\exp(2)+2\,\varrho\,\psi_1\,\psi_2\,\exp(1+\eta)}, \tag{66}$$

$$\varsigma_n \equiv \frac{0.50\,(1+\eta)(1+\psi_n\exp(2))}{2(1+\eta)+\psi_1^2\,\exp(2)+\psi_2^2\,\exp(2)+2\,\varrho\,\psi_1\,\psi_2\,\exp(1+\eta)} < 1, \qquad n = 1,2. \tag{67}$$

Thus the market skew is just a convex combination of the individual skews, and imposes the restriction that at least one of the individual skews be more negative than the market skew. To see this, set $\eta = 0$, and $\varrho = 0$. In this special case, ς_0 is identically zero. Now set $\varrho > 0$ and reexamine (65). In sum, while leverage generates negative skew, its implications for index skewness are diametrically opposite to those originating from risk aversion and fat-tailed physical distributions. □

Proof of Equation (30) in Theorem 4

Although the proof is available in Backus, Foresi, Lai, and Wu (1997), we sketch the basic steps to make our analysis self-contained. To justify the functional form (30), standardize stock returns so that they have mean zero and unit variance. Accordingly, let $x \equiv \frac{R(t,\tau)-\mu}{\bar{\sigma}}$, where, as before, $\mu \equiv \mathcal{E}_t^*[R(t,\tau)]$, and $\bar{\sigma} \equiv \sqrt{\mathcal{E}_t^*\{R(t,\tau)-\mathcal{E}_t^*[R(t,\tau)]\}^2}$. Now return to equation (29), and redefine the exercise region as: $\mathcal{K} \equiv \{\frac{\ln(K)-\ln(S(t))-\mu}{\bar{\sigma}} > x\}$. As a consequence

$$\int_{\mathcal{K}} e^{-r\tau}\,(K - S(t)\exp[x\bar{\sigma}+\mu])\,q[x]\,dx = K e^{-r\tau}\,[1-\mathcal{N}(d_2)] - S(t)\,[1-\mathcal{N}(d_1)]\,. \tag{68}$$

From probability theory, a robust class of density functions can be approximated in terms of its moments and the Gaussian density (see Johnson, Kotz, and Balakrishnan (1994, p. 25)), as in

$$q[x] \approx \Phi[x] - \frac{1}{3!}\frac{\partial^3\Phi[x]}{\partial x^3} \times \text{SKEW}(t,\tau) + \frac{1}{4!}\frac{\partial^4\Phi[x]}{\partial x^4} \times [\text{KURT}(t,\tau)-3], \tag{69}$$

where $\Phi[x] = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ denotes the standard normal density function. Thus, the left-hand side of (68) becomes:

$$\int_{\mathcal{K}} e^{-r\tau} (K - S(t) \exp[x\bar{\sigma} + \mu]) \; q[x] \, dx = \int_{\mathcal{K}} e^{-r\tau} (K - S(t) \exp[x\bar{\sigma} + \mu]) \; \Phi[x] \, dx -$$
$$\frac{1}{3!} \text{SKEW}(t, \tau) \times \int_{\mathcal{K}} e^{-r\tau} (K - S(t) \exp[x\bar{\sigma} + \mu]) \; \frac{\partial^3 \Phi[x]}{\partial x^3} \, dx +$$
$$\frac{1}{4!} [\text{KURT}(t, \tau) - 3] \times \int_{\mathcal{K}} e^{-r\tau} (K - S(t) \exp[x\bar{\sigma} + \mu]) \; \frac{\partial^4 \Phi[x]}{\partial x^4} \, dx, \tag{70}$$

which gives the theoretical put price linearly in terms of the Black-Scholes price (evaluated at the true volatility), the risk-neutral skewness and (excess) risk-neutral kurtosis.

Two remaining steps need some explanation. First, take a Taylor series of $\mathcal{N}(d_1)$ around $\bar{\sigma}$, and use Leibnitz differentiation rule to simplify the expression

$$K e^{-r\tau} [1 - \mathcal{N}(d_2)] - S(t) [1 - \mathcal{N}(d_1)] - \int_{\mathcal{K}} e^{-r\tau} (K - S(t) \exp[x\bar{\sigma} + \mu]) \; \Phi[x] \, dx. \tag{71}$$

Second, $\frac{\partial^3 \Phi[x]}{\partial x^3}$ and $\frac{\partial^4 \Phi[x]}{\partial x^4}$ can be directly computed by differentiating the normal density function. That is

$$\frac{\partial^3 \Phi[x]}{\partial x^3} = \frac{1}{\sqrt{2\pi}} (3x - x^3) \, e^{-x^2/2}$$
$$\frac{\partial^4 \Phi[x]}{\partial x^4} = \frac{1}{\sqrt{2\pi}} (3x - 6x^2 + x^4) \, e^{-x^2/2}.$$

Collecting the remaining terms, and exploiting the moment generating function of the Gaussian (i.e., its translates and derivatives), we achieve the desired result in (30). This result is, however, not observationally equivalent to the counterpart one (i.e., Proposition 2) in Backus, Foresi, Lai, and Wu (1997) (it is unnecessary to approximate $\alpha[y]$, $\beta[y]$, and $\theta[y]$). As the closed-forms for $\alpha[y]$, $\beta[y]$, and $\theta[y]$ are not particularly instructive, they are omitted here. This completes the proof that the structure of option prices, as represented through the Black-Scholes implied volatility curve, is affine in risk-neutral skewness and kurtosis (also see footnote 4 for further clarifications). □

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Table 1: Description of Out of Money Calls and Puts

The table reports the number of observations, and the mid-point price as the average of the bid-ask quotes, for short-term and medium-term out of money calls and puts for 30 stocks and the S&P 100. The ticker, name and the recent weight of the stock in the index (as of 5/98) are also reported. The call (put) is out of money if K/S > 1 (K/S < 1), where S denotes the contemporaneous stock price and K is the strike. Short-term options have remaining days to expiration between 9 and 60 days, and medium-term between 61 and 120 days. Only the last daily quote prior to 3:00 p.m. CST of each option contract are used in our calculations. The sample period extends from January 1, 1991 through December 31, 1995 for a total of 358,851 option quotes (162,046 calls and 196,805 puts).

			Number of Option Quotes				Mid-Point of Option Quote			
		OEX	Short		Medium		Short		Medium	
Ticker	Stock	Wgt.	Call	Put	Call	Put	Call	Put	Call	Put
1. AIG	American Int'l	2.32	3414	3884	1779	2471	1.26	0.99	2.34	1.47
2. AIT	Ameritech	1.24	1902	2199	1260	1570	0.62	0.58	0.96	0.89
3. AN	Amoco	1.09	2112	1942	1491	1435	0.49	0.48	0.79	0.76
4. AXP	American Express	1.27	2325	2367	1458	1696	0.41	0.40	0.66	0.60
5. BA	Boeing Company	1.27	2848	2624	1927	1896	0.56	0.48	0.93	0.71
6. BAC	BankAmerica Corp.	1.53	2640	3023	1576	2007	0.62	0.53	0.99	0.77
7. BEL	Bell Atlantic	1.90	2242	2335	1409	1600	0.47	0.47	0.71	0.74
8. BMY	Bristol-Myers	2.85	3040	3311	1927	2335	0.63	0.57	1.04	0.84
9. CCI	Citicorp	1.82	2545	2983	1512	2007	0.47	0.41	0.79	0.57
10. DD	Du Pont	2.33	2492	2639	1472	1731	0.57	0.53	0.98	0.79
11. DIS	Walt Disney Co.	2.04	4020	4677	2297	2905	1.06	0.87	2.00	1.40
12. F	Ford Motor	1.66	2924	3068	2062	2264	0.56	0.51	0.90	0.80
13. GE	General Electric	7.29	3323	4019	1857	2801	0.67	0.59	1.28	0.93
14. GM	General Motors	1.36	3021	3134	2107	2208	0.58	0.53	0.98	0.78
15. HWP	Hewlett-Packard	1.73	3973	5305	2168	3978	1.29	0.92	2.57	1.38
16. IBM	Int. Bus. Mach.	3.05	5605	4806	3514	2755	0.89	0.84	1.41	1.31
17. JNJ	Johnson & Johnson	2.48	2999	3256	1646	2148	0.81	0.70	1.40	1.00
18. KO	Coca Cola Co.	5.18	2438	3305	1450	2589	0.62	0.50	1.09	0.69
19. MCD	McDonald's Corp.	1.21	2321	2285	1443	1814	0.51	0.40	0.89	0.60
20. MCQ	MCI Comm.	0.99	2437	2311	1503	1508	0.46	0.44	0.74	0.65
21. MMM	Minn Mining	1.01	3532	3730	1946	2175	0.80	0.75	1.32	1.21
22. MOB	Mobil Corp.	1.63	2573	2618	1795	2232	0.71	0.67	1.15	1.00
23. MRK	Merck & Co.	3.75	3283	4163	1865	2639	0.98	0.83	1.69	1.31
24. NT	Northern Telecom	0.89	1916	1788	1213	1176	0.60	0.53	0.94	0.72
25. PEP	PepsiCo Inc.	1.65	2091	2459	1285	1695	0.40	0.36	0.65	0.51
26. SLB	Schlumberger Ltd	1.04	2965	2678	1670	1699	0.77	0.71	1.34	1.06
27. T	AT&T Corp.	2.64	2423	2607	1498	1783	0.45	0.36	0.73	0.50
28. WMT	Wal-Mart Stores	3.31	2539	2959	1868	2036	0.49	0.42	0.80	0.63
29. XON	Exxon Corp.	4.64	2364	2502	1375	1556	0.46	0.44	0.73	0.66
30. XRX	Xerox Corp.	0.89	3665	4615	1927	2921	1.23	0.94	2.13	1.43
31. OEX	S&P 100 Index		12793	22755	10981	16828	2.15	1.86	4.98	4.47

Table 3: Quantifying the Structure of Option Prices

For short-term and medium-term out of money options on 30 stocks and the S&P 100, the table displays the average coefficients for the specification,

$$\ln(\sigma_j) \quad = \quad \Pi_0 + \Pi \ln(y_j) + \epsilon_j \qquad\qquad j = 1, \cdots, J.$$

Here, σ is the Black-Scholes implied volatility of option with moneyness $y \equiv \frac{K}{S}$. The regression is estimated via OLS for each of the 260 weeks in the period of 1:1:91–12:31:95 in which there are a minimum of 8 observations, using out of money puts ($\frac{K}{S} < 1$) and out of money calls ($\frac{K}{S} > 1$). The table reports the estimated (i) at the money implied volatility corresponding to $K/S = 1$ as: $\exp(\Pi_0)$, (ii) the slope of the smile, Π, and (iii) the coefficient of determination, R^2 (in %), as the time-series average over all the weekly regressions (Fama-McBeth (1973)). The reported t-statistic is the time-series average, divided by the standard error (the standard deviation of the estimate normalized by the square-root of the number of estimates). The table also displays (in percentage) the fraction of the weekly estimates of the slope that satisfy $\Pi < 0$.

	Short-term Options						**Medium-term Options**					
	$\exp(\Pi_0)$		**Slope of the Smile,** Π		R^2	$\Pi < 0$	$\exp(\Pi_0)$		**Slope of the Smile,** Π		R^2	$\Pi < 0$
Ticker	Avg.	tstat	Avg.	tstat	Avg.	%	Avg.	tstat	Avg.	tstat	Avg.	%
1. AIG	0.22	124.44	-1.09	-23.95	43.57	97	0.22	119.84	-0.36	-9.21	38.27	76
2. AIT	0.19	109.38	-1.96	-20.26	55.41	96	0.22	108.39	-0.20	-3.28	31.98	57
3. AN	0.19	94.85	-0.96	-9.47	36.08	80	0.19	80.11	-0.79	-9.24	41.46	83
4. AXP	0.31	83.74	-0.26	-6.75	27.62	74	0.29	66.60	-0.70	-14.05	56.95	97
5. BA	0.27	99.59	-0.69	-13.89	33.29	80	0.24	96.42	-1.02	-26.39	73.28	97
6. BAC	0.30	79.40	-1.16	-27.88	56.81	95	0.28	93.15	-0.87	-30.62	74.77	98
7. BEL	0.21	92.78	-1.54	-17.26	48.12	86	0.21	89.94	-0.98	-17.22	59.01	95
8. BMY	0.21	99.26	-1.38	-18.67	46.55	89	0.20	93.39	-1.31	-23.18	71.59	98
9. CCI	0.35	59.92	-0.83	-21.47	42.32	90	0.31	58.59	-0.91	-19.12	75.82	97
10. DD	0.24	132.92	-0.86	-24.93	42.01	95	0.23	129.37	-0.60	-10.25	48.80	90
11. DIS	0.28	133.28	-0.91	-28.49	48.29	95	0.27	146.50	-0.72	-34.03	71.73	99
12. F	0.31	127.62	-0.62	-18.43	37.77	88	0.29	137.45	-0.58	-25.07	57.62	96
13. GE	0.21	107.70	-1.85	-38.86	61.02	99	0.21	117.81	-1.26	-36.17	81.16	98
14. GM	0.31	115.87	-0.52	-16.83	34.86	83	0.28	138.09	-0.70	-38.11	76.70	99
15. HWP	0.33	106.69	-0.83	-27.78	50.95	96	0.31	147.65	-0.50	-25.81	56.87	96
16. IBM	0.29	85.58	-0.36	-7.64	29.85	71	0.26	103.52	-0.61	-28.06	65.53	97
17. JNJ	0.24	93.99	-1.00	-21.27	41.70	93	0.22	96.00	-1.01	-28.84	69.70	99
18. KO	0.24	100.77	-1.62	-37.71	62.87	99	0.22	113.41	-1.17	-38.21	77.63	100
19. MCD	0.25	105.92	-1.16	-20.01	46.17	93	0.23	90.25	-1.26	-15.51	73.17	96
20. MCQ	0.34	101.47	-0.53	-10.48	26.34	74	0.32	81.57	-0.44	-4.26	49.62	68
21. MMM	0.21	135.99	-1.21	-13.42	42.65	90	0.19	151.90	-1.05	-27.48	67.55	96
22. MOB	0.19	117.06	-1.34	-24.87	44.17	94	0.19	124.79	-0.59	-14.69	44.92	80
23. MRK	0.27	88.15	-0.62	-9.14	38.67	72	0.24	98.47	-0.97	-30.19	76.17	98
24. NT	0.31	82.69	-0.31	-4.38	28.40	72	0.30	69.44	-0.37	-7.04	34.64	72
25. PEP	0.26	77.35	-1.13	-15.26	45.50	91	0.23	84.13	-1.27	-39.23	80.38	100
26. SLB	0.25	116.06	-0.54	-9.87	30.84	76	0.24	129.84	-0.29	-7.05	30.07	67
27. T	0.21	103.24	-1.44	-24.20	48.59	95	0.19	104.26	-1.56	-53.61	85.59	100
28. WMT	0.29	96.29	-0.95	-19.77	44.85	88	0.27	93.92	-0.86	-19.82	67.79	92
29. XON	0.17	114.84	-1.47	-21.97	41.97	91	0.16	107.16	-1.58	-24.39	75.58	99
30. XRX	0.26	132.35	-1.31	-33.60	55.73	98	0.25	162.13	-0.52	-16.55	49.87	88
31. OEX	0.14	82.23	-4.42	-70.34	86.08	100	0.14	84.31	-3.45	-79.67	93.82	100

Table 5: Variation in Individual Equity Option Prices Across Time

For each of the 30 stocks and the S&P 100, the table reports the results of a time-series regression: $\text{SLOPE}(t) = \alpha + \beta\,\text{SKEW}(t) + \theta\,\text{KURT}(t) + \delta\,\text{SLOPE}(t-1) + \epsilon(t)$, where SLOPE(t) is the (weekly) slope of the smile (i.e., the previously computed $\text{II}(t)$ in Table 3). SKEW(t) and KURT(t) are the *risk-neutral* skew and kurtosis for each of the 260 weeks in the sample period, 1:1:91–12:31:95. We include SLOPE($t-1$) to correct for the autocorrelation of the dependent variable. The method of estimation is OLS. The t-statistics are computed using the Newey-West (with a lag length of 8 weeks) methodology that corrects for heteroscedasticity and serial correlation. Standard errors with lag length up to 20 are virtually similar. R^2 is the coefficient of determination (in %). The reported $\chi^2(1)$ is the likelihood ratio test statistic for the null hypothesis that $\theta = 0$. The corresponding p-value is presented under the column "p-value." Only the results using *short-term* smiles are shown here.

Panel A: Unrestricted Regressions

Ticker	β	$t(\beta)$	θ	$t(\theta)$	δ	$t(\delta)$	R^2	**LR Test** $\chi^2(1)$	**LR Test** p-value
1. AIG	0.75	7.80	0.03	0.83	0.44	7.18	46.05	0.68	0.41
2. AIT	1.42	6.70	0.24	4.52	0.21	2.20	46.08	23.12	0.00
3. AN	0.73	6.05	-0.06	-2.23	0.21	3.81	44.64	23.12	0.00
4. AXP	0.35	3.59	0.01	0.95	0.33	3.13	43.76	2.00	0.16
5. BA	0.95	10.21	0.03	1.99	0.26	4.14	66.60	6.56	0.01
6. BAC	0.68	10.42	0.04	2.95	0.39	7.73	68.14	8.53	0.00
7. BEL	1.24	12.11	0.13	4.51	0.28	4.22	64.31	31.83	0.00
8. BMY	1.05	10.06	0.05	2.15	0.37	7.69	71.17	8.00	0.00
9. CCI	0.58	9.50	-0.01	-1.25	0.42	8.78	63.39	1.22	0.27
10. DD	0.57	9.26	0.04	3.04	0.30	4.60	43.80	11.27	0.00
11. DIS	0.52	6.58	0.04	2.47	0.52	10.18	55.87	6.13	0.01
12. F	0.41	5.97	0.02	2.31	0.45	8.25	57.68	7.25	0.01
13. GE	1.07	11.76	0.10	5.62	0.46	9.51	62.34	30.01	0.00
14. GM	0.49	6.90	-0.01	-0.79	0.46	9.22	58.91	0.97	0.33
15. HWP	0.47	8.10	0.06	2.57	0.52	8.54	61.50	10.02	0.00
16. IBM	0.80	6.47	-0.03	-1.12	0.53	8.77	74.85	2.57	0.11
17. JNJ	0.49	4.13	0.01	0.90	0.43	6.74	49.56	1.60	0.21
18. KO	0.86	10.30	0.08	4.57	0.27	4.55	58.65	37.00	0.00
19. MCD	0.87	11.10	0.09	4.62	0.29	3.81	57.54	36.67	0.00
20. MCQ	0.81	7.63	0.03	1.56	0.18	3.10	47.02	3.61	0.06
21. MMM	0.87	5.14	0.08	1.96	0.63	8.17	47.28	3.45	0.06
22. MOB	0.95	8.96	0.05	2.02	0.31	6.31	54.20	6.43	0.01
23. MRK	0.77	7.67	0.07	5.34	0.45	5.77	74.64	32.59	0.00
24. NT	0.58	5.74	0.04	1.53	0.27	6.04	48.61	5.10	0.02
25. PEP	0.78	9.65	0.04	3.43	0.28	6.02	58.67	13.61	0.00
26. SLB	0.95	10.43	0.10	4.25	0.27	7.64	60.76	22.21	0.00
27. T	1.00	10.03	0.08	4.85	0.27	5.09	71.32	33.58	0.00
28. WMT	0.59	13.24	0.04	2.88	0.43	6.65	66.34	11.09	0.00
29. XON	1.11	9.17	0.08	4.13	0.15	3.40	54.49	22.14	0.00
30. XRX	0.57	4.50	0.08	2.71	0.51	10.62	43.09	9.79	0.00
31. OEX	1.83	5.65	0.21	5.02	0.58	7.68	74.82	28.43	0.00

Panel B of Table 5: Restricted Regressions

	Restricted $\theta \equiv 0$ (Skewness Alone)					Restricted $\beta \equiv 0$ (Kurtosis Alone)				
Ticker	β	$t(\beta)$	δ	$t(\delta)$	R^2	θ	$t(\theta)$	δ	$t(\delta)$	R^2
1. AIG	0.71	7.22	0.44	7.12	45.91	-0.12	-3.03	0.53	8.69	32.52
2. AIT	0.77	4.19	0.29	3.28	34.23	-0.04	-1.39	0.37	4.19	15.19
3. AN	0.60	2.87	0.24	4.32	38.94	0.00	0.09	0.33	4.90	10.94
4. AXP	0.32	3.85	0.34	3.92	43.29	-0.02	-2.39	0.50	5.34	25.94
5. BA	0.88	9.16	0.27	4.28	65.74	-0.05	-2.36	0.54	7.39	33.50
6. BAC	0.58	10.32	0.39	8.23	67.06	-0.07	-4.93	0.57	11.05	46.62
7. BEL	0.84	10.70	0.29	4.38	59.44	-0.11	-5.32	0.44	5.99	36.56
8. BMY	0.91	7.17	0.37	7.56	70.26	-0.09	-3.29	0.61	10.58	53.23
9. CCI	0.59	10.85	0.43	8.76	63.21	-0.07	-5.31	0.57	10.16	45.56
10. DD	0.47	7.71	0.31	4.58	41.29	-0.01	-1.80	0.44	6.08	20.92
11. DIS	0.49	6.48	0.52	10.67	54.82	0.12	0.94	0.64	14.92	42.43
12. F	0.36	4.78	0.47	8.45	56.46	-0.01	-1.09	0.63	14.54	40.28
13. GE	0.74	7.87	0.49	9.83	57.72	-0.06	-3.30	0.58	11.82	38.70
14. GM	0.49	7.02	0.45	9.01	58.75	-0.03	-2.81	0.63	13.39	41.20
15. HWP	0.41	7.14	0.54	9.30	59.98	0.00	0.09	0.69	12.67	48.80
16. IBM	0.78	7.40	0.54	9.16	74.60	0.00	0.18	0.77	19.31	63.70
17. JNJ	0.45	4.58	0.44	6.99	49.25	-0.04	-2.74	0.58	10.88	37.57
18. KO	0.63	10.01	0.25	4.00	52.27	-0.02	-0.88	0.51	7.83	29.58
19. MCD	0.61	5.97	0.32	4.67	50.81	-0.04	-3.05	0.47	6.67	25.83
20. MCQ	0.77	7.33	0.19	3.15	46.27	-0.03	-1.32	0.35	4.32	13.21
21. MMM	0.70	5.78	0.65	8.73	46.57	-0.03	-1.22	0.66	9.11	42.26
22. MOB	0.87	8.83	0.32	6.38	53.27	-0.05	-2.53	0.48	7.80	24.36
23. MRK	0.53	8.35	0.58	11.36	71.19	-0.03	-1.80	0.77	13.77	58.16
24. NT	0.47	8.56	0.28	6.61	47.41	-0.07	-5.89	0.39	6.84	28.85
25. PEP	0.66	8.58	0.28	5.18	56.33	-0.04	-3.88	0.45	6.01	27.88
26. SLB	0.77	6.60	0.31	8.79	57.23	-0.07	-2.15	0.52	12.71	29.96
27. T	0.65	10.53	0.32	6.43	67.32	-0.07	-5.84	0.53	8.60	49.38
28. WMT	0.51	11.68	0.47	8.25	64.82	-0.03	-1.33	0.68	14.83	47.11
29. XON	0.85	7.33	0.17	3.65	50.40	-0.06	-4.24	0.35	6.95	21.31
30. XRX	0.42	3.70	0.51	10.18	40.90	-0.01	-0.59	0.57	12.13	33.34
31. OEX	0.69	5.38	0.64	9.97	71.90	-0.06	-3.82	0.76	14.57	68.78

Table 6: The Character of Individual and Index Risk-Neutral Skewness

For each of the 30 stocks and the S&P 100, the table reports three sets of numbers, relating to the weekly risk-neutral moments estimated. In the first two columns, we report (i) the percentage of observations for which $\mathrm{SKEW}_n < 0$, and (ii) the percentage of observations for which the risk-neutral skewness of the stock, SKEW_n, is *more* than the risk-neutral skewness of the market, $\mathrm{SKEW}_{\mathrm{m}}$ (i.e., less negative than the risk-neutral index skewness). The next five columns present the results of an OLS regression: $\mathrm{SKEW}_n(t) = \Psi_0 + \Psi_1\,\mathrm{SKEW}_{\mathrm{m}}(t) + \epsilon(t)$, where Ψ_0, Ψ_1 are the intercept and sensitivity coefficients, respectively; $t(\Psi_0), t(\Psi_1)$ are the t-statistics, and R^2 is the coefficient of determination (in %). The last three columns display the average estimate of the risk-neutral volatility, skew and kurtosis (with one exception, all moments are statistically significant and omitted). The volatility is the square-root of the variance contract, reported in %. All moments used are of *short-term* maturity. The sample period is 1:1:91–12:31:95.

	Sign of Skewness		Univariate Regression $\mathrm{SKEW}_n(t) = \Psi_0 + \Psi_1\,\mathrm{SKEW}_{\mathrm{m}}(t) + \epsilon(t)$					Price of Moments		
Ticker	$\mathrm{SKEW}_n < 0$	$\mathrm{SKEW}_n > \mathrm{SKEW}_{\mathrm{m}}$	Ψ_0	$t(\Psi_0)$	Ψ_1	$t(\Psi_1)$	R^2	$\sqrt{V}$	SKEW	KURT
1. AIG	68	96	0.11	1.43	0.29	4.46	7.15	7.98	-0.21	2.20
2. AIT	83	77	-0.54	-3.77	0.11	0.85	0.31	6.59	-0.65	4.18
3. AN	69	84	0.35	1.79	0.67	3.92	5.91	6.59	-0.38	5.00
4. AXP	48	90	-0.08	-0.50	0.04	0.28	0.03	10.93	-0.12	4.51
5. BA	58	94	0.17	1.65	0.29	3.21	3.83	9.16	-0.14	4.54
6. BAC	77	88	0.13	1.13	0.52	5.29	9.86	10.48	-0.44	3.99
7. BEL	79	72	-0.89	-4.86	-0.21	-1.28	0.65	7.09	-0.68	5.62
8. BMY	74	86	0.22	1.73	0.63	5.68	11.13	7.42	-0.46	4.46
9. CCI	69	92	-0.01	-0.06	0.25	2.98	3.34	12.71	-0.28	3.88
10. DD	69	92	-0.00	-0.04	0.24	2.77	2.89	8.39	-0.26	3.87
11. DIS	62	98	-0.09	-1.28	0.04	0.61	0.15	10.17	-0.13	3.18
12. F	58	93	0.05	0.39	0.16	1.55	0.93	11.02	-0.13	3.98
13. GE	88	87	-0.08	-0.97	0.41	5.37	10.04	7.60	-0.53	3.90
14. GM	56	95	-0.01	-0.15	0.07	1.00	0.39	11.07	-0.09	3.53
15. HWP	61	96	0.18	2.48	0.32	5.06	9.03	11.85	-0.17	2.33
16. IBM	43	98	0.27	3.92	0.20	3.47	4.47	10.49	0.04	2.89
17. JNJ	65	91	0.28	2.36	0.52	5.20	9.55	8.49	-0.30	4.12
18. KO	87	82	-0.21	-1.93	0.32	3.44	4.39	8.27	-0.56	4.48
19. MCD	71	85	-0.34	-2.22	0.07	0.51	0.10	8.51	-0.41	5.18
20. MCQ	53	91	-0.09	-0.81	0.05	0.48	0.09	12.18	-0.15	3.78
21. MMM	85	95	0.03	0.40	0.36	5.55	10.66	7.27	-0.36	3.28
22. MOB	77	88	-0.15	-1.43	0.22	2.54	2.43	6.47	-0.39	3.47
23. MRK	51	86	-0.43	-2.65	-0.24	-1.76	1.19	9.38	-0.16	4.41
24. NT	37	93	0.16	1.07	0.18	1.40	0.82	10.44	-0.04	4.03
25. PEP	72	87	-0.04	-0.26	0.33	2.78	2.98	8.67	-0.39	5.87
26. SLB	50	94	0.13	1.11	0.19	1.82	1.27	8.74	-0.07	3.09
27. T	78	76	-0.76	-4.75	-0.14	-1.01	0.39	7.28	-0.61	6.10
28. WMT	70	88	0.20	1.53	0.53	4.78	8.23	10.34	-0.38	4.18
29. XON	83	82	-0.25	-1.42	0.31	2.07	1.64	5.93	-0.58	5.49
30. XRX	77	93	-0.09	-1.19	0.22	3.40	4.29	9.27	-0.33	2.50
31. OEX	100							5.56	-1.09	3.99

Table 7: GMM Tests of the Market Skew Equation

Consider the restrictions imposed by the power utility pricing kernel: $\widehat{\epsilon}(t+1) \equiv \text{SKEW}_{\text{m}}(t+1) - \overline{\text{SKEW}}_{\text{m}}(t+1) + \gamma\left(\overline{\text{KURT}}_{\text{m}}(t+1) - 3\right)\overline{\text{STD}}_{\text{m}}(t+1)$, which is another way to express (13) of Theorem 2. The risk aversion parameter, γ, is estimated by generalized method of moments (GMM), as described in the text. In Panel A and Panel B, we report the GMM results when the risk-neutral market skew, SKEW_{m}, is recovered from 86 day and 58 day options, respectively. Over the entire sample of January 1988 through December 1995, there are, thus, 32 (48) non-overlapping observations for 86 (58) day options. We build the time-series of higher-order physical return moments, $\overline{\text{STD}}$, $\overline{\text{SKEW}}$, and $\overline{\text{KURT}}$, from daily returns on the OEX. Thus, a lag length (denoted LAGS) of 350 days means that we go backward 350 days to construct the moments. For consistency, each variable has been annualized. The degrees of freedom, df, are the number of instruments, $\mathcal{Z}(t)$, minus one. In **SET 1**, the instrumental variables are a constant plus $\text{SKEW}_{\text{m}}(t)$. Likewise, **SET 2** (**SET 3**), contains SET 1 (SET 2) plus $\text{SKEW}_{\text{m}}(t-1)$ ($\text{SKEW}_{\text{m}}(t-2)$). For robustness, other information sets were tried; they yielded similar implications. The minimized value (multiplied by T) of the GMM criterion function, $\mathcal{J}_T$, is chi-squared distributed with degrees of freedom, df. The impact of physical skews on risk-neutral skews is studied by considering the ad-hoc specification: $\tilde{\epsilon}(t+1) \equiv \text{SKEW}_{\text{m}}(t+1) - \gamma_0 \overline{\text{SKEW}}_{\text{m}}(t+1)$.

Panel A: Risk-Neutral OEX Skews from 86 Day Options

			$E\{\widehat{\epsilon}(t+1)\otimes\mathcal{Z}(t)\}=0$				$E\{\tilde{\epsilon}(t+1)\otimes\mathcal{Z}(t)\}=0$			
$\mathcal{Z}(t)$	LAGS	df	γ	t(γ)	$\mathcal{J}_T$	p-value	γ_0	t(γ_0)	$\mathcal{J}_T$	p-value
SET 1	350	1	2.26	2.11	7.60	0.005	12.01	4.46	7.33	0.006
	400	1	2.08	2.32	4.69	0.030	11.20	2.93	4.87	0.027
	450	1	1.76	2.48	3.77	0.052	9.82	3.09	3.99	0.045
SET 2	350	2	2.29	1.97	10.93	0.004	15.99	2.66	8.86	0.011
	400	2	2.25	2.22	6.96	0.030	12.08	2.85	6.52	0.038
	450	2	1.99	2.40	4.26	0.118	10.85	3.01	4.52	0.104
SET 3	350	3	11.39	2.67	7.01	0.071	22.32	2.78	7.44	0.059
	400	3	1.76	2.16	11.15	0.010	20.23	2.95	6.17	0.103
	450	3	1.89	2.35	6.70	0.082	11.52	2.97	5.59	0.133

Panel B: Risk-Neutral OEX Skews from 58 Day Options

			$E\{\widehat{\epsilon}(t+1)\otimes\mathcal{Z}(t)\}=0$				$E\{\tilde{\epsilon}(t+1)\otimes\mathcal{Z}(t)\}=0$			
$\mathcal{Z}(t)$	LAGS	df	γ	t(γ)	$\mathcal{J}_T$	p-value	γ_0	t(γ_0)	$\mathcal{J}_T$	p-value
SET 1	350	1	2.09	2.64	13.97	0.000	11.95	3.63	8.90	0.000
	400	1	1.91	2.80	7.81	0.005	9.35	3.64	6.66	0.009
	450	1	1.36	3.05	8.90	0.052	7.25	3.99	7.77	0.005
SET 2	350	2	3.21	2.60	14.63	0.000	16.78	3.76	7.53	0.023
	400	2	2.12	2.67	14.23	0.000	12.29	3.67	8.56	0.013
	450	2	1.44	2.93	11.20	0.003	8.01	3.85	9.04	0.010
SET 3	350	3	5.98	2.66	9.48	0.023	20.87	3.90	5.66	0.129
	400	3	2.60	2.60	16.95	0.000	16.51	3.77	7.65	0.053
	450	3	1.59	2.89	11.40	0.009	8.84	3.78	8.75	0.032

Table 2: Black-Scholes Implied Volatilities versus American Option Implied Volatilities

For a sample of 10 stocks and the OEX, the table reports the Black-Scholes (denoted BS) and the American (denoted AM) option implied volatilities, obtained by inverting the Black-Scholes and the American option price, respectively. The American option price is estimated by Richardson extrapolation of a 50 and 100 step binomial trees, accounting for lumpy dividends (see Broadie and Detemple (1996)). The implied volatilities of individual options are then averaged within each moneyness-maturity category and across days. Two categories of out of money options are used corresponding to the intervals [-10%,-5%) and [-5%,0). Short-term options have remaining days to expiration between 9 and 60 days, and medium term between 61 and 120 days. All numbers correspond to the period of January 1, 1995 through December 31, 1995. As out of money call options have the same implieds as in the money puts at a given moneyness level, the four columns representing the implieds of out of money puts and calls may be also viewed as the entire smile ranging from out of money puts to in the money puts.

	Short-term Options								Medium-term Options							
	OTM Puts				OTM Calls				OTM Puts				OTM Calls			
	-10% to -5%		-5% to 0%		0% to -5%		-5% to -10%		-10% to -5%		-5% to 0%		0% to -5%		-5% to -10%	
Ticker	**BS**	**AM**	**BS**	**AM**	**BS**	**AM**	**BS**	**AM**	**BS**	**AM**	**BS**	**AM**	**BS**	**AM**	**BS**	**AM**
AIG	21.59	21.51	19.98	19.79	18.86	18.86	19.13	19.13	19.88	19.67	19.84	19.38	18.41	18.41	18.46	18.46
BA	26.73	26.64	23.61	23.41	21.78	21.77	22.87	22.87	24.50	24.28	23.39	23.02	21.93	21.90	20.89	20.88
DIS	26.40	26.31	23.97	23.76	22.79	22.79	24.04	24.05	25.54	25.33	24.33	23.93	22.68	22.68	22.32	22.32
GE	22.78	22.70	20.04	19.85	17.79	17.79	18.48	18.48	20.53	20.33	20.11	19.75	17.10	17.09	16.50	16.49
GM	27.20	27.10	25.38	25.17	25.46	25.16	26.24	26.07	26.23	25.95	25.83	25.35	25.49	25.46	25.33	25.33
HWP	34.26	34.16	33.11	32.89	31.64	31.64	33.23	33.23	33.42	33.12	33.31	32.82	32.09	32.09	33.19	33.19
IBM	28.76	28.67	27.24	27.03	25.88	25.88	26.71	26.71	26.59	26.33	26.42	25.94	24.95	24.95	24.80	24.80
JNJ	22.45	22.37	20.22	20.03	18.64	18.60	19.72	19.70	20.74	20.52	20.13	19.67	17.78	17.78	17.64	17.64
MMM	21.35	21.26	20.19	20.00	18.53	18.25	19.81	19.74	19.27	19.03	18.65	18.19	16.71	16.71	17.62	17.62
XRX	26.32	26.24	24.78	24.58	23.09	23.09	22.97	22.98	24.69	24.43	24.68	24.20	22.18	22.18	21.72	21.72
OEX	18.52	18.49	13.45	13.36	10.72	10.72	10.76	10.76	16.01	15.93	13.31	13.10	10.73	10.73	10.08	10.08

Table 4: Structure of Option Prices and Moments in the Stock Cross-section

For short-term and medium-term options on 30 stocks and the S&P 100, the table reports the average coefficients of weekly cross-sectional regression,

$$\text{SLOPE}_n = \alpha + \beta\,\text{SKEW}_n + \theta\,\text{KURT}_n + \epsilon_n \qquad n = 1, \cdots, N(t),$$

where, corresponding to stock n, SLOPE_n is the slope of the smile, Π_n, as described in Table 3, and SKEW_n, KURT_n are the risk-neutral skews and kurtosis, respectively. For each day in the week, the risk-neutral skewness and kurtosis are estimated from the cross-section of out of money calls and puts (as in Theorem 1). The weekly estimate is then derived as the time-average of the daily estimates. The regression is estimated by OLS for each week in the sample period. The table reports the coefficients and the coefficient of determination (R^2, in %), as the time-series average over all the weekly regressions (as in Fama-McBeth (1973)) for both restricted and unrestricted regressions. The reported t-statistic is the time-series average divided by the time-series standard error. $N(t)$ is the number of stocks in the cross-section in week t. Each row of the table shows the results for a specific maturity (short or medium) and time-period. "Full" refers to the entire period from January 1, 1991 through December 31, 1995.

	Unrestricted Regression							**Restricted $\theta \equiv 0$ (Skewness Alone)**					**Restricted $\beta \equiv 0$ (Kurtosis Alone)**				
	α		β		θ		R^2	α		β		R^2	α		θ		R^2
Year	Avg.	tstat	Avg.	tstat	Avg.	tstat	Avg.	Avg.	tstat	Avg.	tstat	Avg.	Avg.	tstat	Avg.	tstat	Avg.
Short-Full	-1.56	-29.66	1.45	55.88	0.46	16.54	51.37	-0.75	-72.28	1.26	46.39	46.54	-0.79	-30.42	-0.08	-12.72	5.60
Short-1991	-1.56	-14.39	1.29	26.68	0.48	8.08	49.70	-0.77	-35.05	1.11	19.17	43.20	-0.98	-17.98	-0.02	-1.86	3.70
Short-1992	-1.49	-14.37	1.41	28.28	0.42	7.30	52.08	-0.76	-32.18	1.23	22.46	47.60	-0.76	-14.19	-0.10	-6.87	6.79
Short-1993	-1.83	-15.21	1.62	26.42	0.63	9.75	52.84	-0.72	-38.38	1.40	22.60	48.03	-0.81	-15.74	-0.08	-5.66	3.26
Short-1994	-1.79	-17.52	1.55	27.54	0.58	10.75	54.83	-0.73	-34.22	1.30	23.29	49.12	-0.80	-16.33	-0.08	-6.55	4.98
Short-1995	-1.12	-8.78	1.37	22.05	0.21	3.32	47.39	-0.74	-26.08	1.24	18.62	44.74	-0.61	-8.65	-0.14	-8.42	9.27
Med.-Full	-1.56	-24.53	1.04	57.63	0.55	15.59	56.29	-0.57	-70.37	1.01	52.09	48.12	-1.13	-27.38	0.07	4.79	9.25
Med.-1991	-1.87	-16.93	0.88	38.64	0.74	11.90	60.48	-0.57	-44.42	0.81	25.23	47.00	-1.18	-17.36	0.10	4.64	7.56
Med.-1992	-2.17	-20.67	0.92	26.04	0.88	15.11	58.54	-0.57	-39.78	0.92	27.55	44.59	-1.42	-16.90	0.18	5.97	14.28
Med.-1993	-1.41	-12.58	1.10	30.68	0.47	7.64	54.35	-0.53	-41.03	1.10	28.17	50.18	-1.25	-13.71	0.10	3.84	5.47
Med.-1994	-1.59	-9.37	1.28	41.29	0.54	5.72	59.73	-0.59	-47.36	1.26	36.22	53.79	-1.10	-9.00	0.05	1.17	9.63
Med.-1995	-0.78	-5.97	1.03	21.25	0.12	1.66	48.37	-0.60	-19.74	0.96	20.02	45.03	-0.72	-12.37	-0.10	-7.69	9.32