Complexity, Parallel Computation and Statistical Physics

Jon Machta

Measures of Complexity workshop
Santa Fe Institute
January 13, 2011

SANTA FE INSTITUTE

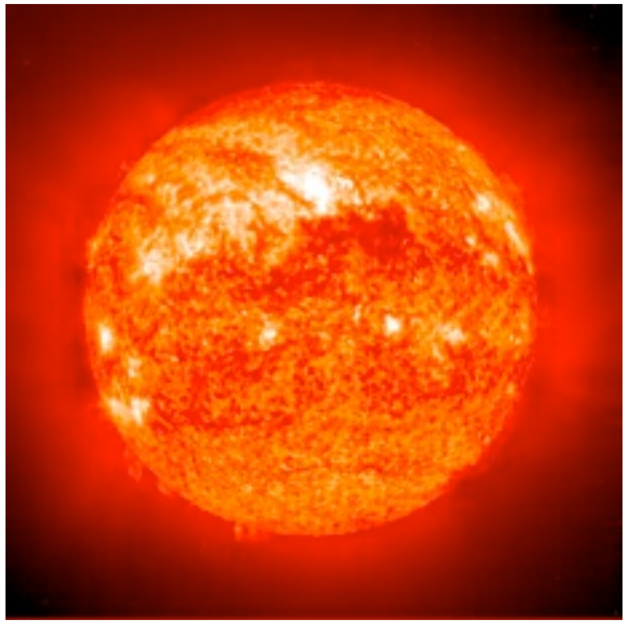
Outline

- Overview and motivation: What aspect of natural complexity are we trying to formalize?
- Background: statistical physics and parallel computational complexity.
- Depth: a useful proxy for complexity?
- Examples: the simple and complex in statistical physics
- Conclusions

Collaborators

- Ray Greenlaw
- Cris Moore
- Dan Tillberg
- Ben Machta
- Ken Moriaty
- Xuenan Li

What's the difference?



from NASA

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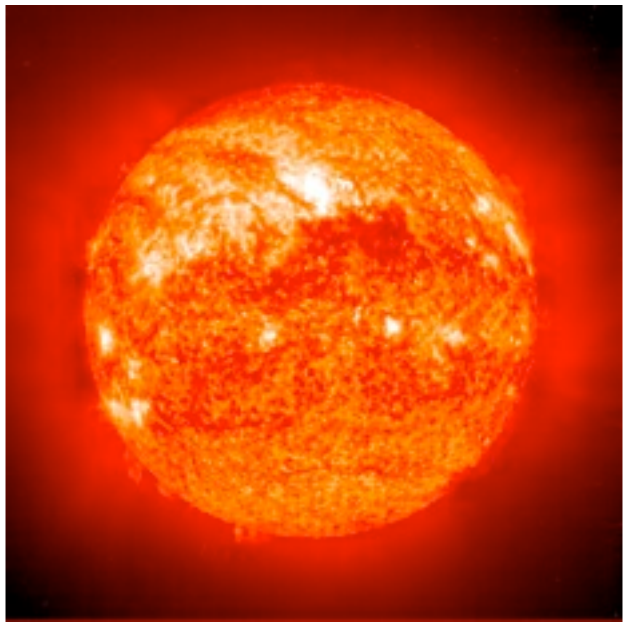

from NASA

- Mass
- Temperature
- Entropy
- Entropy production

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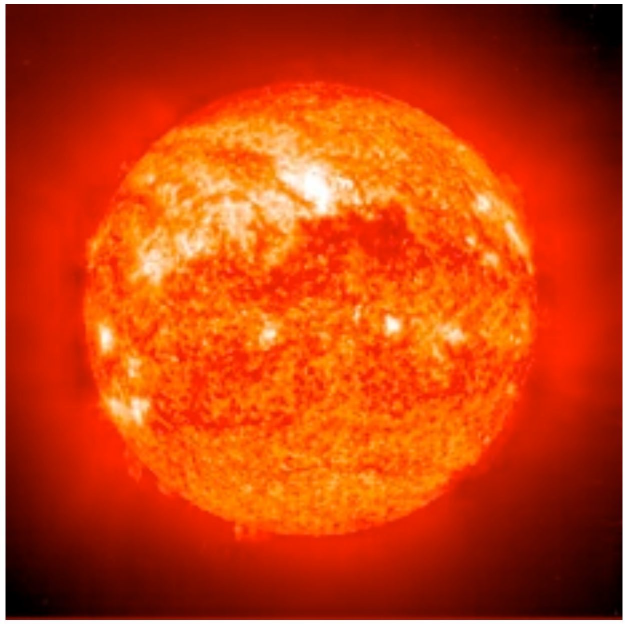

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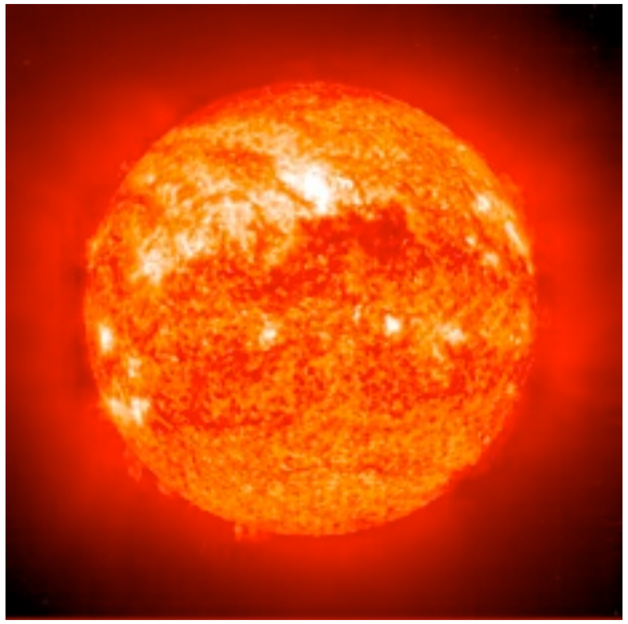

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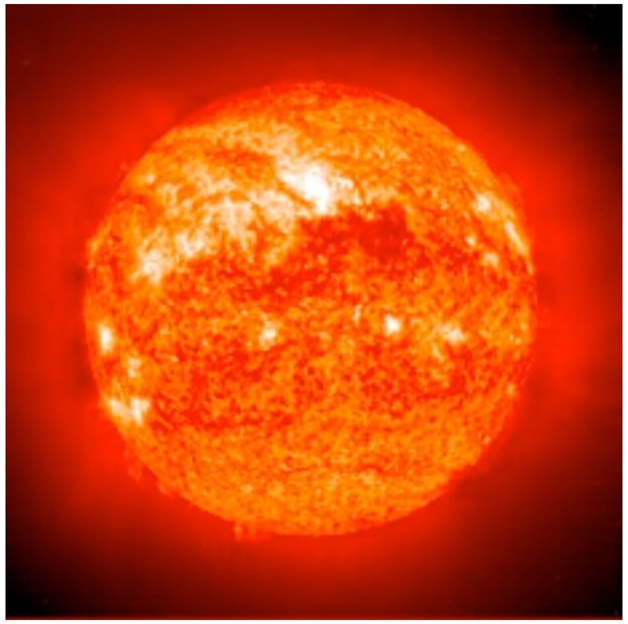

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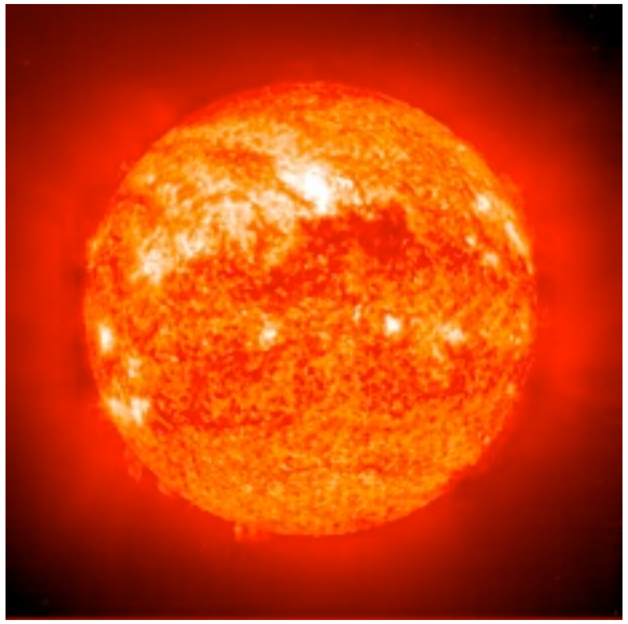

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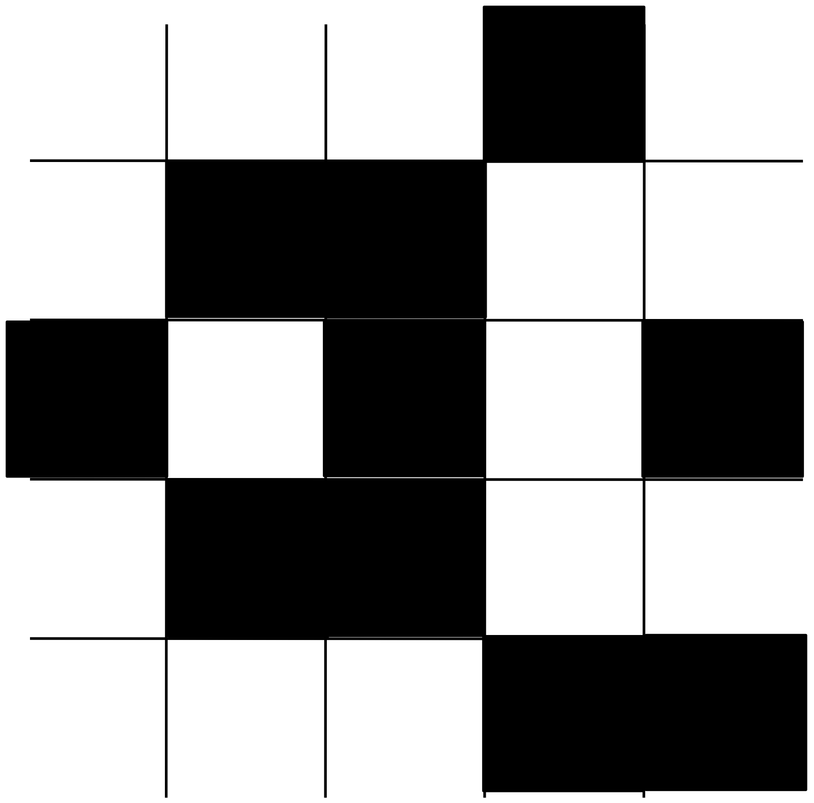
- Complexity

Ising Model

- A more tractable example from statistical physics.
- System states described by “spins” $s_i = \pm 1$ on a lattice.
- Probability of system states described by the Gibbs distribution

Ising Model

$$\mathcal{H} = -\sum_{(i,j)} s_i s_j$$



$$P[\mathbf{s}] = e^{-\mathcal{H}[\mathbf{s}]/kT}/Z$$

Gibbs Distribution

Ising Model

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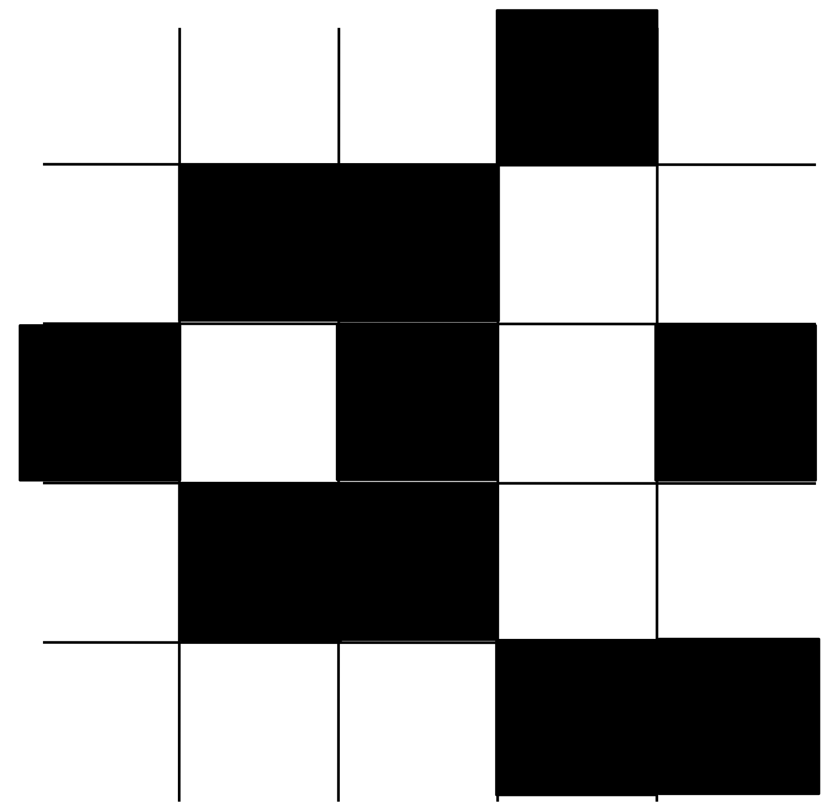

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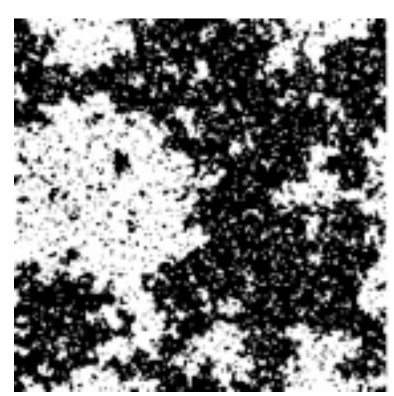
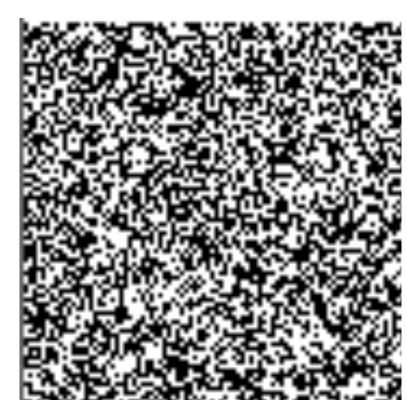
Gibbs Distribution

$T = 0$



$T = T_c$



$T = \infty$

Ising Model

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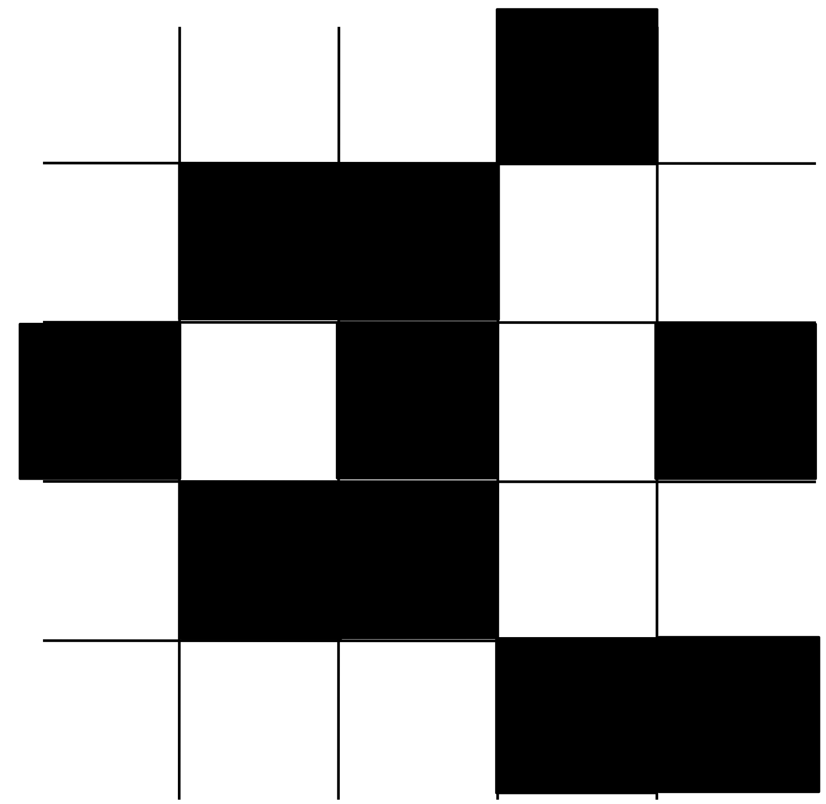
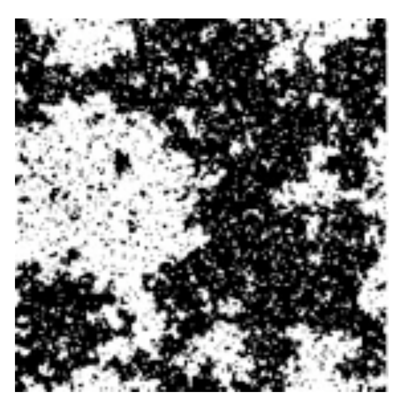


$$P[\mathbf{s}] = e^{-\mathcal{H}[\mathbf{s}]/kT}/Z$$

Critical point

$T = 0$



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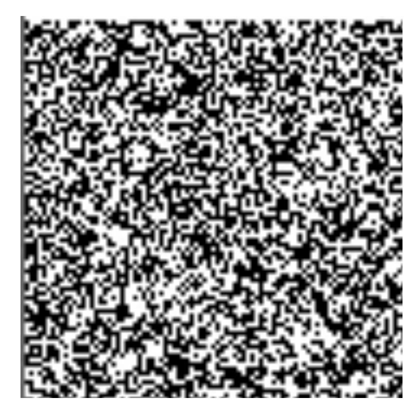
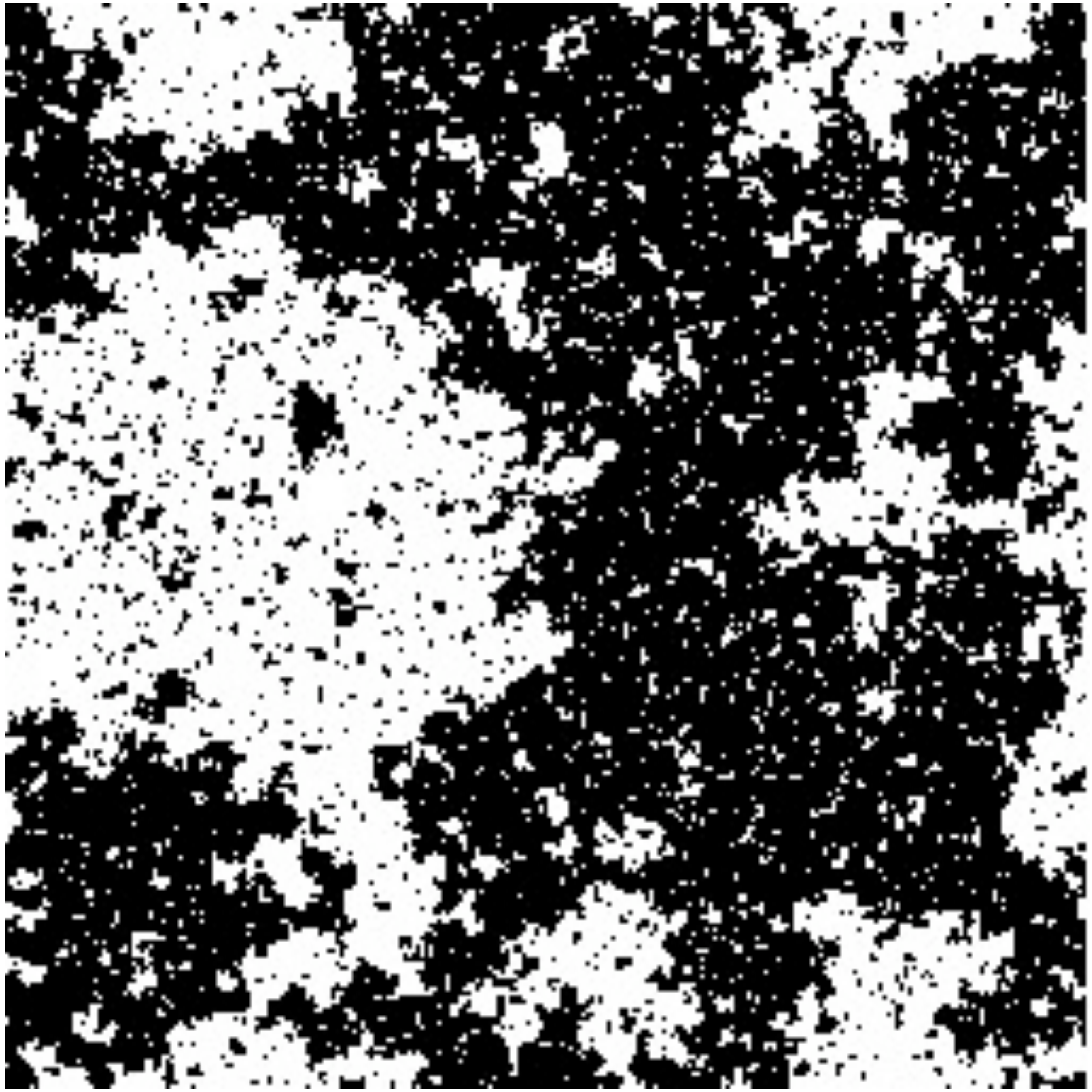


$T = \infty$

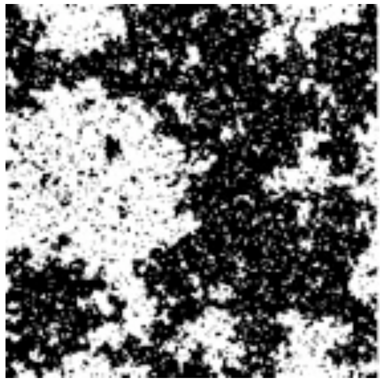
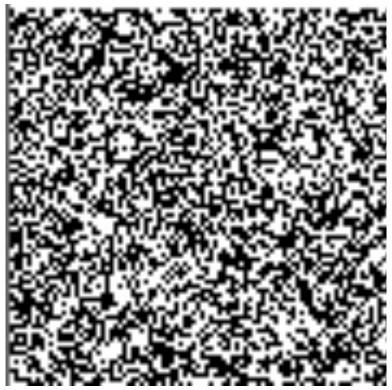
The Ising Critical Point

- Long range correlations
- Fractal clusters of like spins
- Structure on all length scales.
- Difficult to simulate numerically and analyze theoretically.



Ising Model

Temperature

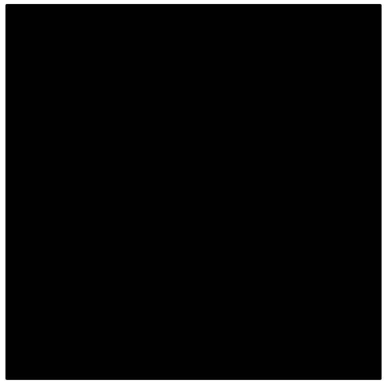
 <  < 

$T = 0$ $T = T_c$ $T = \infty$

Ising Model

Temperature

Entropy



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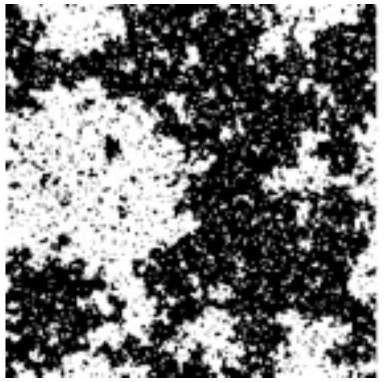

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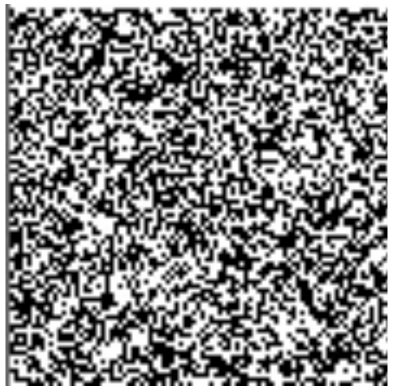


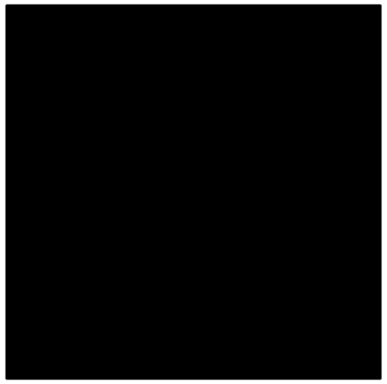
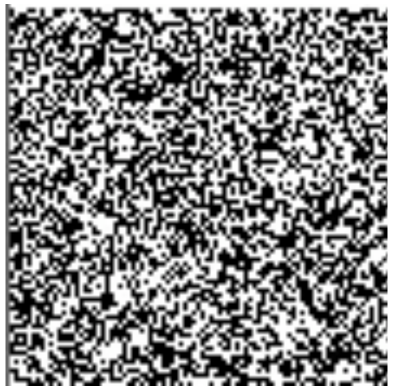
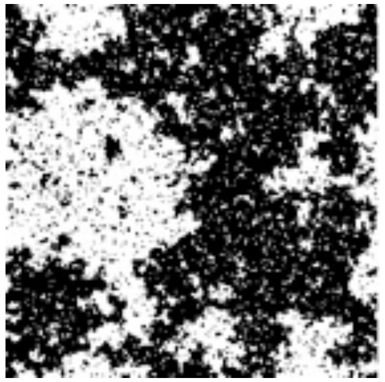
completely ordered

completely disordered

Ising Model

Temperature

Entropy

Complexity  ,  << 

- *Question*: What makes the Earth more complex than the Sun and the Ising critical point more complex than its high or low temperature phases?

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- *One answer*: A long history.

History and Complexity

Charles Bennett

–in SFI Studies in the Sciences of Complexity, Vol. 7 (1990)

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- The emergence of a complex system from simple initial conditions requires a long history.

- The Earth and the Sun are both 4.5 billions years old...

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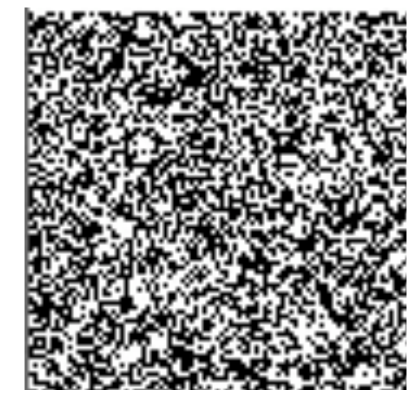
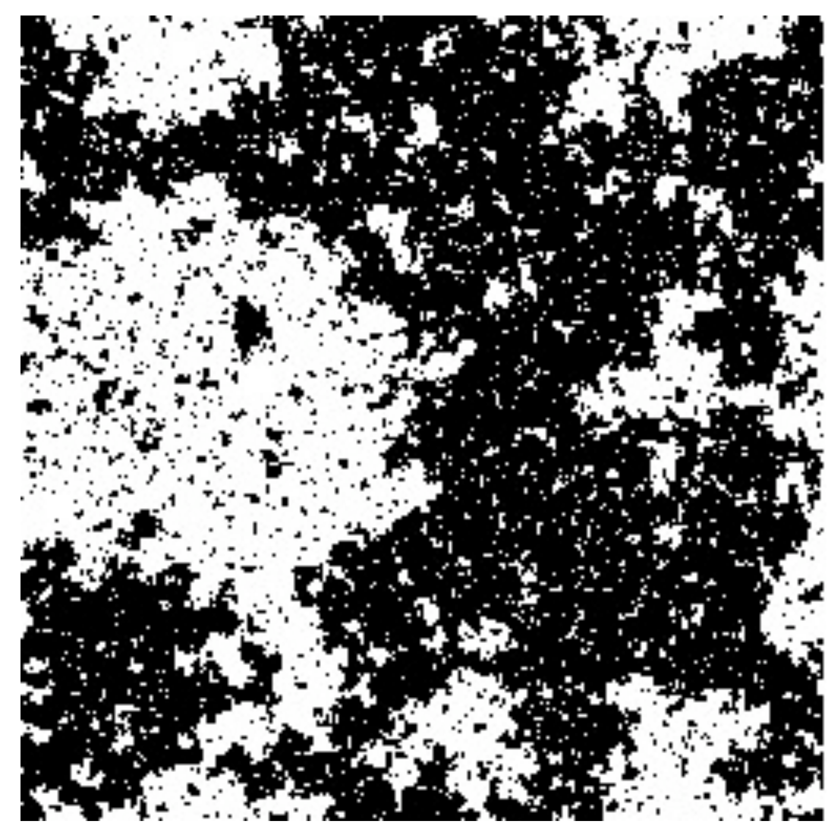
- ...but, the present state of the Sun does not remember the full 4.5 billion year history (except via conserved quantities) while the present state of the Earth (biosphere) is contingent on a very long evolutionary process. The Earth does remember its past.

- High and low temperature states of the Ising model can be sampled using small number of sweeps of a Monte Carlo algorithm.



- The critical state of the Ising model requires a number of sweeps of MC algorithm that scales as a power of the system size (*critical slowing down*).



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- The emergence of a complex system from simple initial conditions requires a long history.
- History can be quantified in terms of the computational complexity (running time) of simulating states of the system.

What computational complexity measure best measures physical complexity?

- Complexity emerges from interactions, not from signal propagation ➔ discount communication.
- Size alone should not contribute to physical complexity ➔ discount hardware.
- These considerations suggest *parallel time* as the appropriate computational complexity measure.

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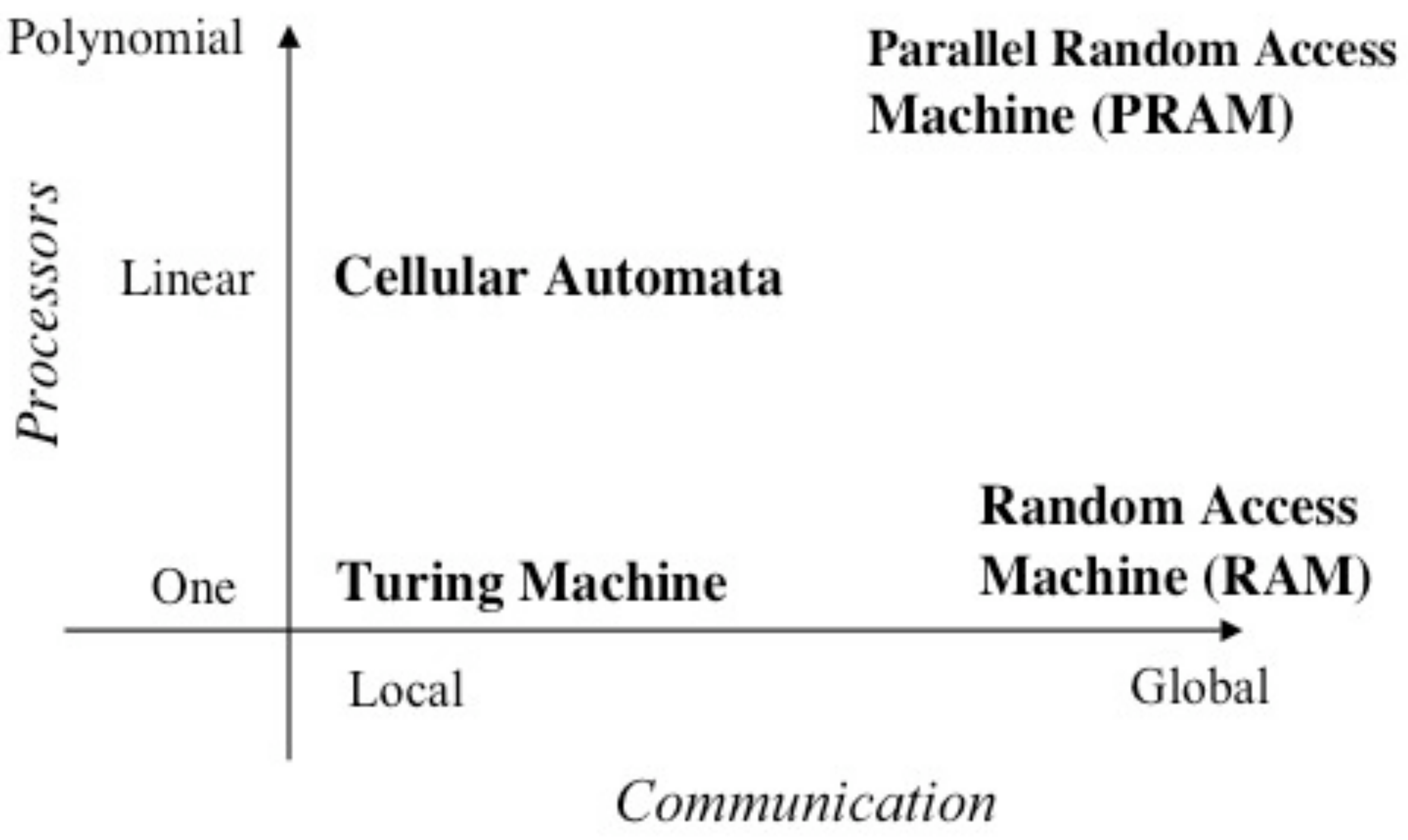
- History can be quantified in terms of the computational complexity of simulating states of the system.
- The appropriate computational measure of history is parallel time.

Models of Computation



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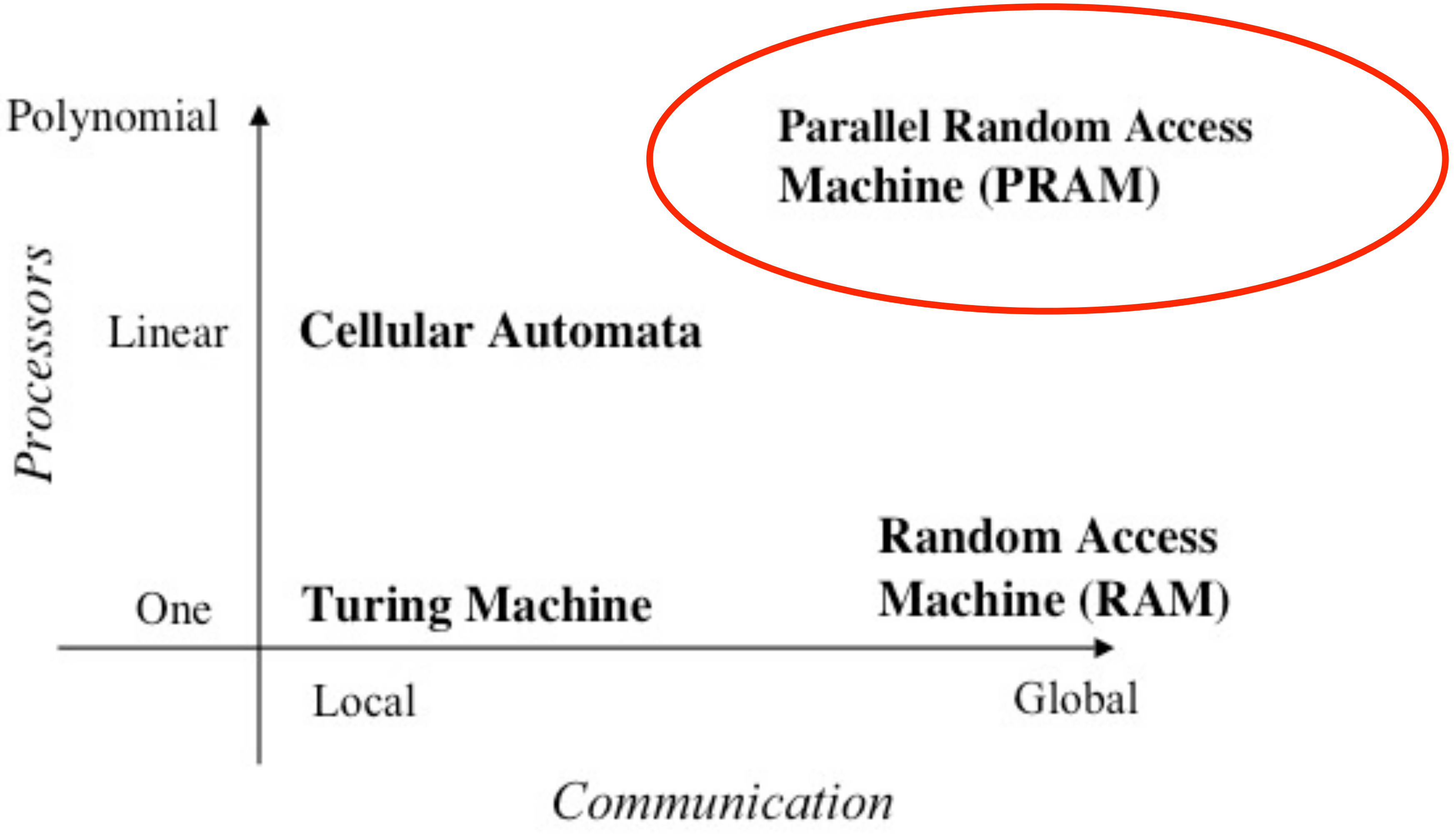
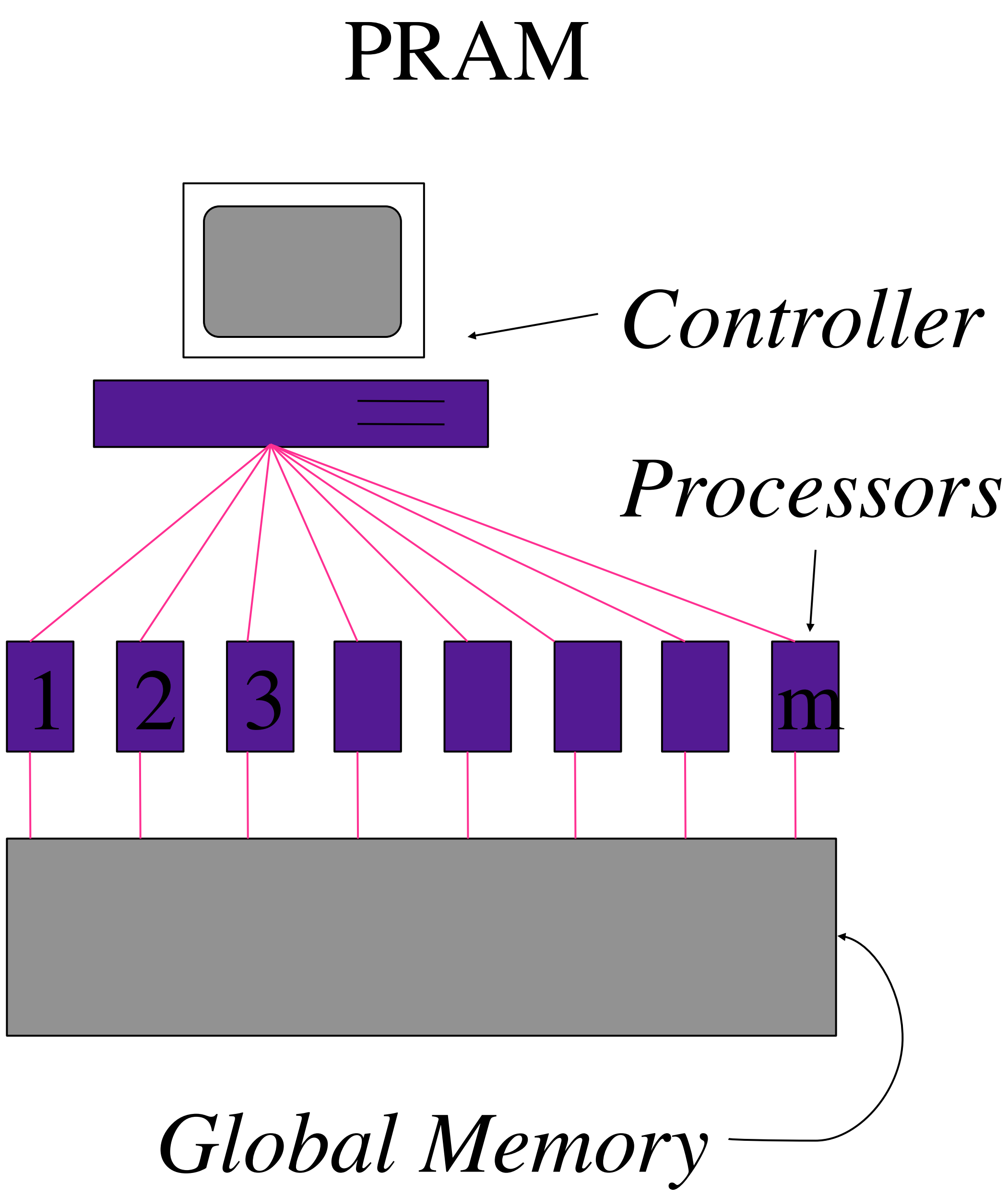
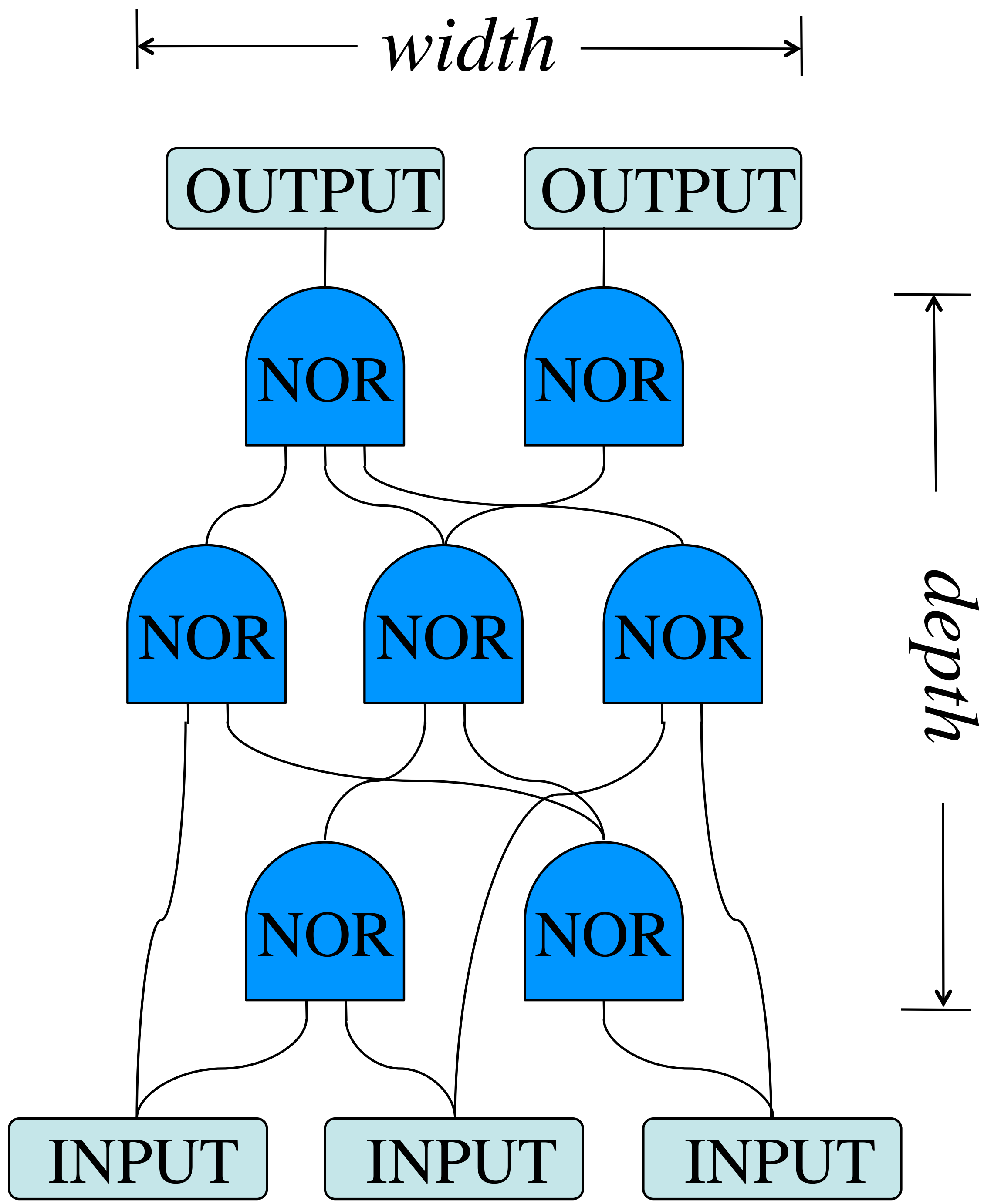


Parallel Random Access Machine



- Each processor runs the same program but has a distinct label
- Each processor communicates with any memory cell in a single time step.
- Primary resources:
  - *Parallel time*
  - *Number of processors*

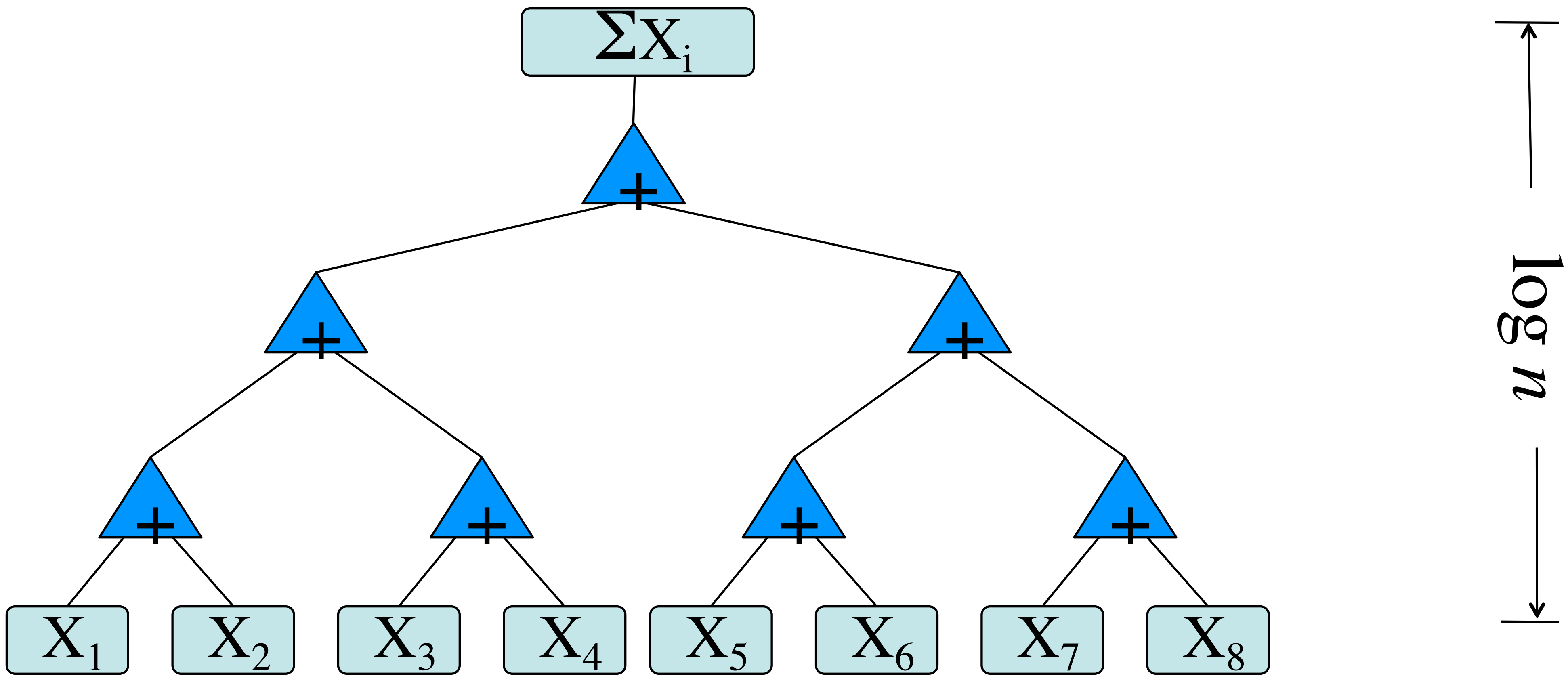
Boolean Circuit Family



- Gates evaluated one level at a time from input to output with no feedback.
- One hardwired circuit for each problem size.
- Primary resources
  - ***Depth***=number of levels
    ≈ ***parallel time***
  - *Width*=maximum number of gates in a level
    ≈*number of processors*
  - *Work*=total number of gates

Parallel Computing

Adding n numbers can be carried out in $O(\log n)$ steps using $O(n)$ processors.



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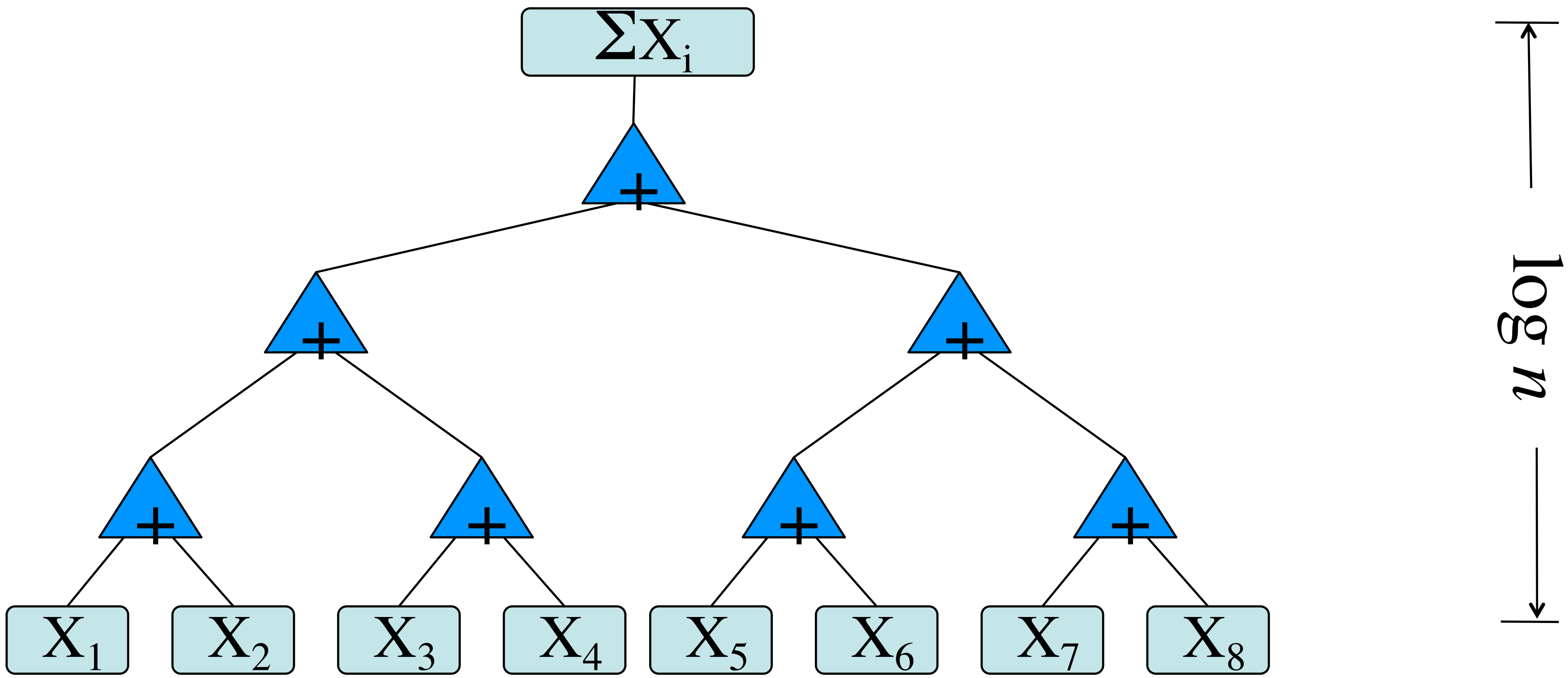

Connected components of a graph can be found in $O(\log^2 n)$ time using polynomially many processors.

Complexity Classes and P-completeness

- **P** is the class of *feasible* problems: solvable with polynomial work.
- **NC** is the class of problems efficiently solved in parallel (polylog depth and polynomial work, **NC** $\subseteq$ **P**).
- Are there feasible problems that cannot be solved efficiently in parallel (**P**≠**NC**)?
- **P**-complete problems are the hardest problems in **P** to solve in parallel. It is believed they are *inherently sequential:* not solvable in polylog depth.
- The Circuit Value Problem is **P**-complete.

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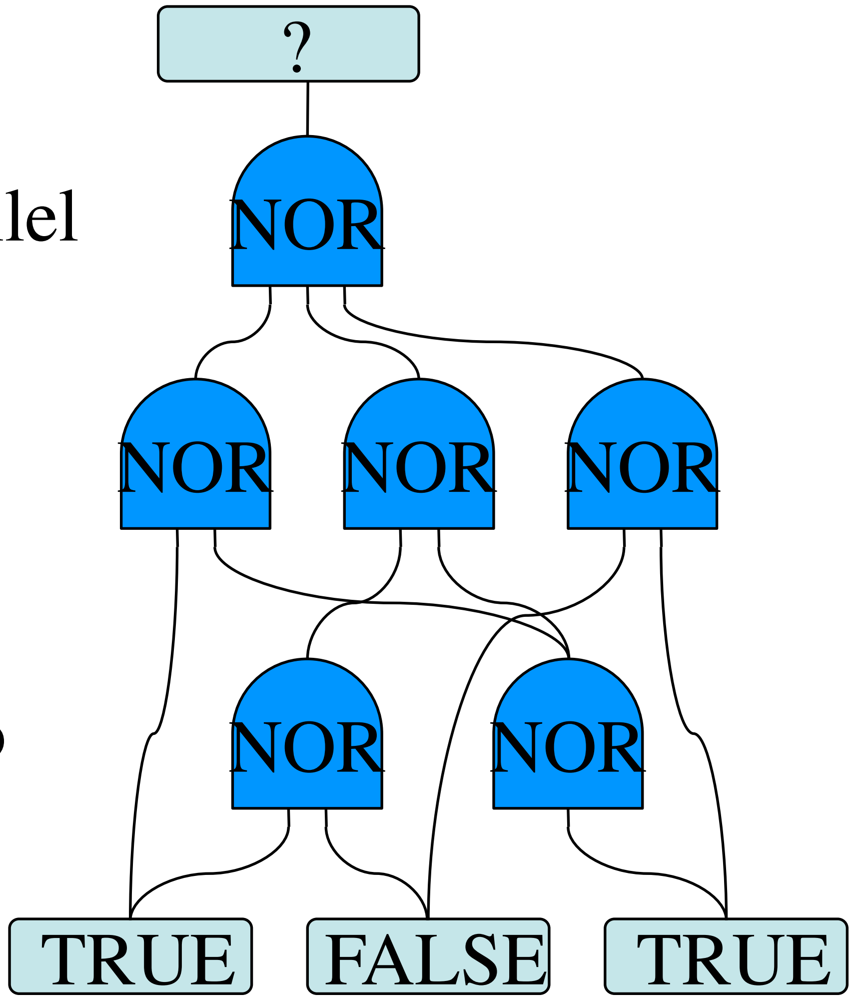
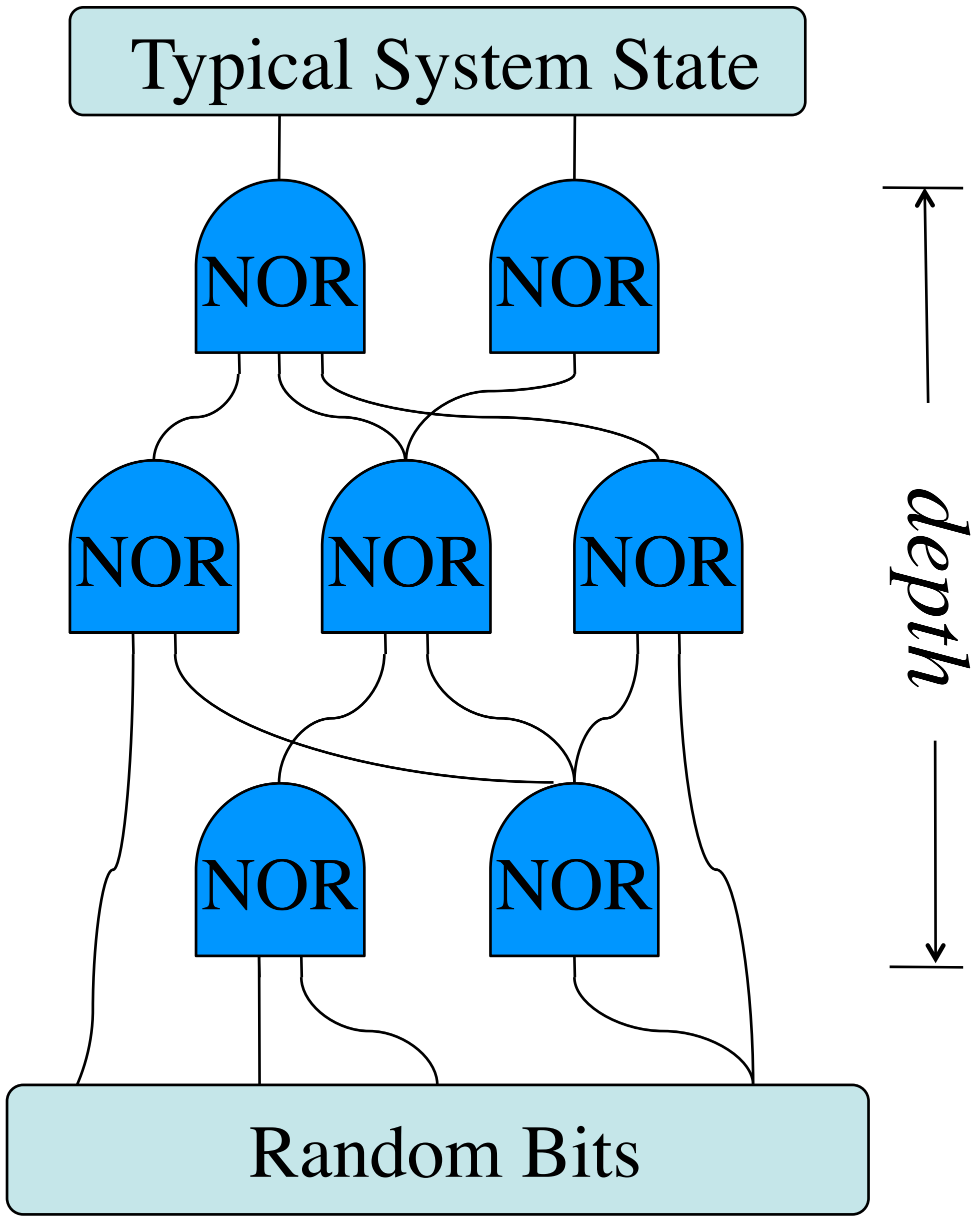


Sampling Complexity



•Monte Carlo simulations convert random bits into descriptions of a typical system states.

•**What is the depth of the shallowest feasible circuit (running time of the fastest PRAM program) that generates typical states**?

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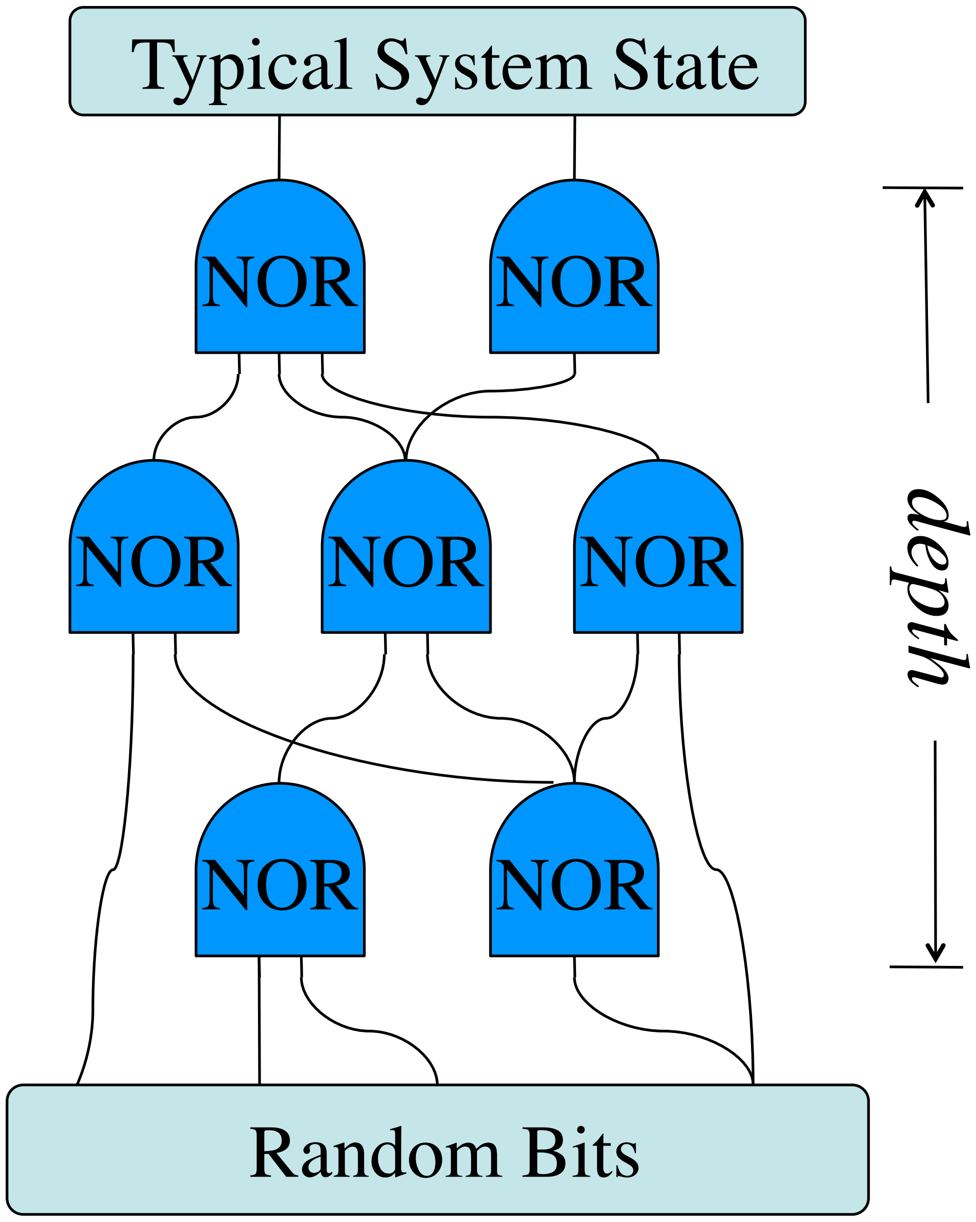

- Monte Carlo simulations convert random bits into descriptions of a typical system states.
- **What is the depth of the shallowest feasible circuit (running time of the fastest PRAM program) that generates typical states**?

Depth is a property of systems in statistical physics

Depth of Natural Systems

The *depth* of a natural system is the time complexity of the fastest parallel Monte Carlo algorithm (PRAM or Boolean circuit family with random inputs) that generates typical system states (or histories) with polynomial hardware.

Comments on fastest

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- A natural system should not be called complex because it emerges slowly via an inefficient process.
  - Many systems that appear to have a long history do not, in fact, have much depth.

Comments on fastest

- A natural system should not be called complex because it emerges slowly via an inefficient process.
  - Many systems that appear to have a long history do not, in fact, have much depth.
- Depth is uncomputable. Upper bounds can be found by demonstrating specific parallel sampling algorithms but lower bounds are difficult to establish.
  - A necessary feature, not a bug!

Maximal Property of Depth

For a system AB composed of independent subsystems A and B, the depth of the whole is the maximum over subsystems:

$$\mathcal{D}(AB) = \max\{\mathcal{D}(A), \mathcal{D}(B)\}$$

Follows immediately from parallelism.

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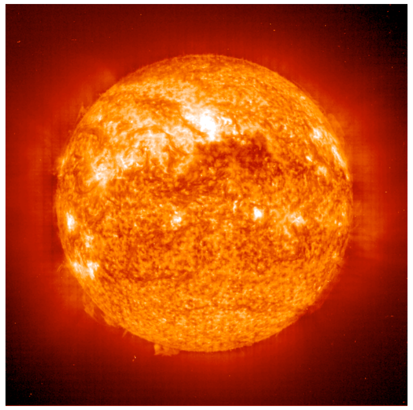
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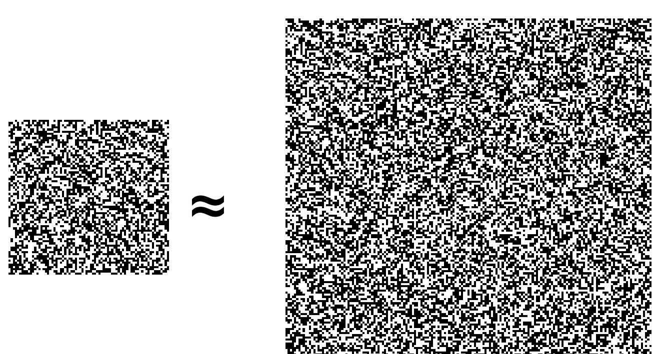
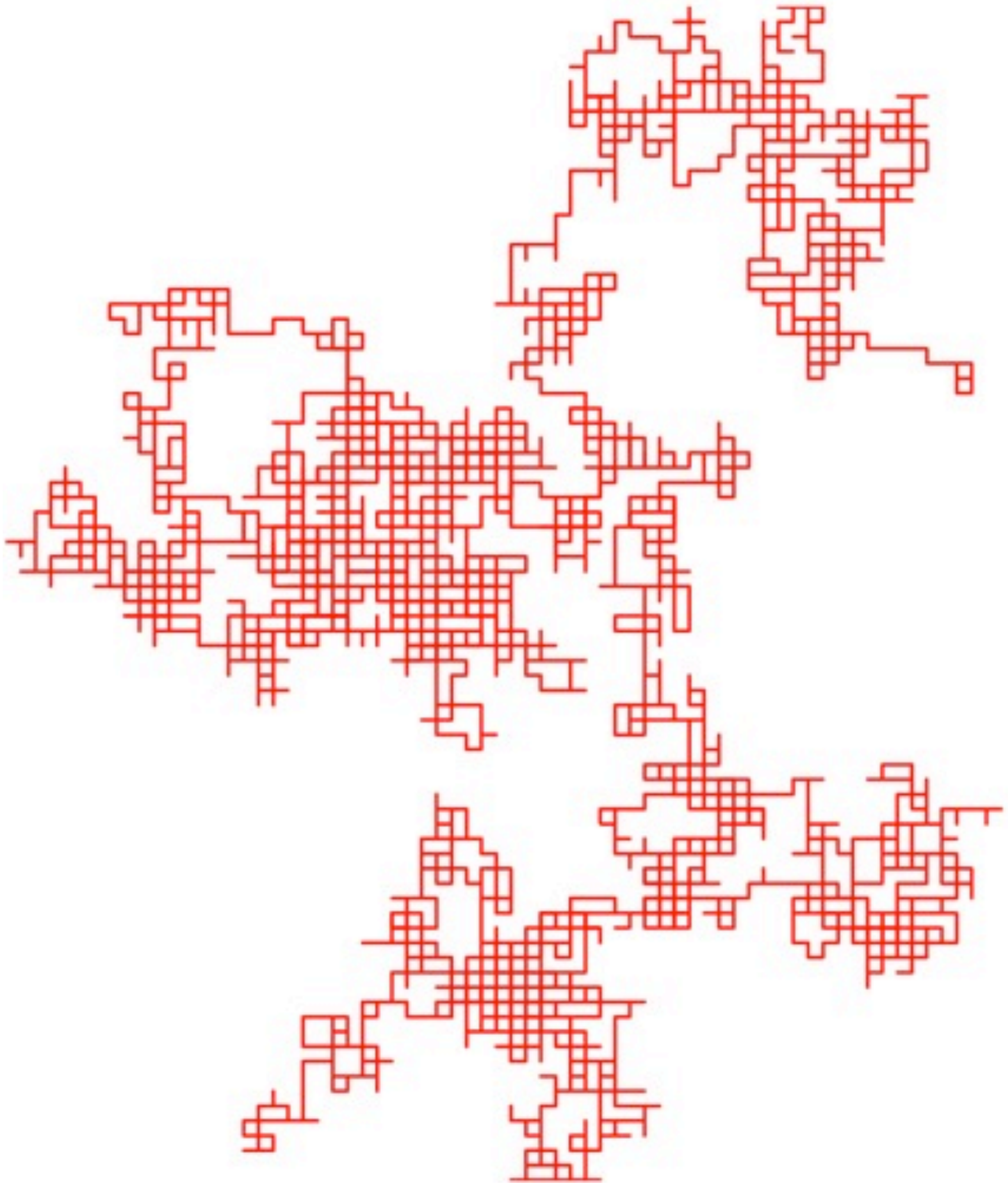


Depth is *intensive* (nearly independent of size) for homogeneous systems with short range correlations.

Examples from statistical physics

- Random walks
- Preferential attachment networks
- The Ising model
- Diffusion limited aggregation

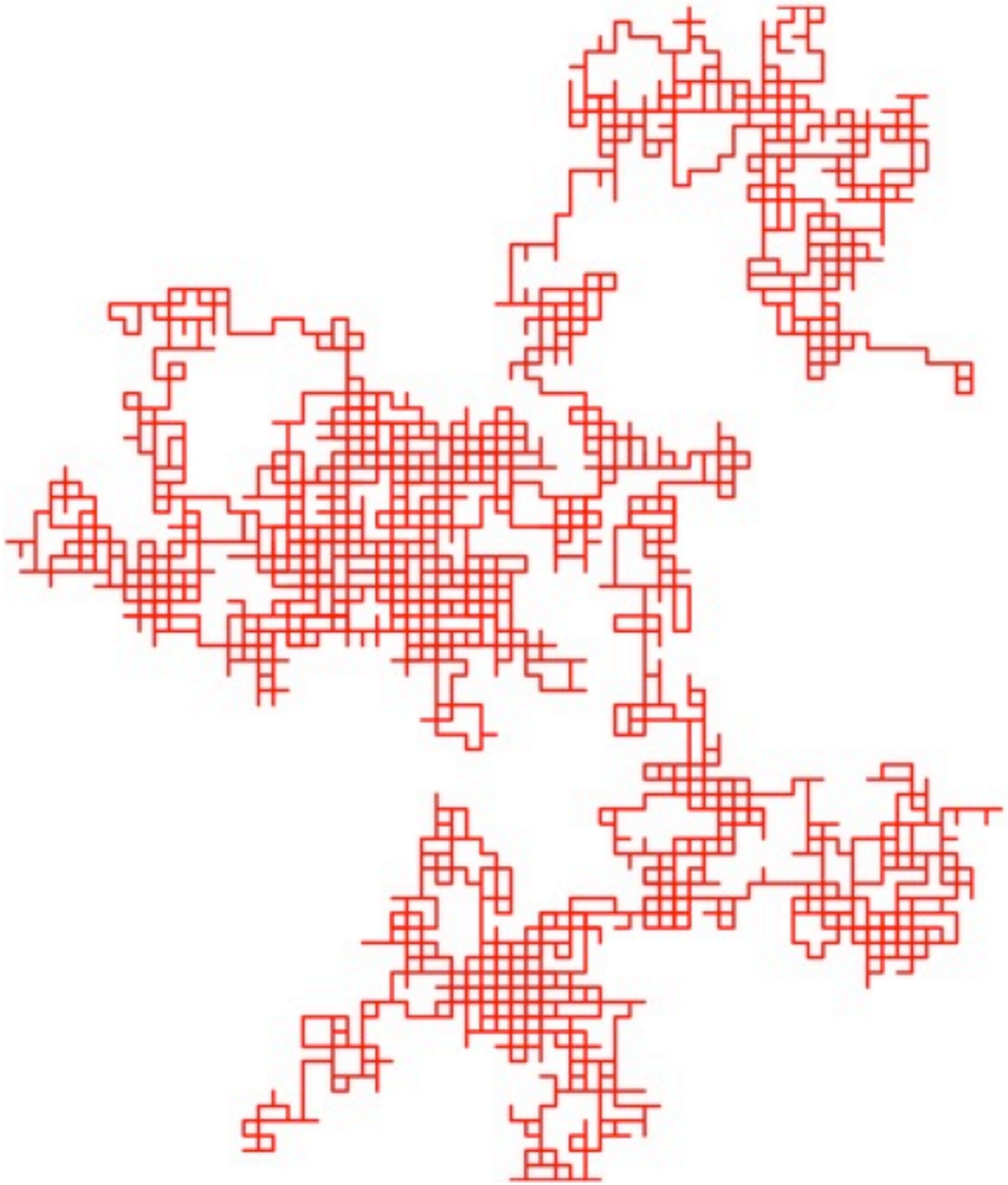
Random Walks



from wikipedia

Random Walks

- There is apparent history in the random walk since its position at time $t+1$ is obtained from the position at time t by adding a random step.



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Random Walks

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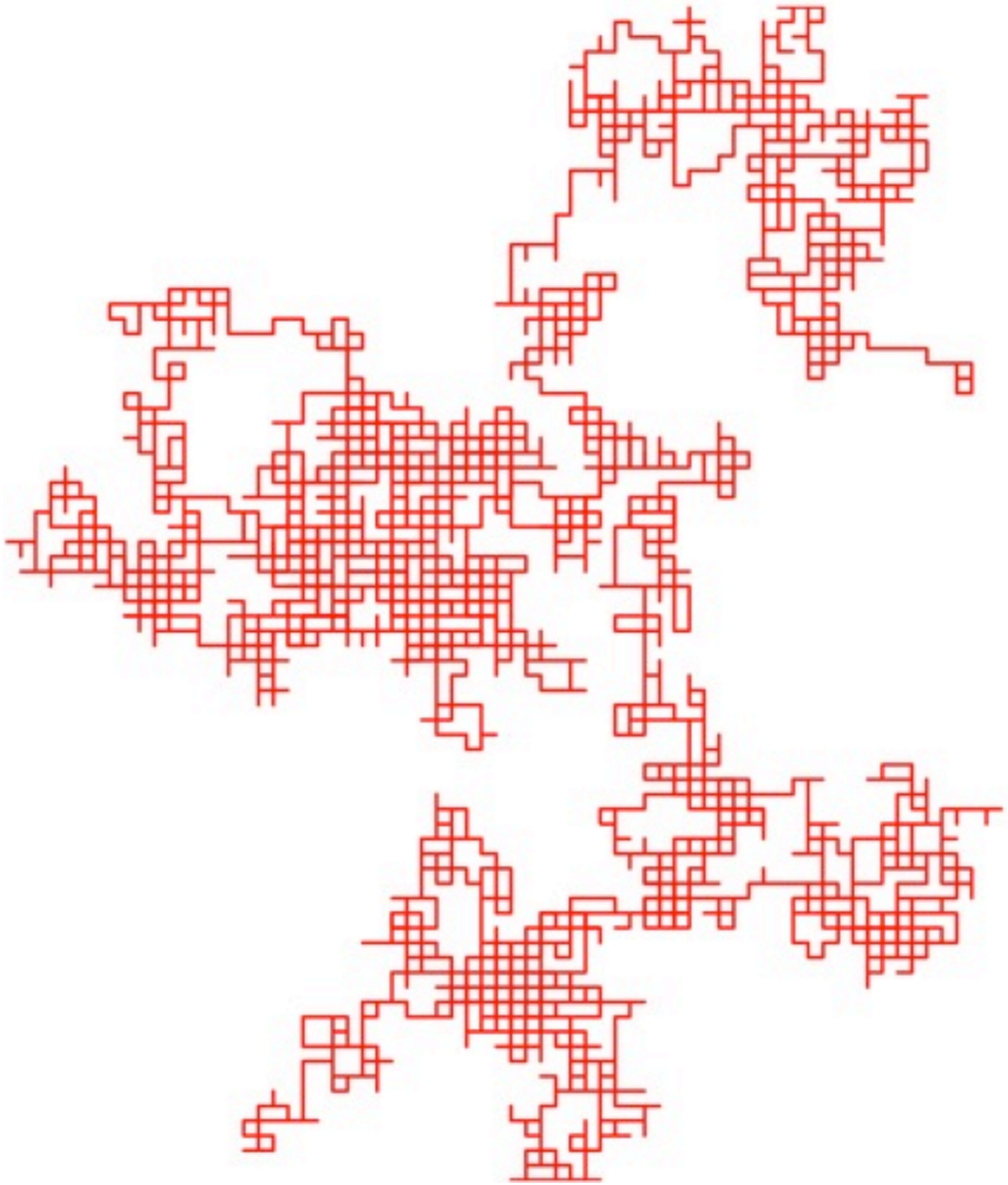
- Since addition can be carried out in log parallel time, a random walk of length t has log t depth.



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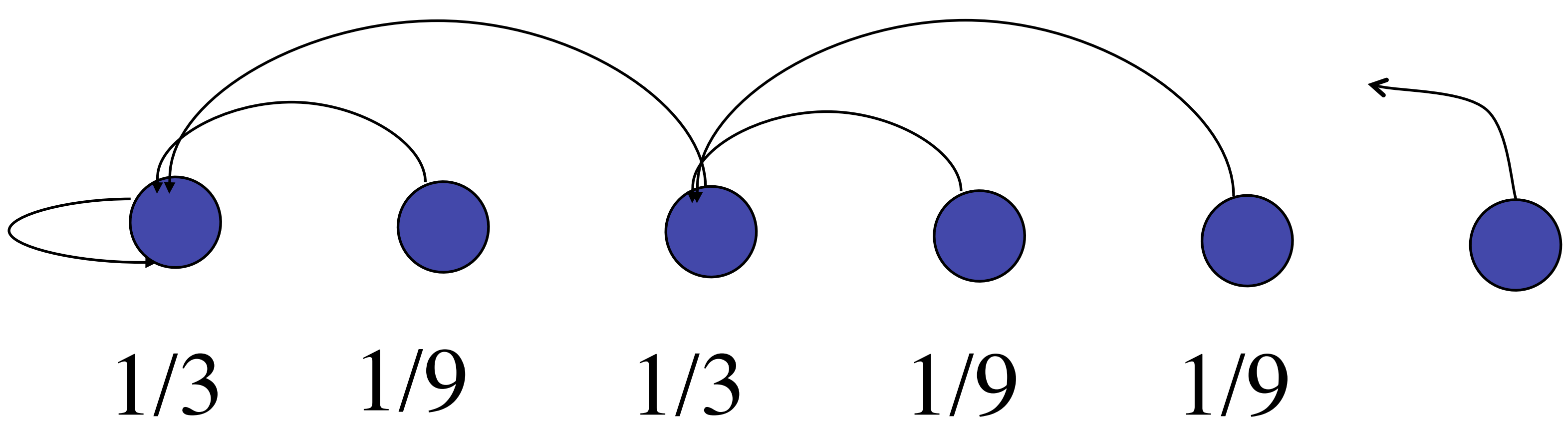
Preferential Attachment Networks

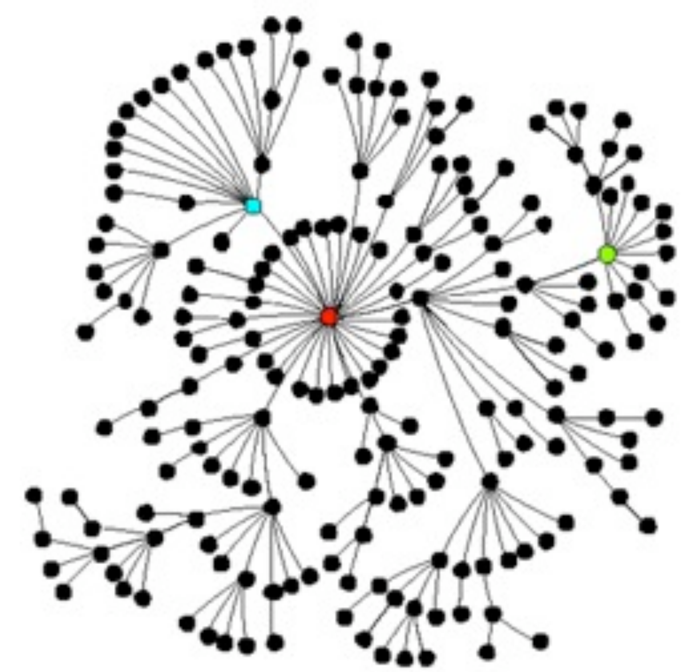
Barabasi, Albert, *Science* **286**, 509 (1999)
Krapivsky, Redner, Leyvraz, *PRL* **85**, 4629 (2000)

Add nodes one at a time, connecting new nodes to old nodes according to a "rich get richer" preferential attachment rule:

$$\pi_n(t) = \mathrm{Prob}[t \text{ connects to } n] \propto k_n(t)^\alpha$$

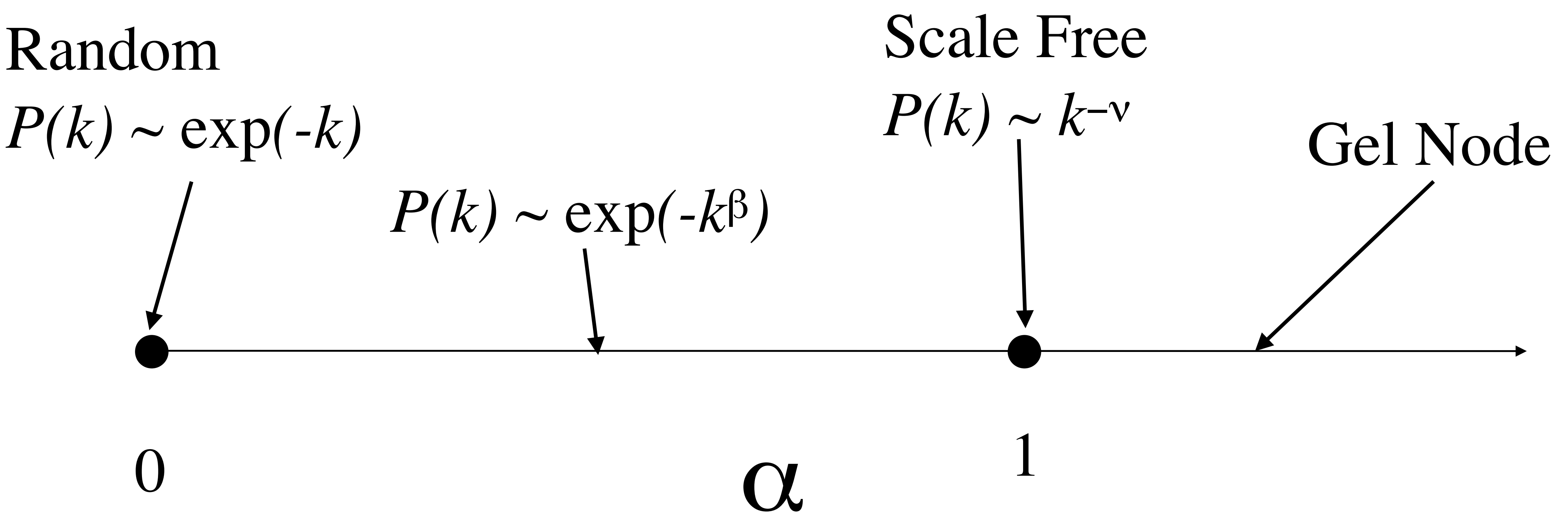
where $k_n(t)$ is the degree of node n at time t.





Behavior of Growing Networks

$$\pi_n(t) = \mathrm{Prob}[t \text{ connects to } n] \propto k_n(t)^\alpha$$



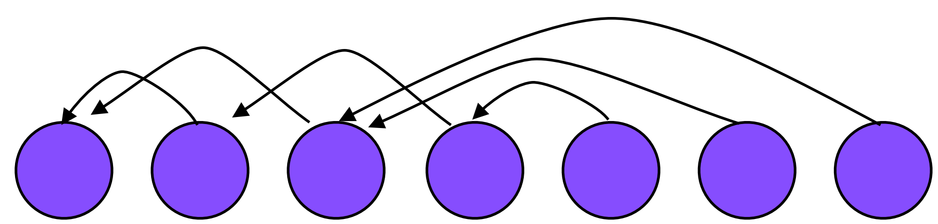
P(k) is the degree distribution

Discontinuous structural transition at $\alpha=1$

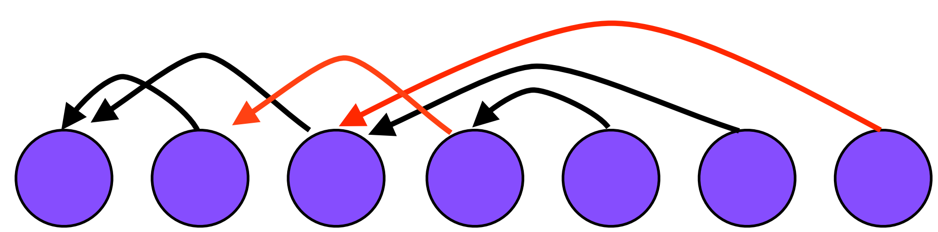
Redirection

Krapivsky, Redner, Leyvraz, *PRE* **63**, 066123 (2001)

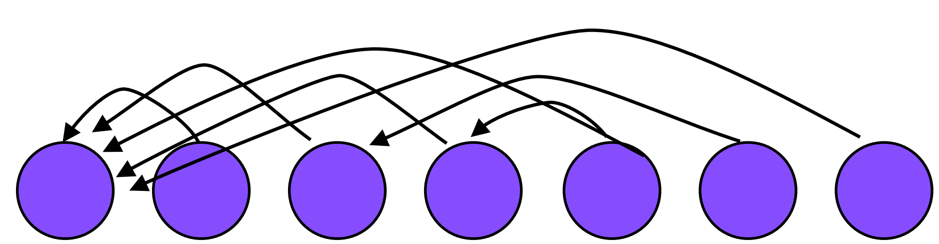
I. Generate a random sequential network.



II. With probability r, color edge **R** (redirect) and with probability $1-r$ color edge **T** (terminal).



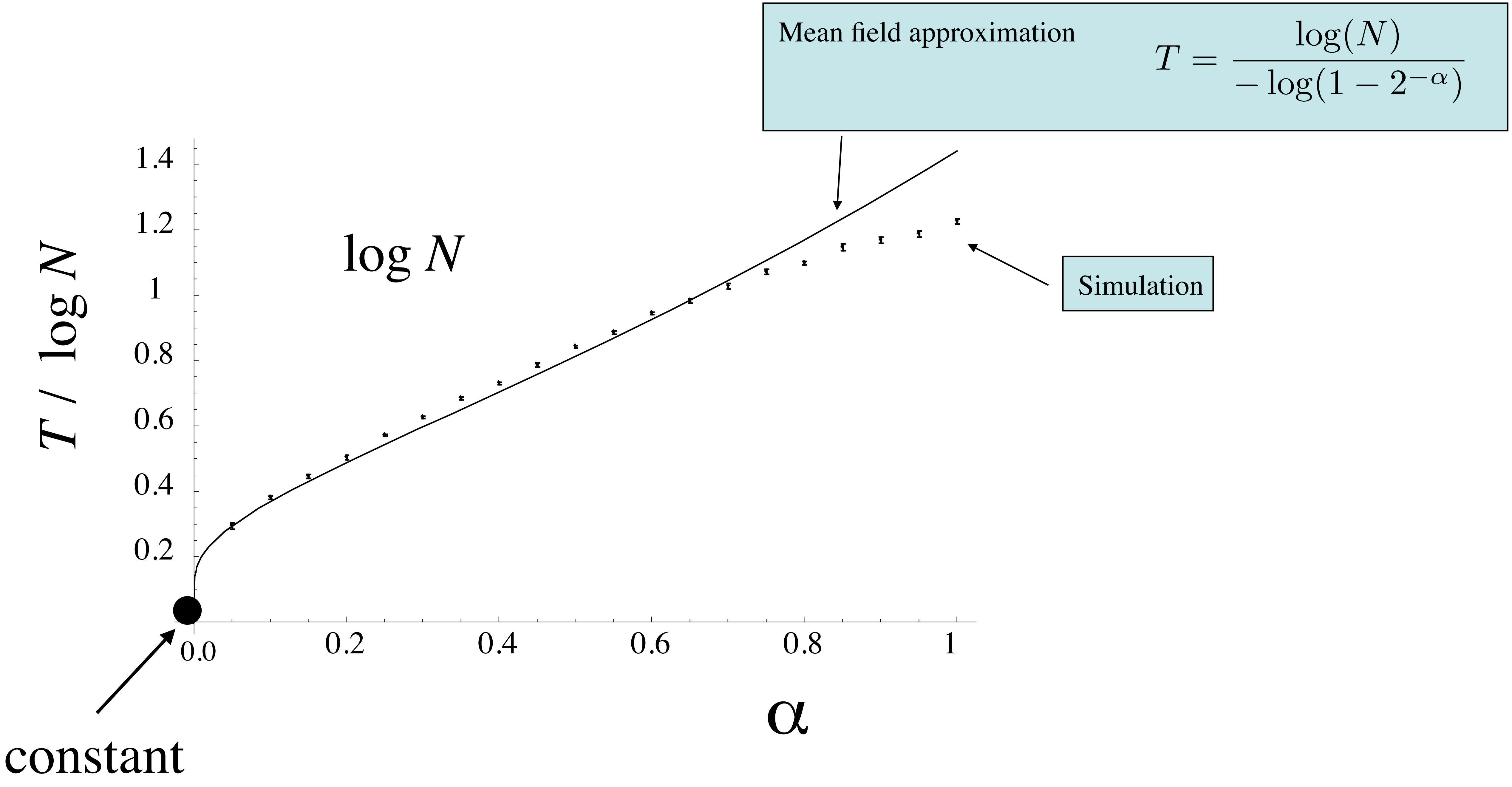
III. New links obtained by tracing **R** edges and stopping after traversing a **T** edge.



Parallel Algorithm for Scale Free Networks

- Redirection provides a fast parallel algorithm for the scale free case.
- The longest redirected path ~log N
- Tracing such a path in parallel ~log log N
- Depth of scale free networks ~log log N

Depth of PA Networks



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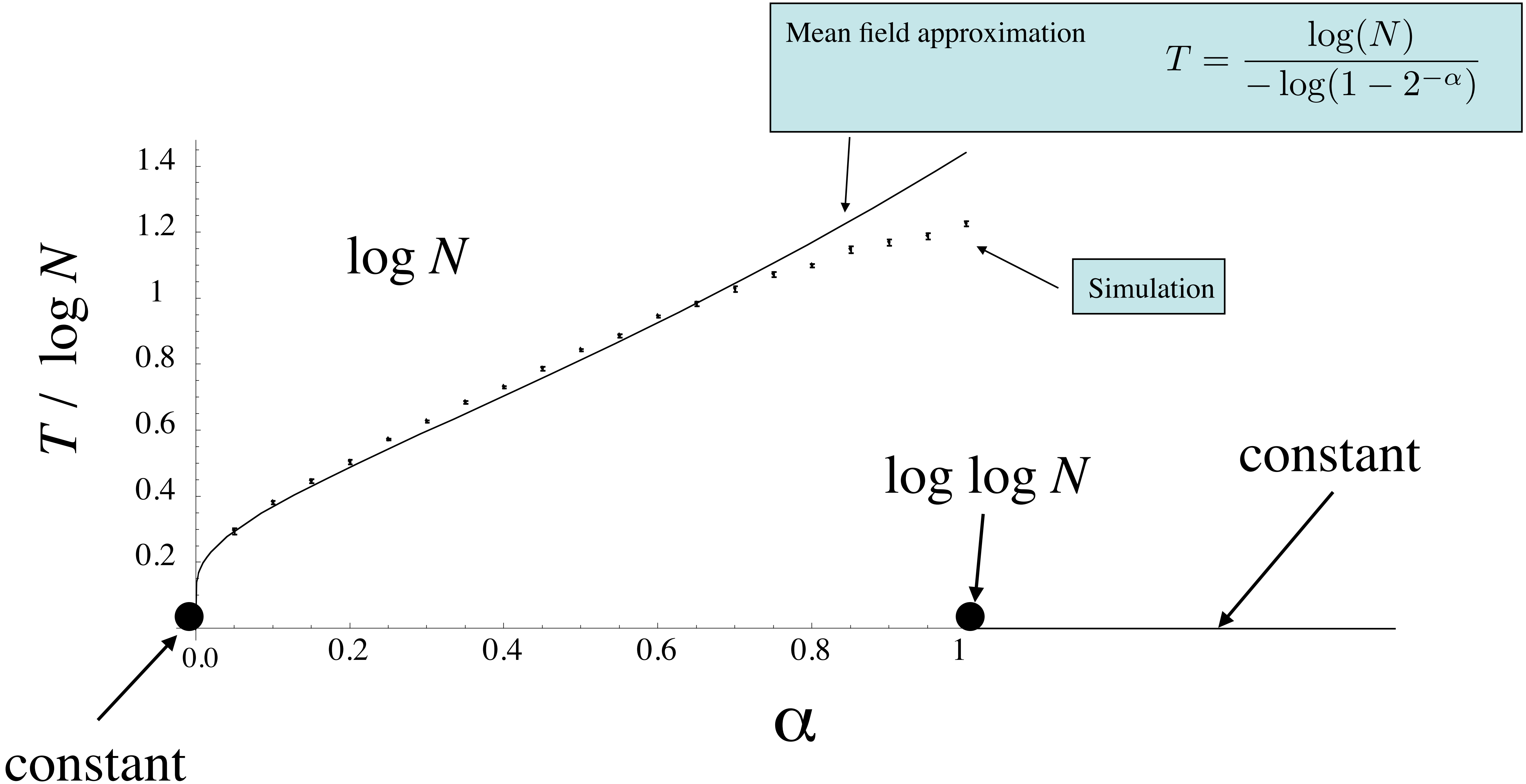

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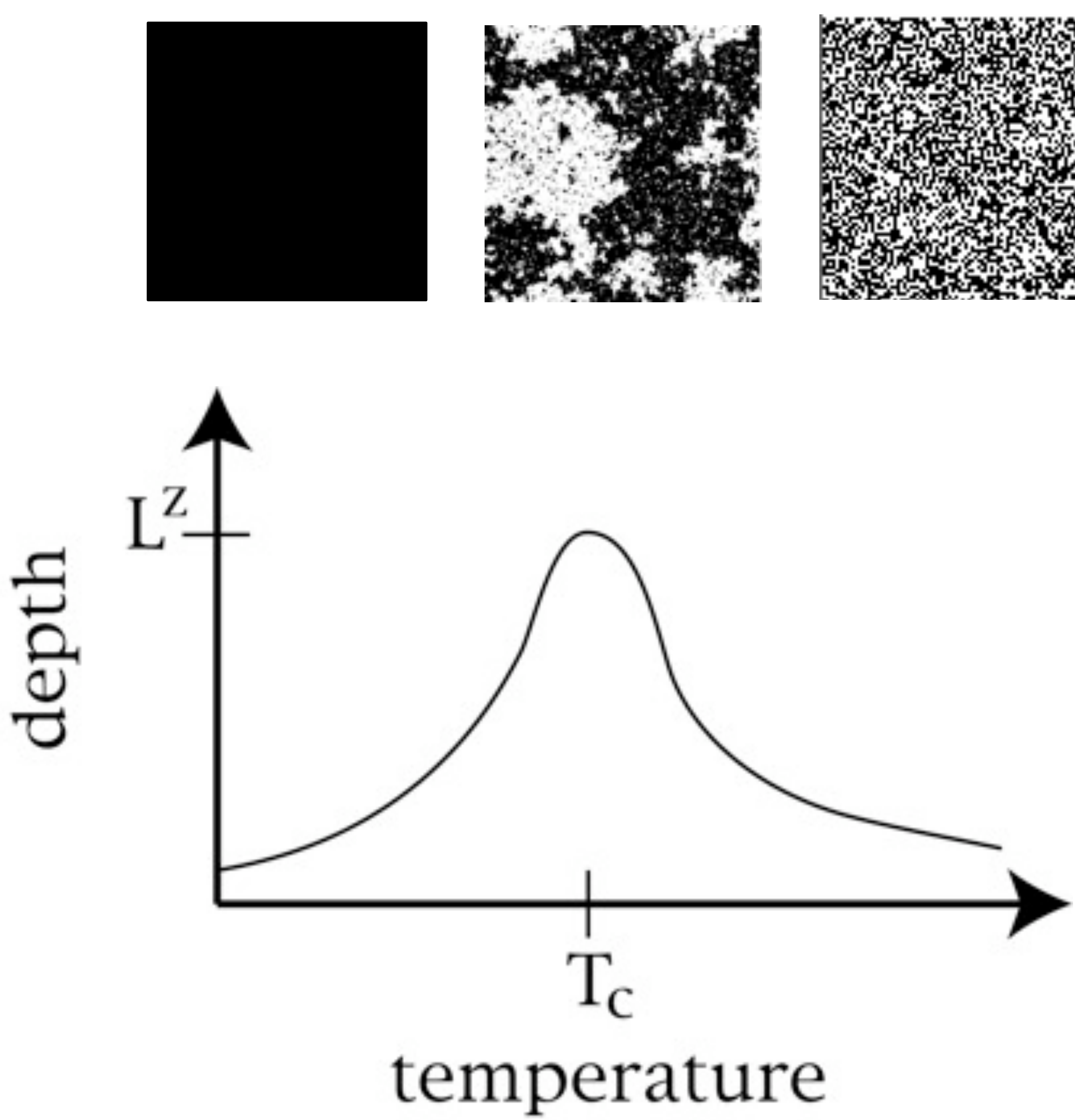
Ising model

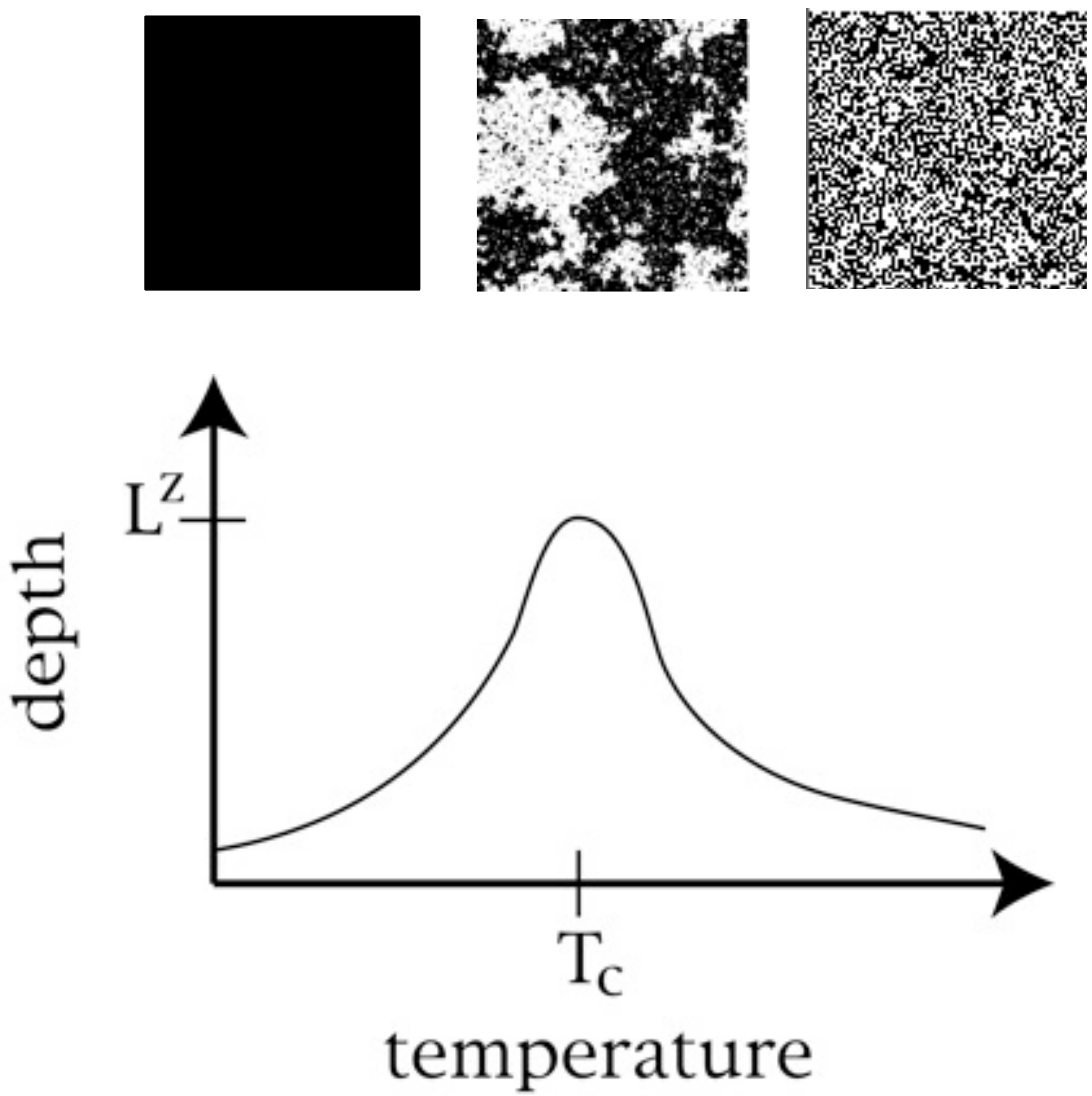


The best known parallel algorithm for the (3D) Ising model (the Swendsen-Wang algorithm) equilibrates at the critical point in a time that scales as a small power of the system size.

$$z \approx 0.5 \text{ at } T = T_c$$

$$z = 0(\log) \text{ for } T \neq T_c$$

Ising model



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$$z \approx 0.5 \text{ at } T = T_c$$

$$z = 0(\log) \text{ for } T \neq T_c$$

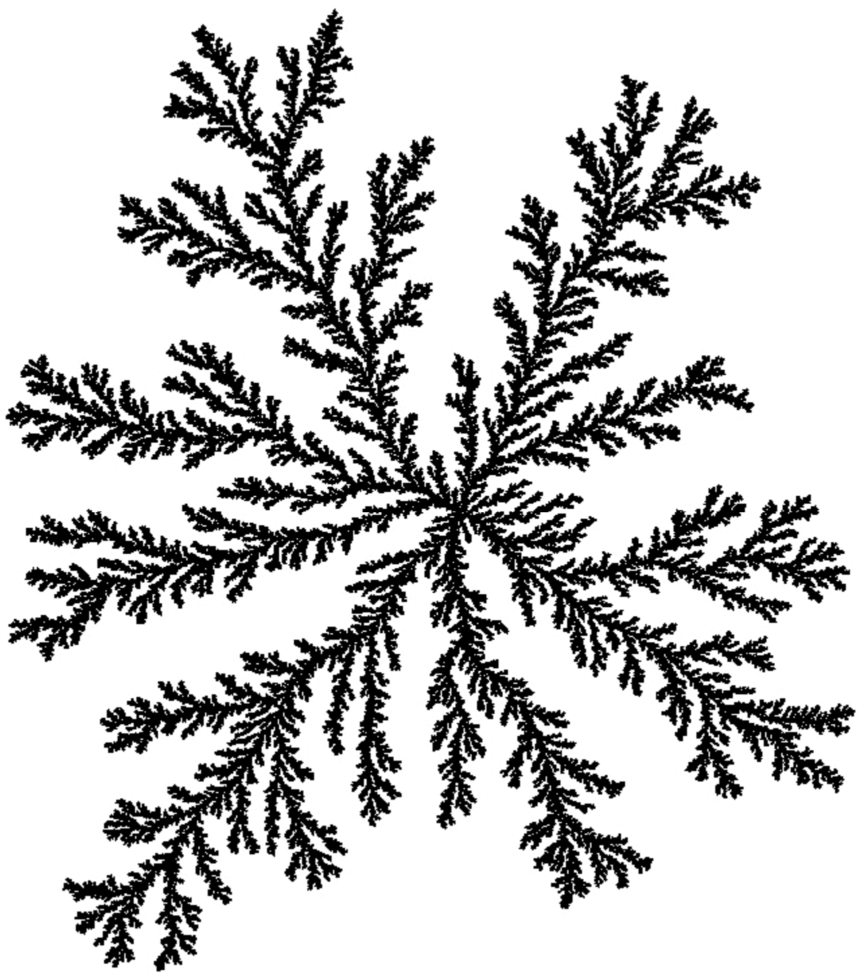
More generally, depth tends to be a maximum at transitions between ordered and disordered states.

Examples from statistical physics

- Random walks
- Preferential attachment networks
- The Ising model
- Diffusion limited aggregation

Diffusion Limited Aggregation

Witten and Sander, *PRL* 47, 1400 (1981)



•Particles added *one at a time* with sticking probabilities given by the solution of Laplace's equation.

•Self-organized fractal object

d_f=1.715… (2D)

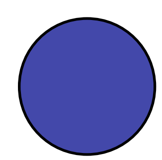
•Physical systems:

Fluid flow in porous media

Electrodeposition

Bacterial colonies

Random Walk Dynamics for DLA



Random Walk Dynamics for DLA

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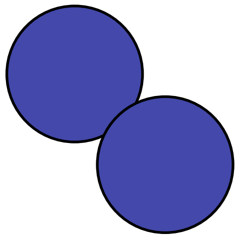

Random Walk Dynamics for DLA

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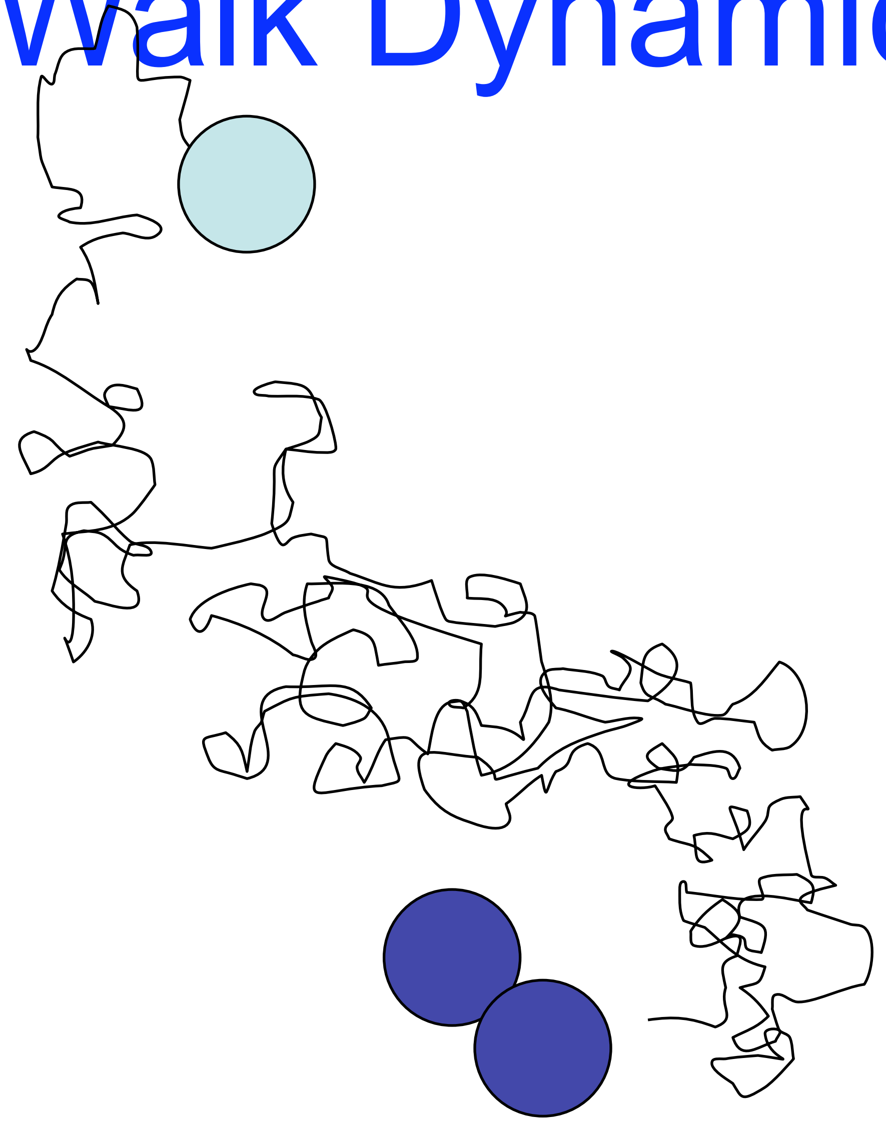
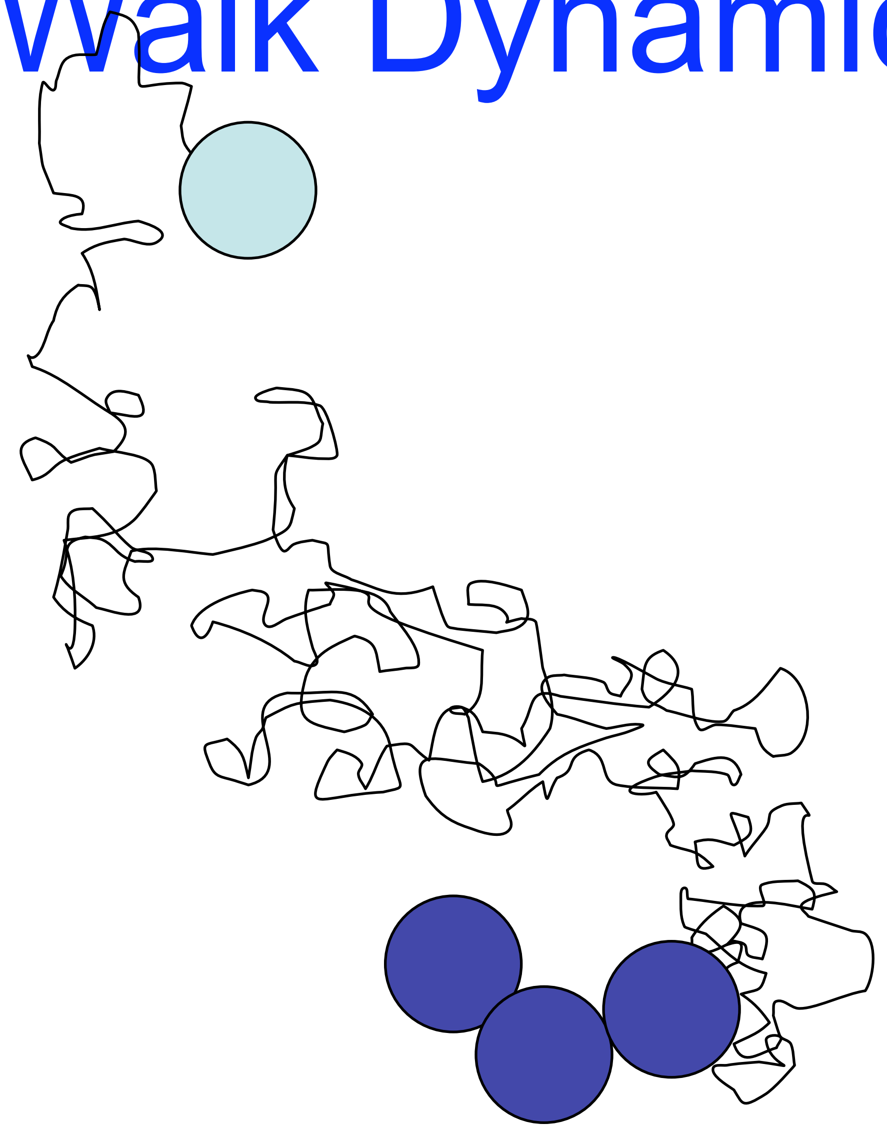


Random Walk Dynamics for DLA



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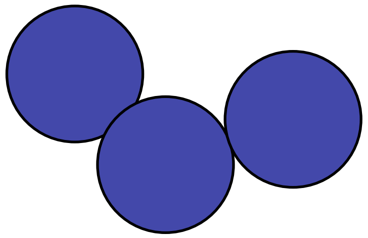
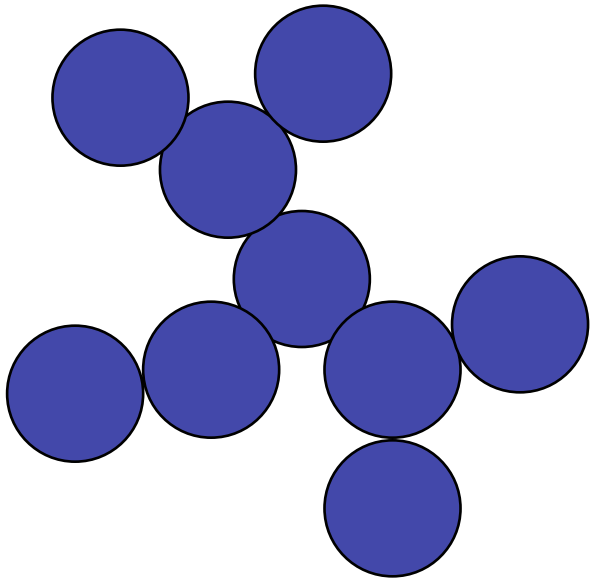
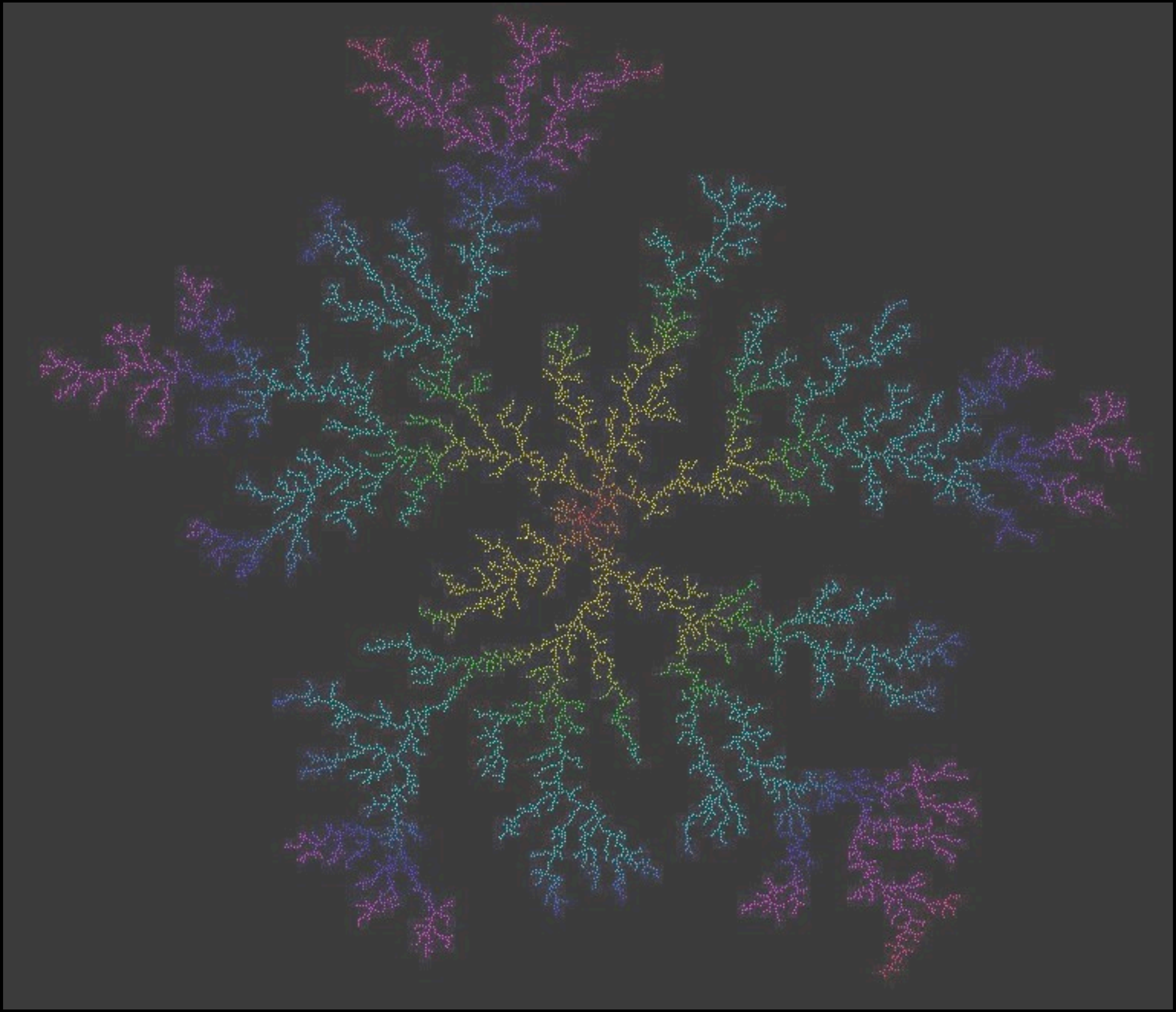


Random Walk Dynamics for DLA

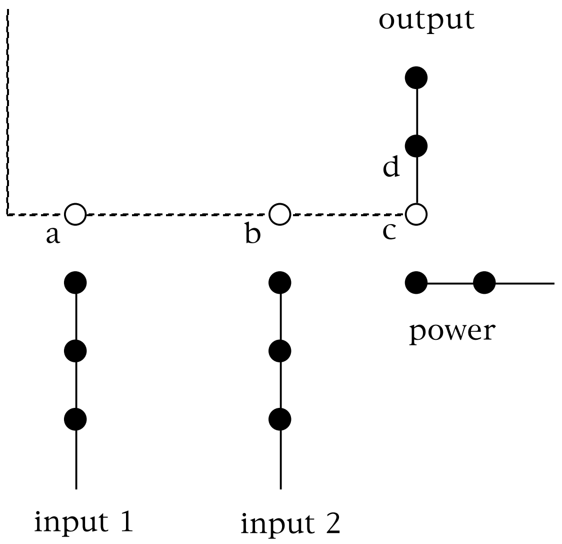




Depth of DLA

Theorem: Determining the shape of an aggregate from the random walks of the constituent particles is a **P**-complete problem.

Proof sketch: Reduce the Circuit Value Problem to DLA dynamics.



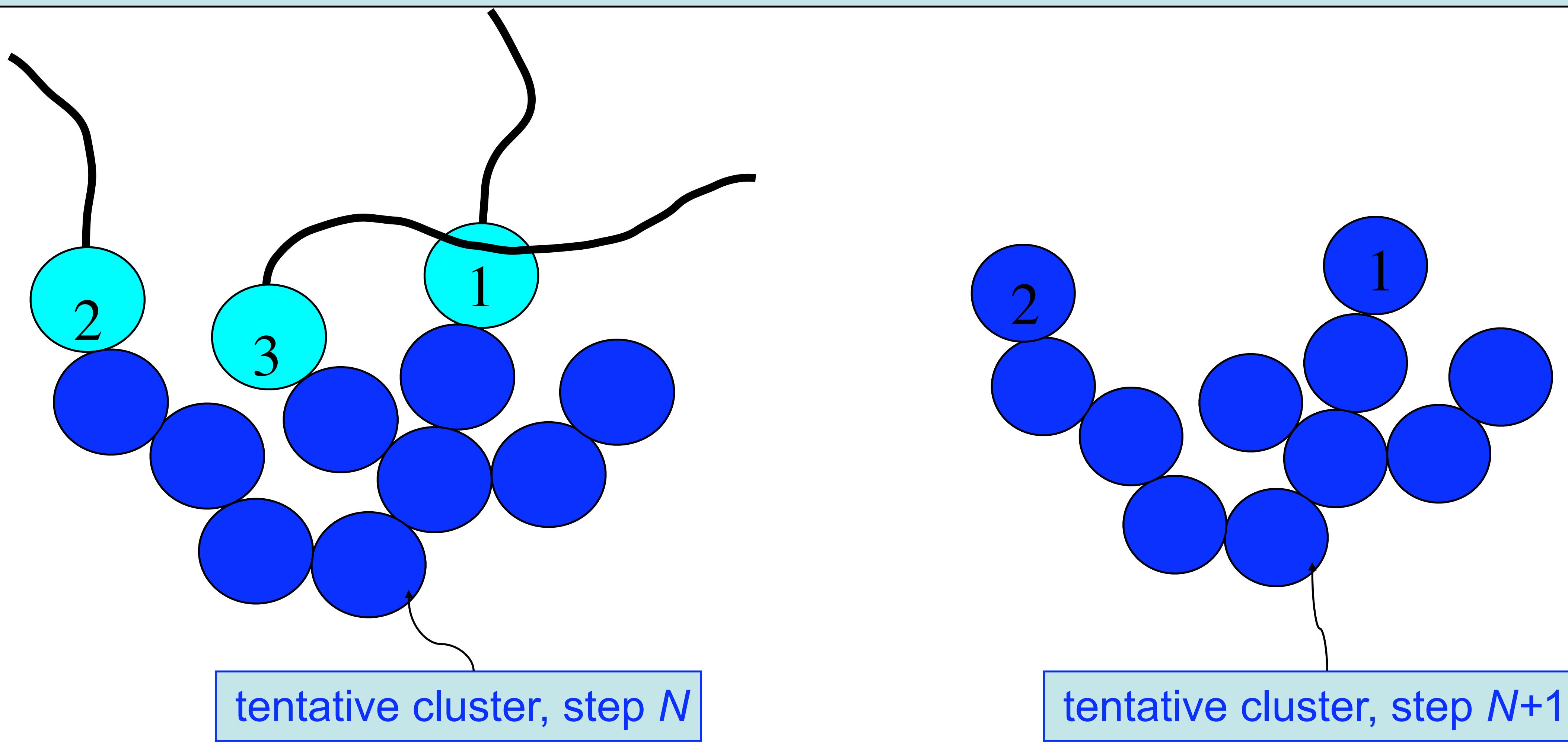
Caveats:

1. **P≠NC** not proven
2. Average case may be easier than worst case
3. Alternative dynamics may be faster than random walk dynamics

Parallel Algorithm for DLA

D. Tillberg and JM, *PRE* **69**, 051403 (2004)

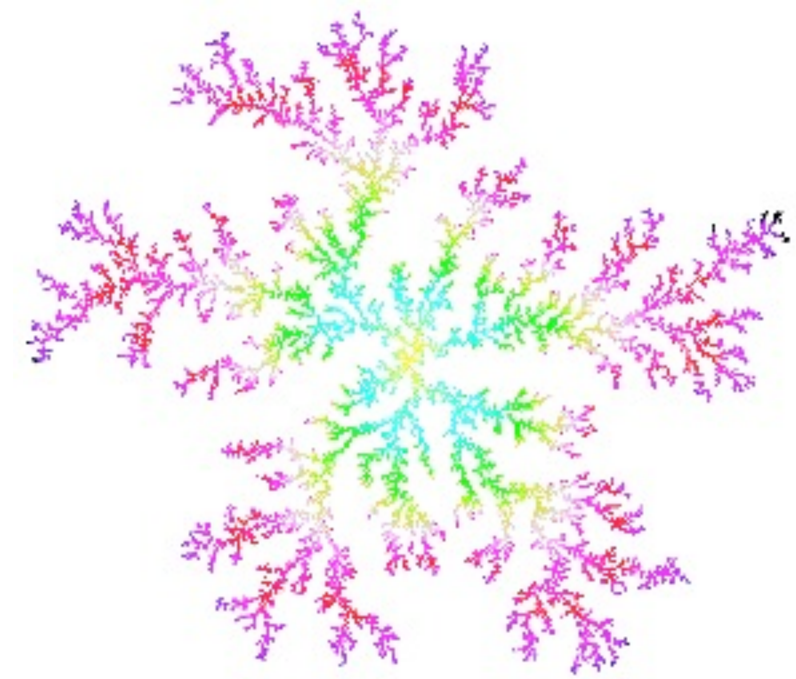
1. Start with seed particle at the origin and *N* walk trajectories
2. In parallel move all particles along their trajectories to tentative sticking points on tentative cluster, which is initially the seed particle at the origin.
3. New tentative cluster obtained by removing all particles that interfere with earlier particles.
4. Continue until all particles are correctly placed.

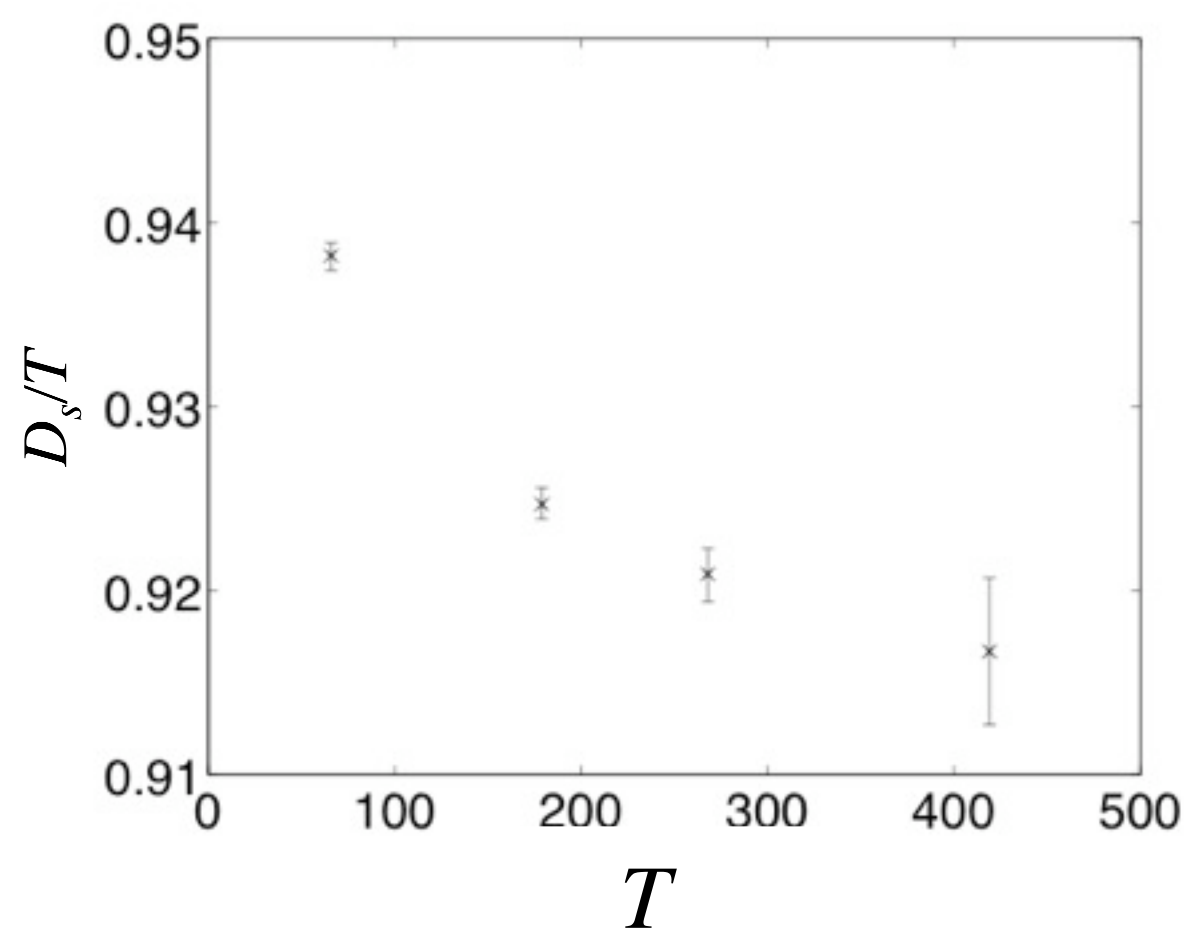


Efficiency of the Algorithm

- DLA is a tree whose structural depth, D_s scales as the radius of the cluster.
- The running time, T of the algorithm is asymptotically proportional to the structural depth.

$$T \sim D_s \sim N^{1/d_f}$$





Summary

- Depth (parallel time complexity of sampling distributions) is a property of any natural system described in the framework of statistical physics.
- Depth captures some salient features of the intuitive notion of complexity.