Complexity in Statistical Physics and Computer Science

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Outline

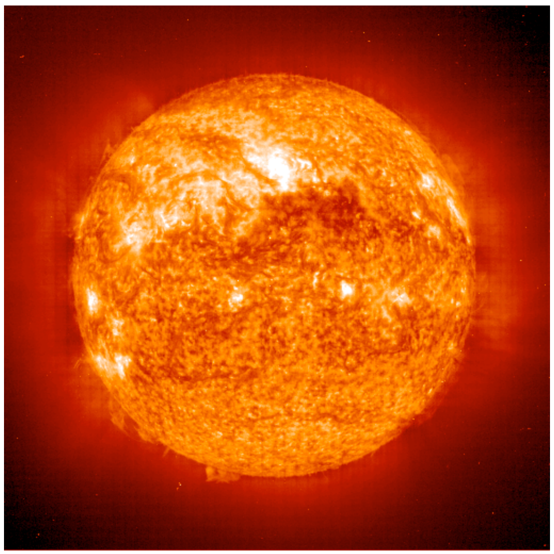
- What is ‘physical complexity?’
- Statistical physics
- Computational complexity
- Computational complexity in statistical physics
- Conclusions

Physical Complexity

- Is there a formally defined quantity applicable to physical systems that captures significant aspects of the intuitive notion of complexity?

- "I shall not today attempt further to define the kinds of material I understand to be embraced with that shorthand description. ... But I know it when I see it."
  - Justice Potter Stewart on pornography

The Sun and the Earth



$R=6.96 \times 10^8$ m

$M=1.99 \times 10^{30}$ kg

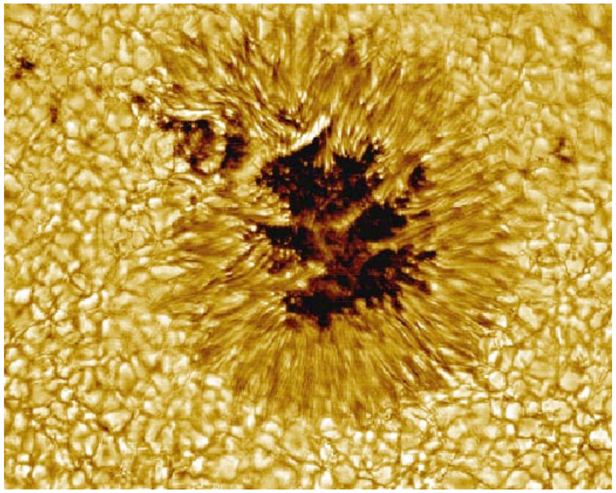
$T=5800$ K

$R=6.38 \times 10^6$ m

$M=5.98 \times 10^{24}$ kg

$T=255$ K

Which is more complex?



Vacuum Tower Telescope, NSA, NOAO

The Dream, Henri Rousseau, 1910

Algorithmic Complexity

Kolmogorov, Chaitin, Solomonoff

Minimum number of bits needed to fully describe the state of the system.

=

ENTROPY

Effective Complexity

–*The Quark and the Jaguar*, Gell-Mann

Minimum number of bits needed to describe the *regularities* of the system.

- Nuclear/particle physics
- Solar physics

SUN

- Nuclear/particle physics
- Condensed matter physics
- Geophysics
- Chemistry
- Biochemistry
- Biology
- Anthropology
- Economics
- Political Science

EARTH

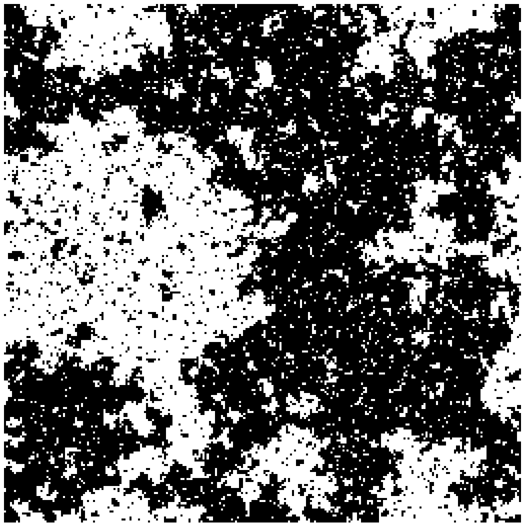
History and Complexity

–Charles Bennett

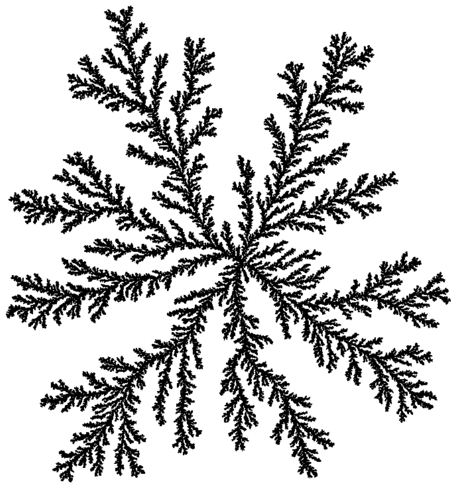
- The emergence of a complex system from a simple initial condition requires a long history.
- The length of the history of a system is measured in terms of the computational complexity of simulating the history.

Statistical Physics

Probabilistic description of systems with many interacting degrees of freedom.



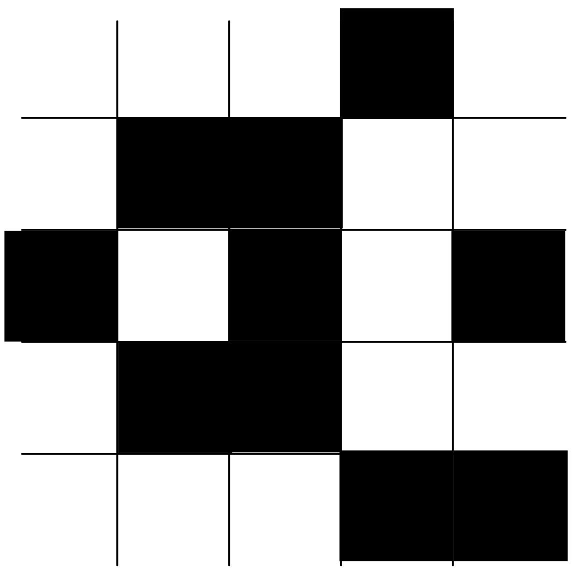
Ising Model



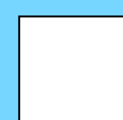
Diffusion Limited Aggregation

Ising Model

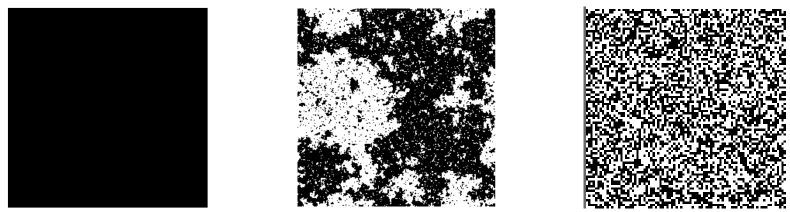
$$\mathcal{H} = -J \sum_{<i,j>} s_i s_j \qquad P[s_i] = e^{-\mathcal{H}/kT} \Big/ Z$$



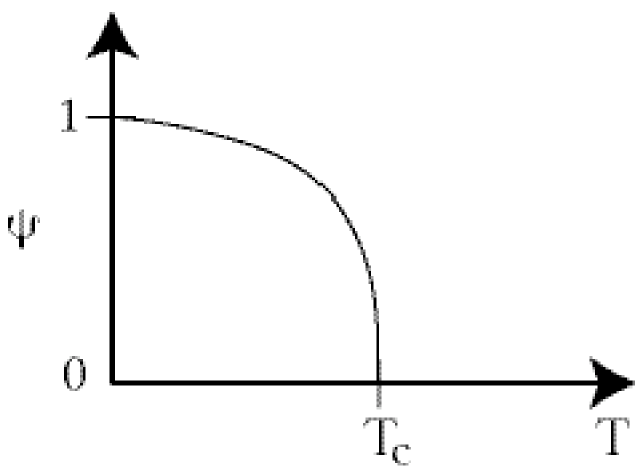
$s_i = 1$ 

$s_i = -1$ 

Behavior of Ising Model



$T \to 0$ $\quad$ $T = T_c$ $\quad$ $T \to \infty$



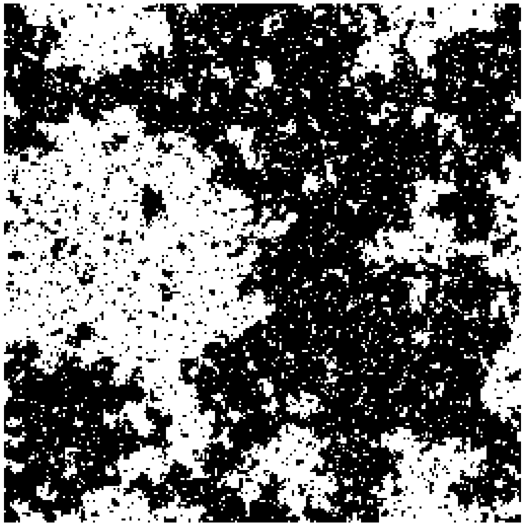
order parameter

$$\psi = \left| \frac{1}{N} \sum_i s_i \right|$$

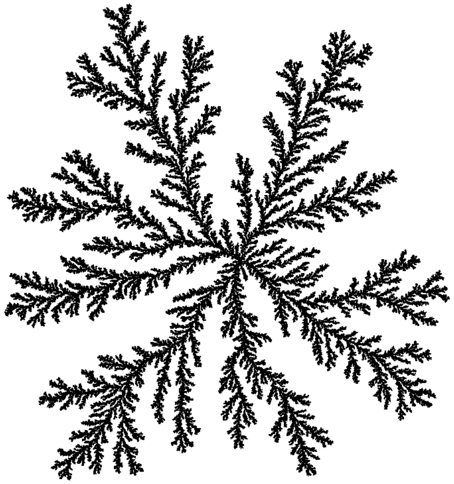
Statistical Physics

Equilibrium

Non-equilibrium

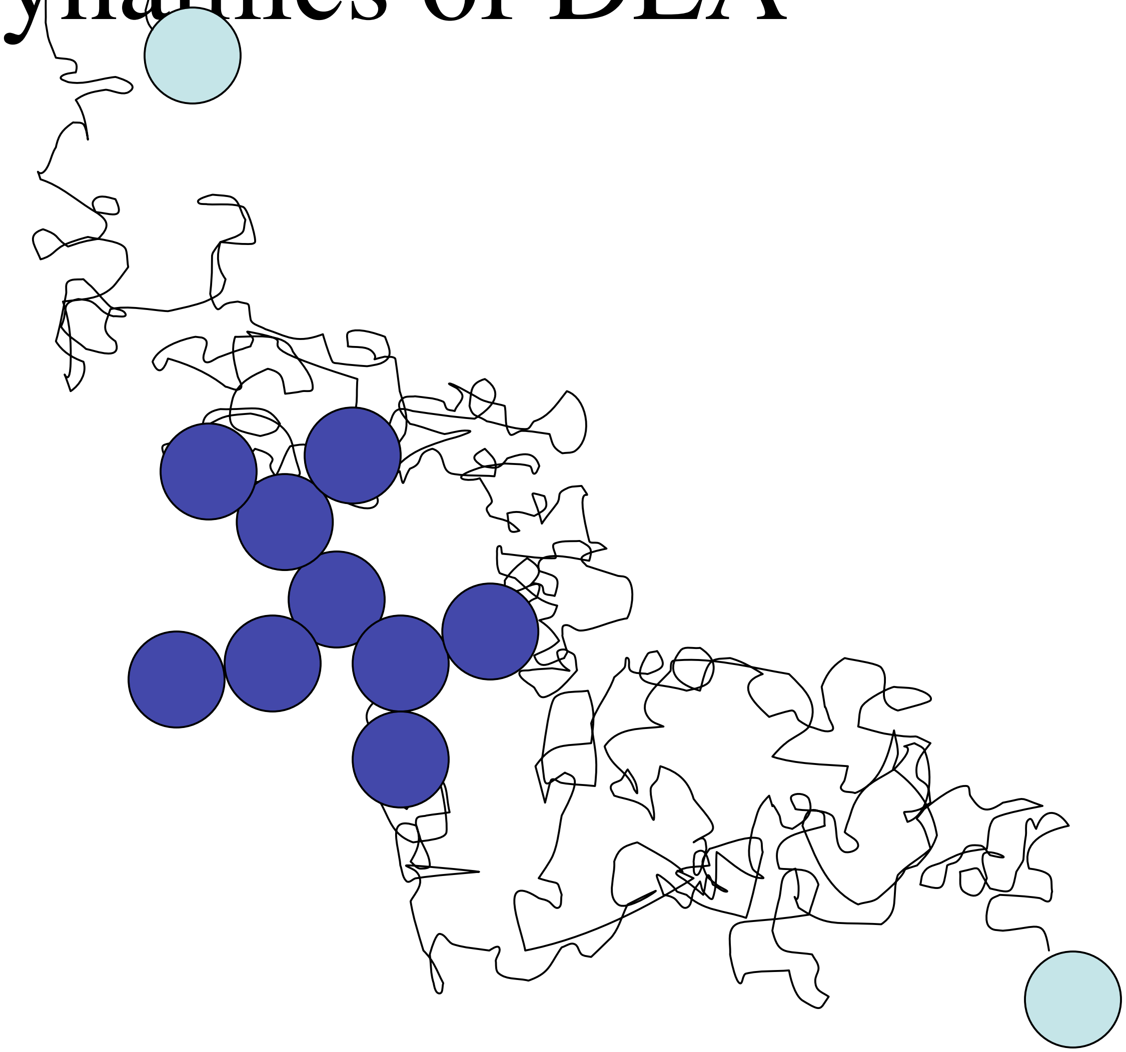


Ising Model

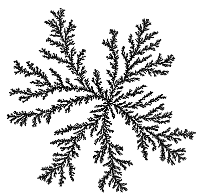


Diffusion Limited Aggregation

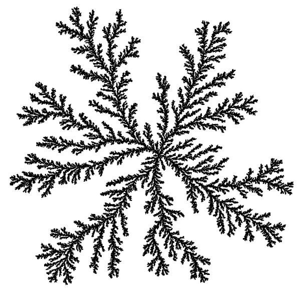
Dynamics of DLA



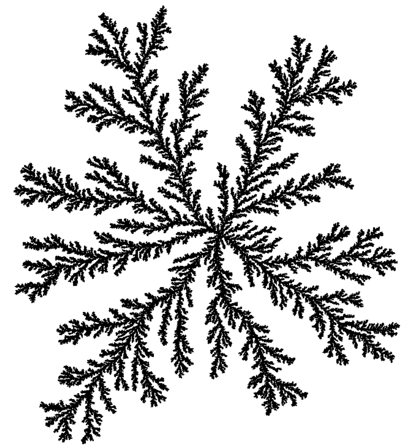
Self-Similar Growth of DLA



1.6×10^6



3.7×10^6



8.7×10^6

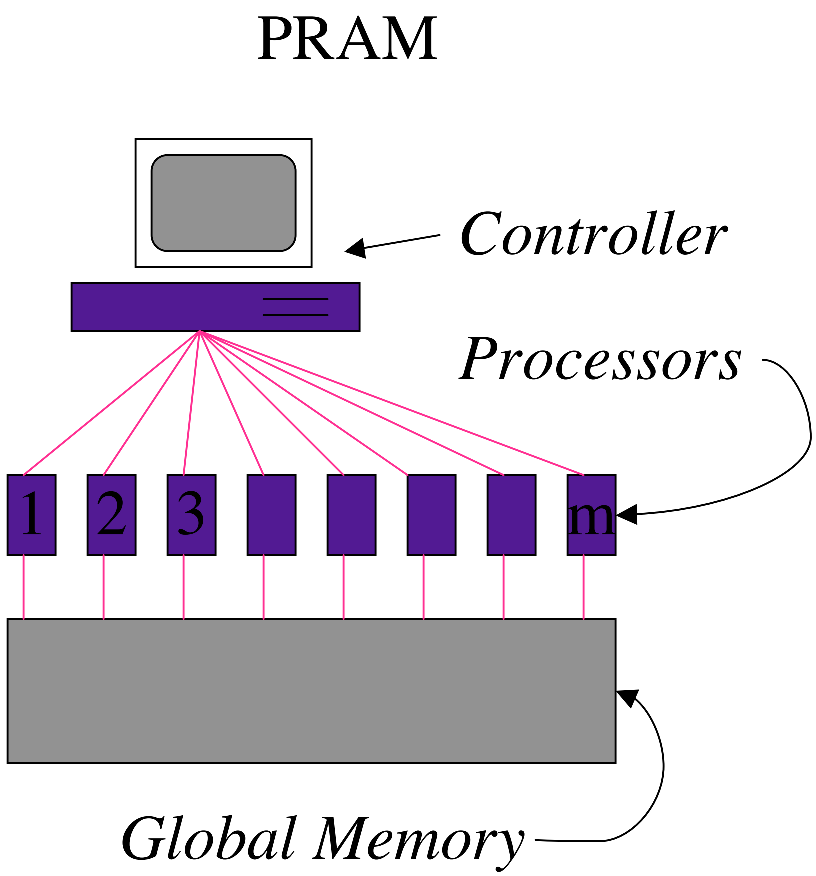
$M = R^d, \quad d = 1.71$

Computational Complexity

- How do computational resources scale with the size of the problem, *n*?
  - Time
  - Hardware

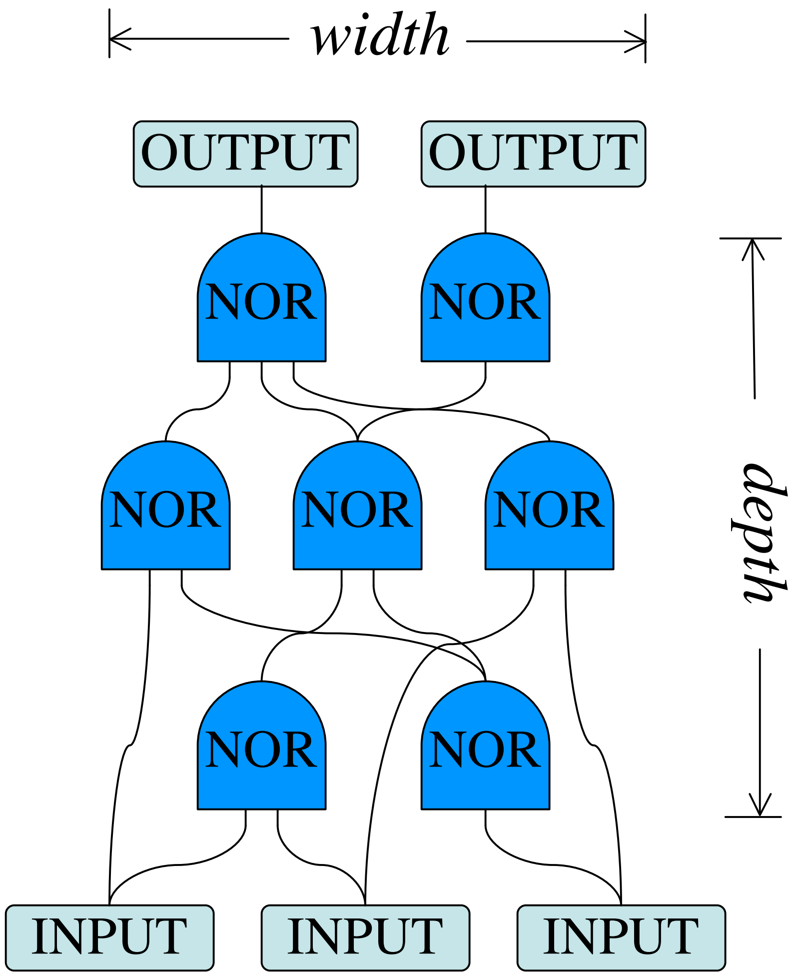
- Equivalent results independent of the model of computation.
  - Turing machine
  - Random access machine
  - ✓ Parallel random access machine
  - ✓ Boolean circuit family
  - Formal logic

Parallel Random Access Machine



- Each processor runs the same program but has a distinct label
- Each processor communicates with any memory cell in a single time step.
- Primary resources:
  - *Number of processors*
  - ✓ *Parallel Time*

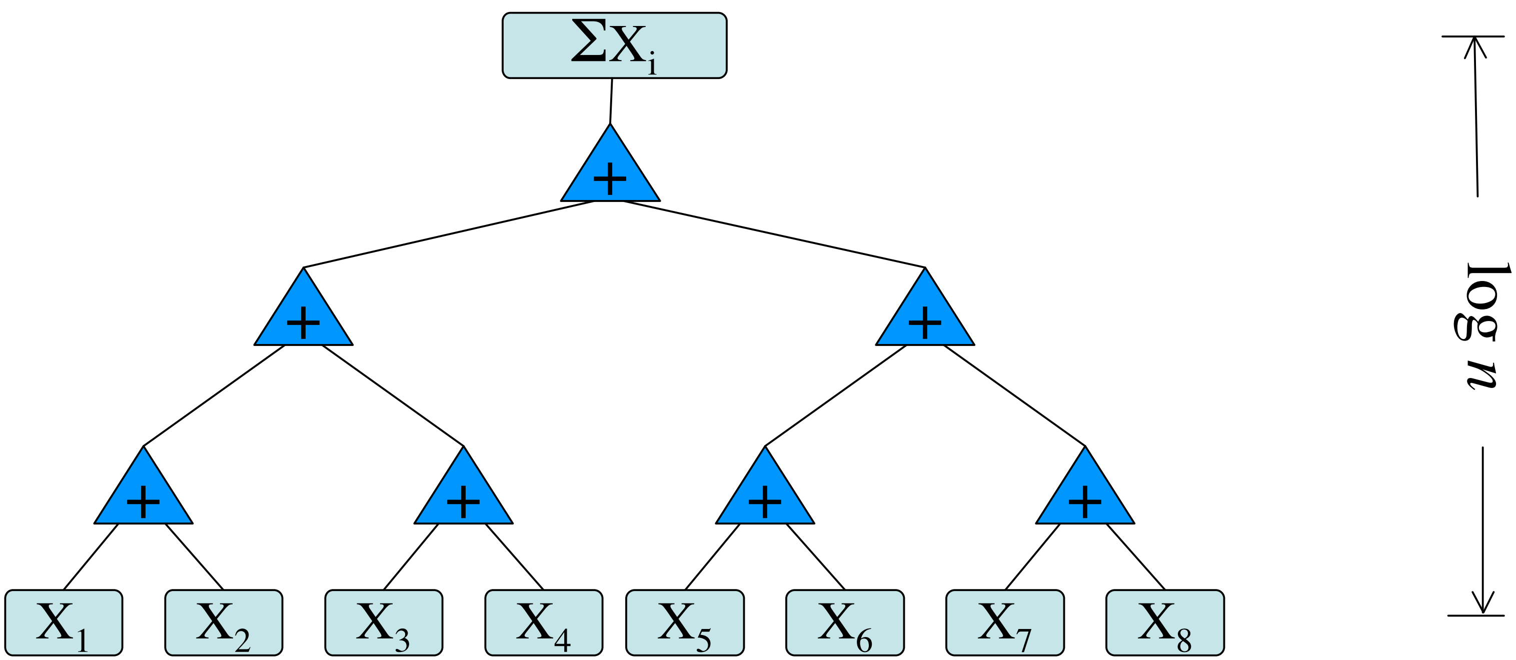
Boolean Circuit Family



- Gates are evaluated one level at a time from input to output.
- One circuit in the family for each problem size.
- Primary resources are
  - *Width*=maximum number of gates in a level~number of processors
  - ✓ ***Depth***=number of levels~parallel time

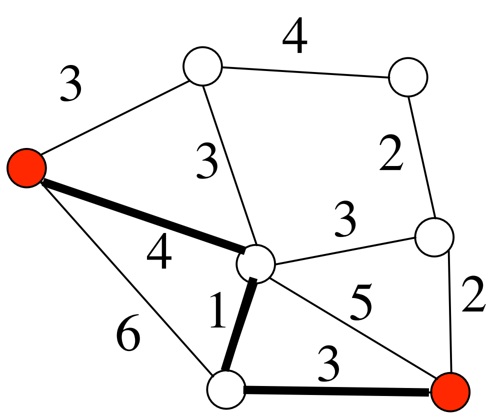
Parallel Computing

Adding n numbers can be carried out in $O(\log n)$ depth (parallel time) using $O(n)$ width (processors).



Minimum Weight Path

Find the minimun weight path between two point on a graph with n weighted edges.

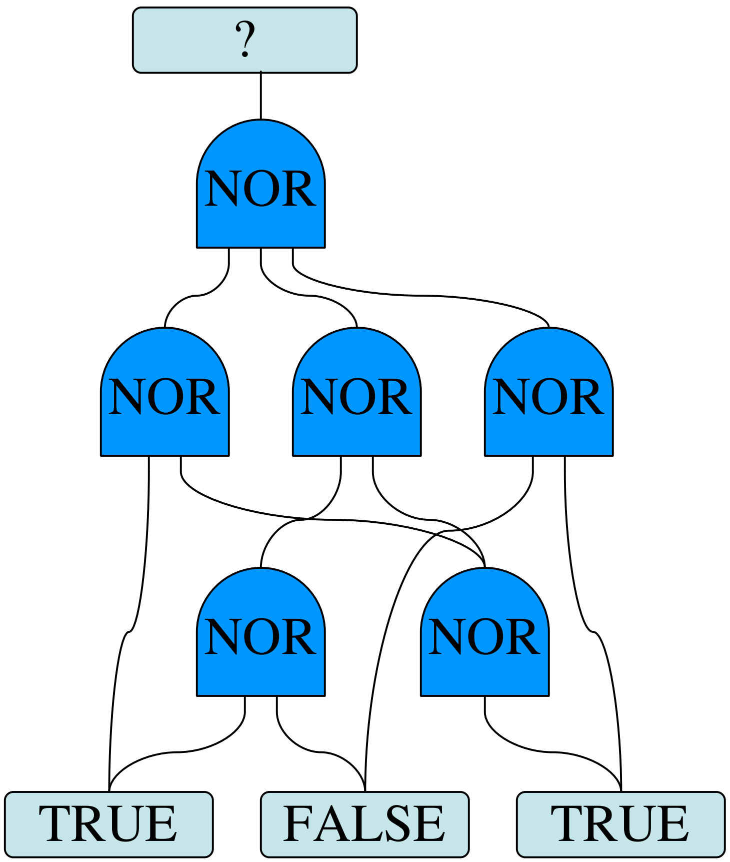


Solution can be found in depth $O(\log^2 n)$ using n^3 hardware.

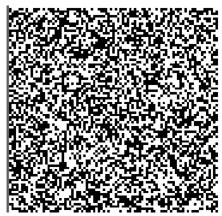
Addition and minimum weight path are in the class **NC**, problems that can be solved in polylog depth with polynomial hardware.

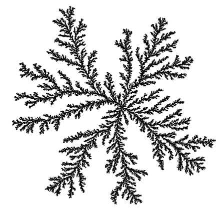
P-completeness

- **P** is the class of *feasible* problems: solvable in polynomial width and depth.
- Can all feasible problems be solved efficiently in parallel (**P=NC**)?
- P-complete problems are the hardest problems in P to solve in parallel. It is believed they are *inherently sequential:* not solvable in polylog depth.
- The circuit value problem is P-complete.



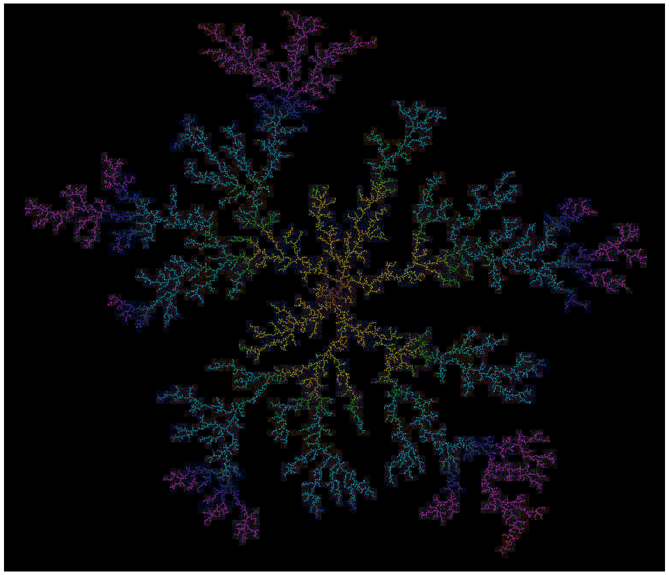
Measuring History: DEPTH





The **DEPTH** of a physical object chosen from a statistical distribution with *n* degrees of freedom is the *parallel time (circuit depth)* required to simulate a typical object with the *most efficient* algorithm using a PRAM (circuit family) with a source of random bits and polynomial in *n* hardware (width).

Depth of DLA



Theorem: Determining the shape of an aggregate from the random walks of the constituent particles is a P-complete problem.

Proof sketch: Reduce the circuit value problem to DLA dynamics.

Q: Is there a faster parallel algorithm than the defining sequential algorithm?

A: Yes, attach many particles in the same step:

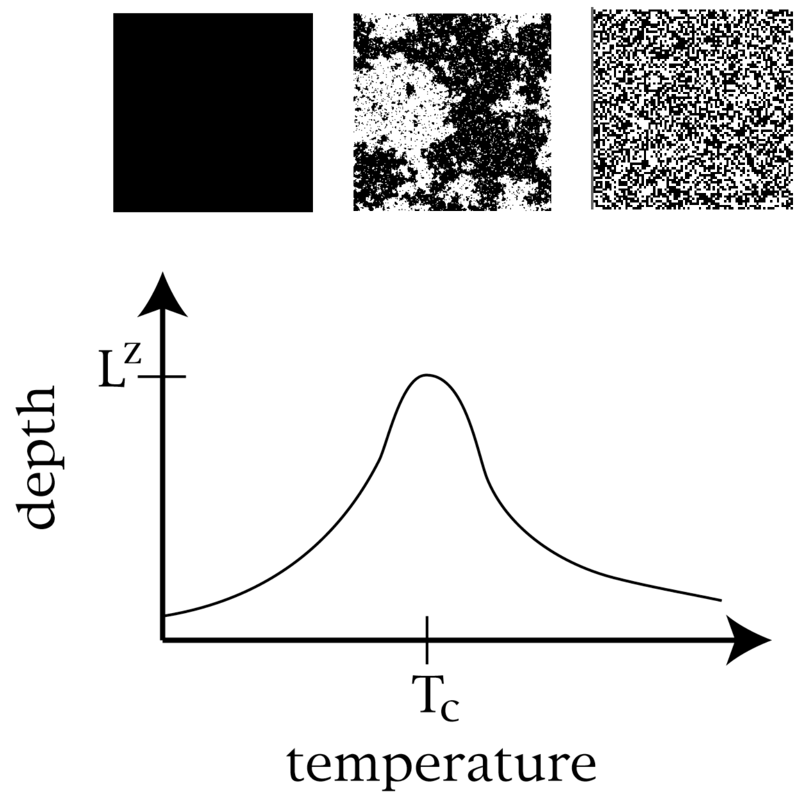
$$D \leq N^{1/d}$$

Depth of Ising Model

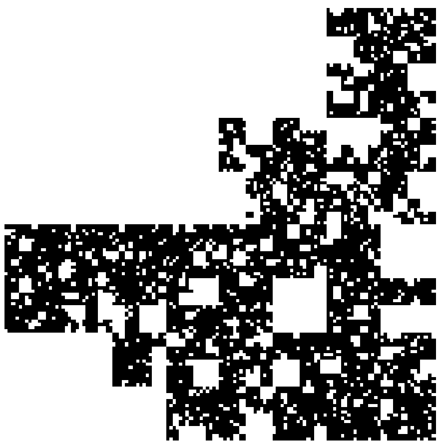
- Away from the critical point depth is constant independent of system size L (Metropolis algorithm).
- Near T_c the best known algorithms are cluster algorithms so that

$$D \leq L^z$$

with z the 'dynamic exponent' ($z \approx 0.2$ for 3D Ising).

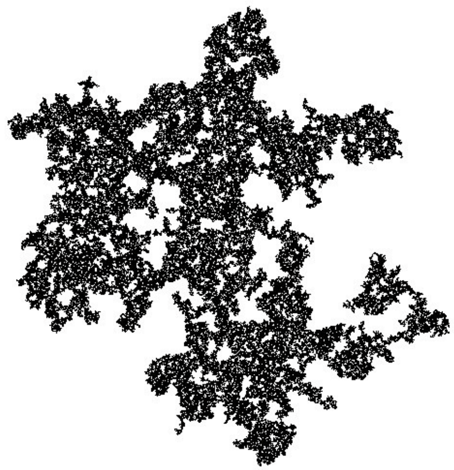


Are Fractals Complex?



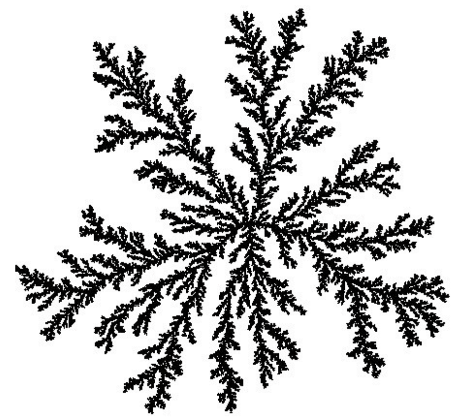
Mandelbrot Percolation

DEPTH ~ constant



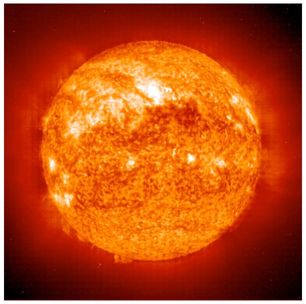
Invasion Percolation

DEPTH ~ $\log^2 N$



DLA

DEPTH ~ $N^{1/d}$



Conclusions

- Computational complexity theory provides interesting perspectives on physical systems.
- Depth, defined as the minimum number of parallel steps needed to simulate a system, is a robust measure widely applicable in statistical physics.
- Depth is correlated with intuitive notions of "physical complexity."