Computational Complexity and Growth Models in Statistical Physics

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Supported by the National Science Foundation

Outline

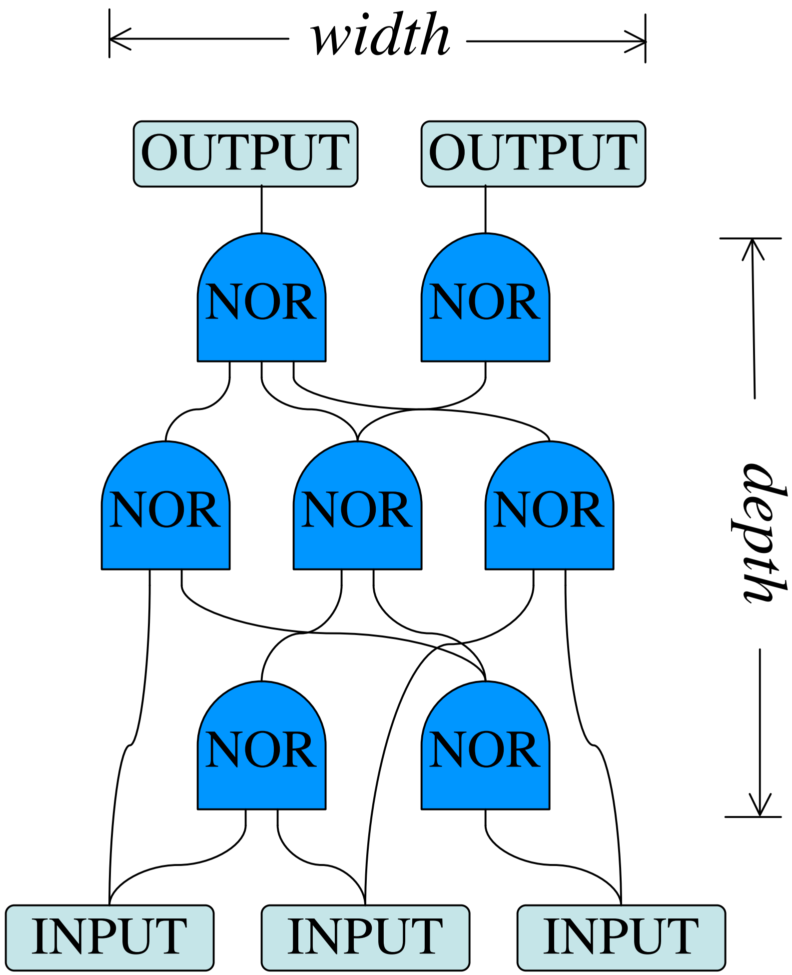
- Parallel computing and computational complexity
- Diffusion limited aggregation
- Growing networks
- Physical complexity and computational complexity

Computational Complexity

- How do computational resources scale with the size of the problem?
  - Time
  - Hardware

- Equivalent results independent of the model of computation.
  - Turing machine
  - ✓**Parallel random access machine**
  - ✓**Boolean circuit family**
  - Formal logic

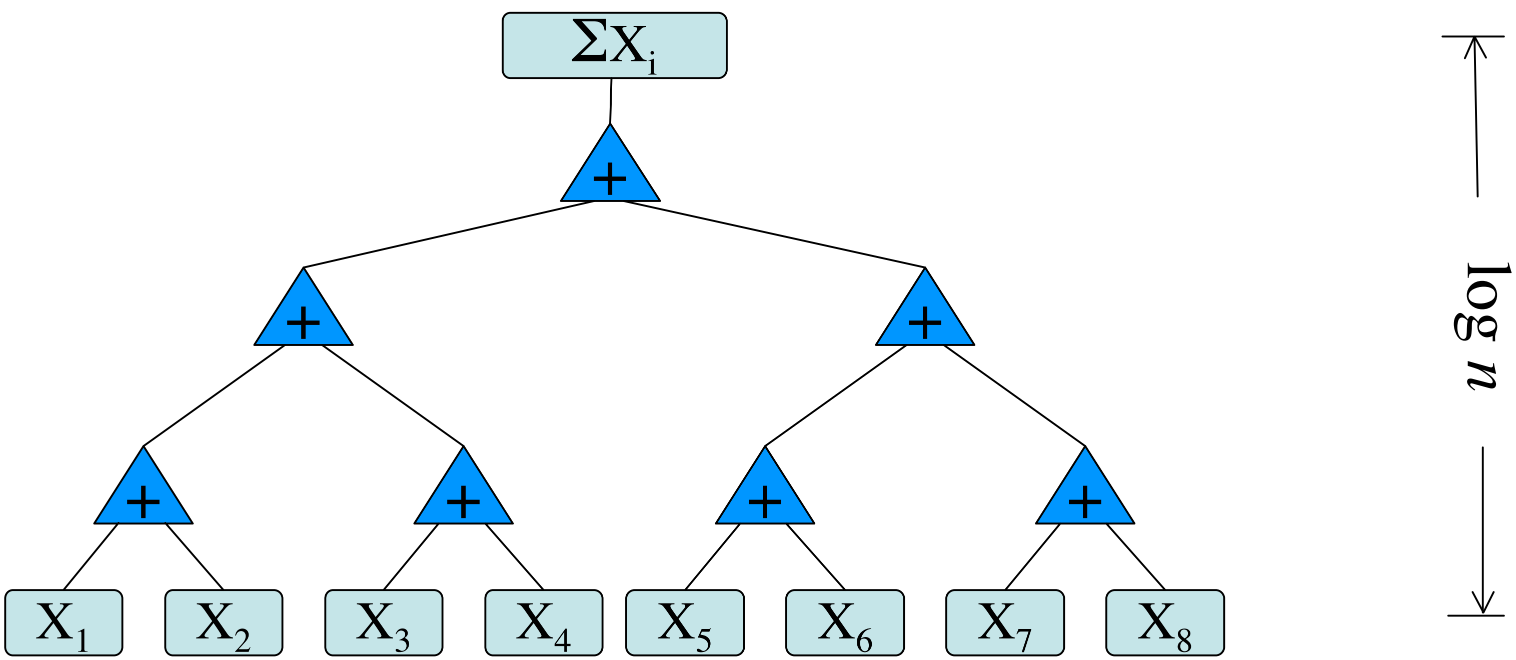
Boolean Circuit Family



- Gates evaluated one level at a time from input to output with no feedback.
- One hardwired circuit for each problem size.
- Primary resources
  - ***Depth***=number of levels, D_c
  - *Width*=maximum number of gates in a level
  - *Work*=total number of gates

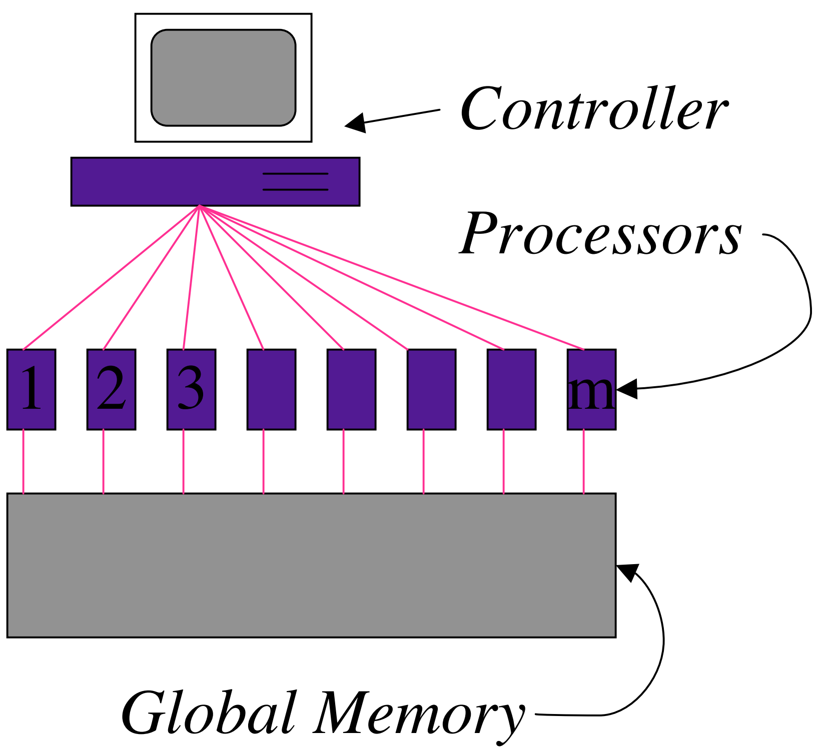
Parallel Computing

Adding n numbers can be carried out in $O(\log n)$ steps using $O(n)$ processors.



Parallel Random Access Machine

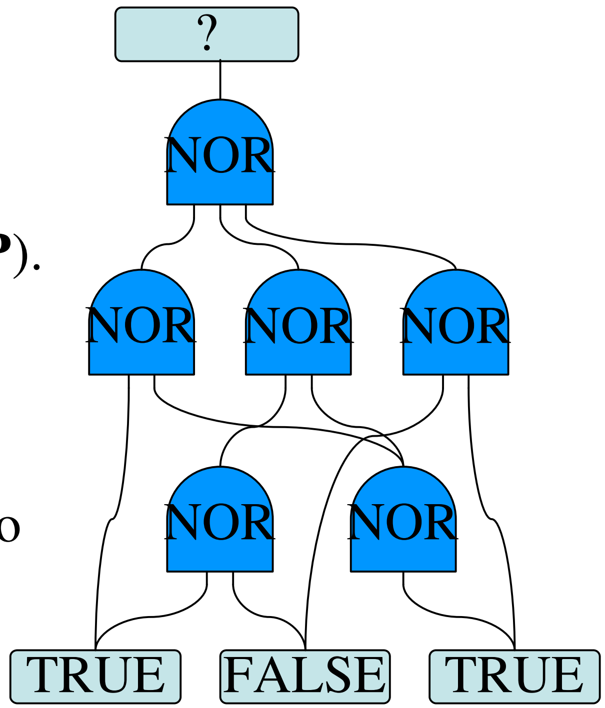
PRAM



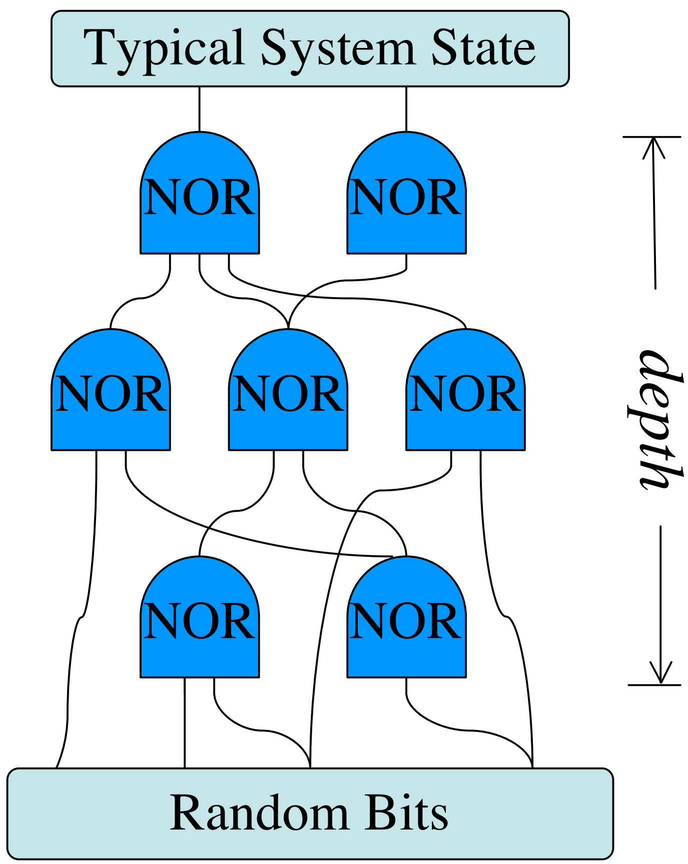
- Each processor runs the same program but has a distinct label
- Each processor communicates with any memory cell in a single time step.
- Primary resources:
  - ***Parallel time ~ depth***
  - *Number of processors~width*

Complexity Classes and P-completeness

- **P** is the class of *feasible* problems: solvable with polynomial work.
- **NC** is the class of problems efficiently solved in parallel (polylog depth and polynomial work, **NC** $\subseteq$ **P**).
- Are there feasible problems that cannot be solved efficiently in parallel (**P≠NC**)?
- **P**-complete problems are the hardest problems in **P** to solve in parallel. It is believed they are *inherently sequential:* not solvable in polylog depth.
- The Circuit Value Problem is **P**-complete.



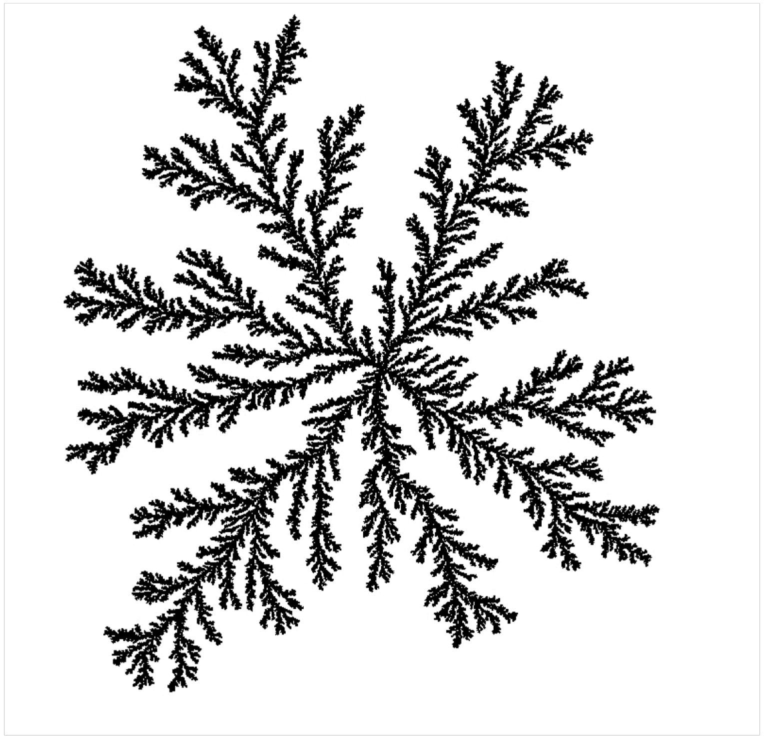
Sampling Complexity



- Monte Carlo simulations convert random bits into descriptions of a typical system states.
- **What is the depth of the shallowest circuit (running time of the fastest PRAM program) that generates typical states**?

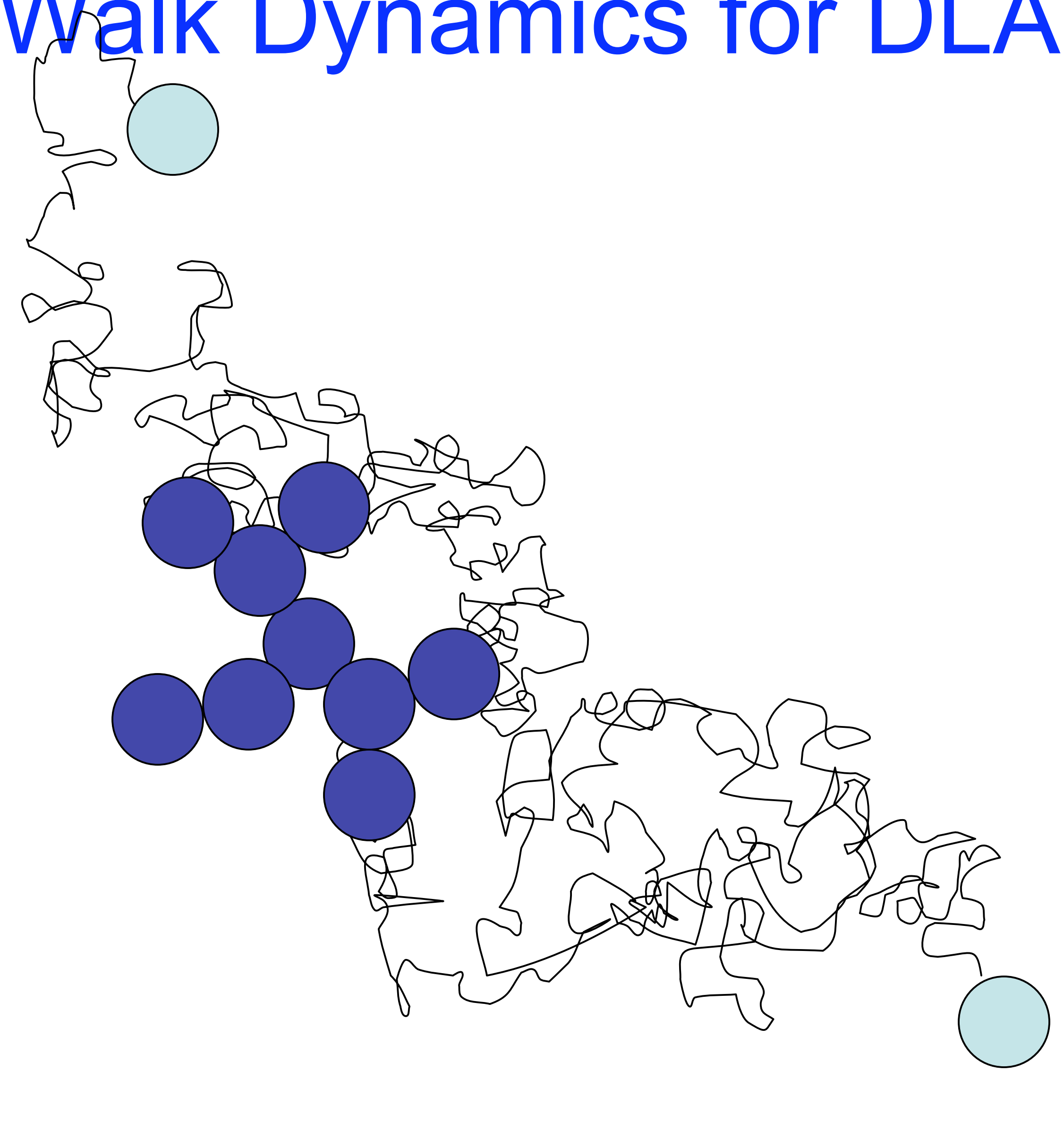
Diffusion Limited Aggregation

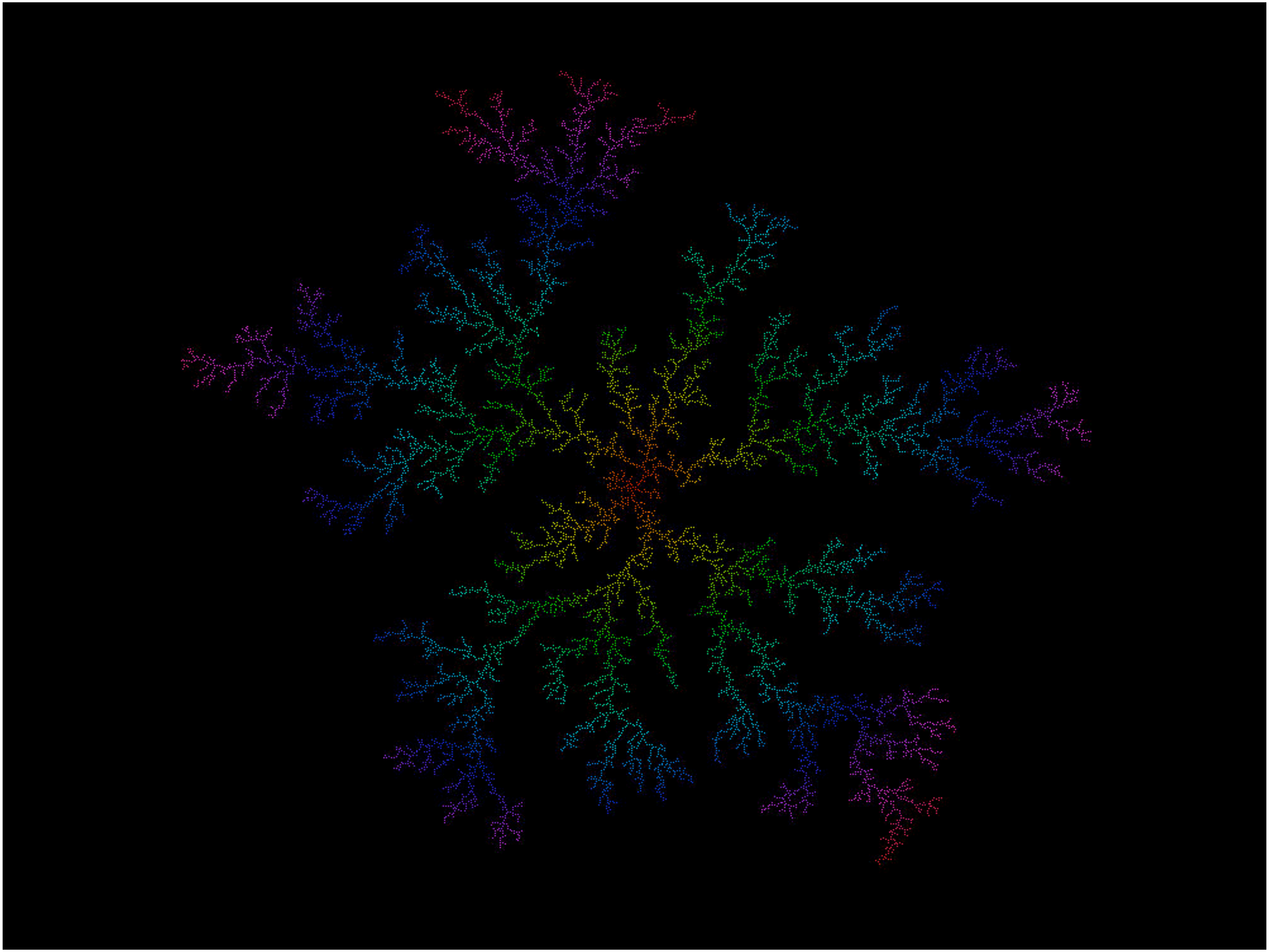
Witten and Sander, PRL 47, 1400 (1981)



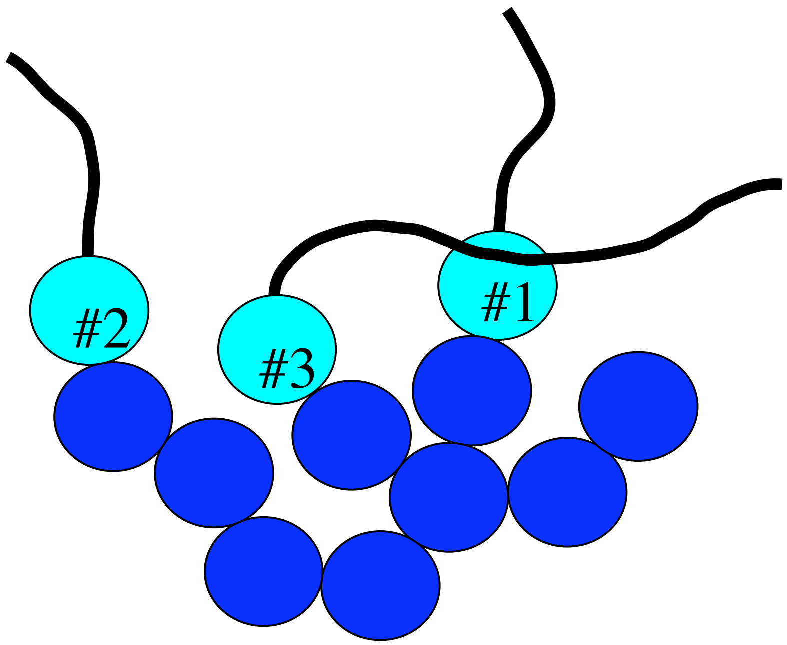
- Particles added *one at a time* with sticking probabilities given by the solution of Laplace's equation.
- Self-organized critical object
  d_f=1.715… (2D)
- Physical systems:
  Fluid flow in porous media
  Electrodeposition
  Bacterial colonies

Random Walk Dynamics for DLA

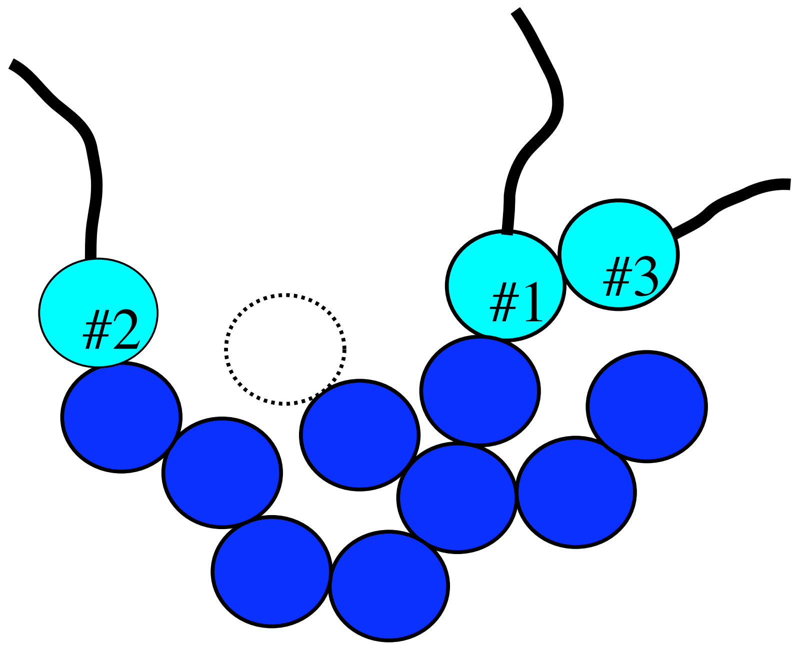




The Problem with Parallelizing DLA



Parallel dynamics ignores *interference* between 1 and 3

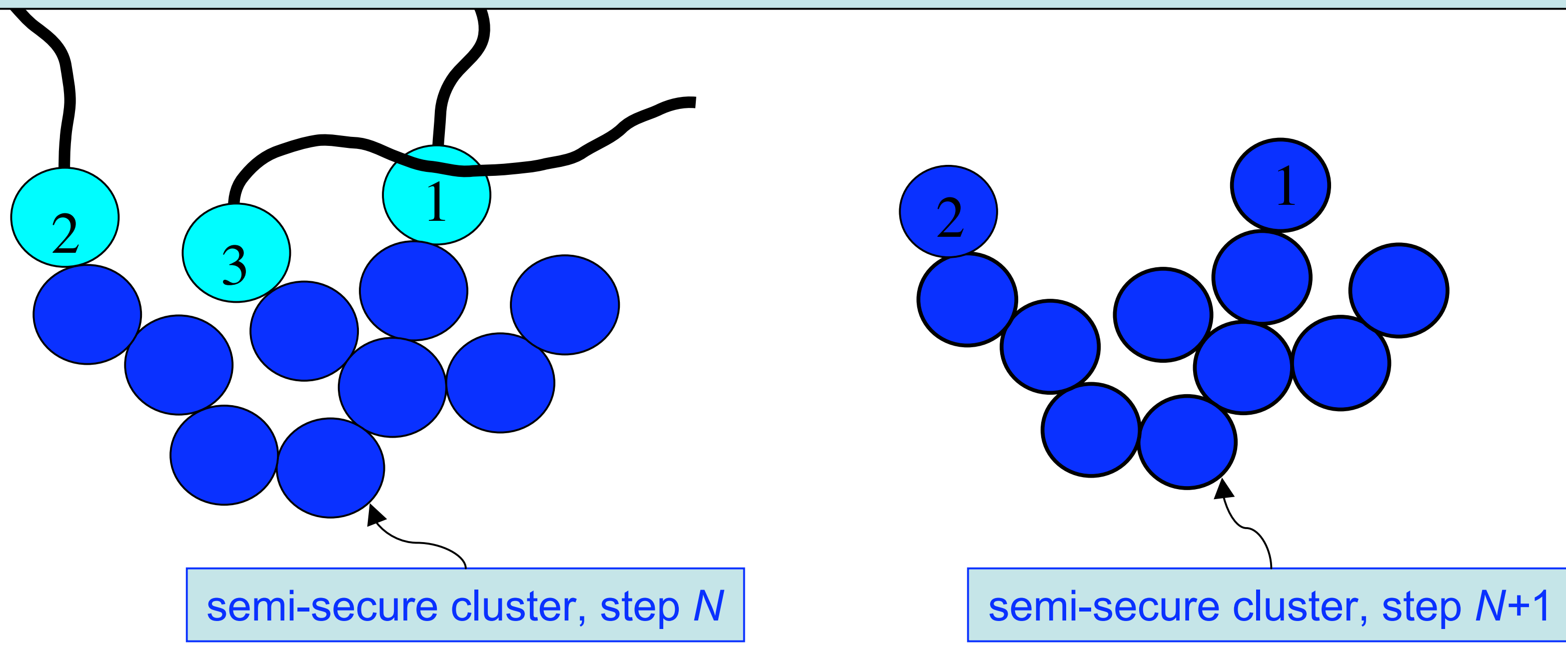


Sequential dynamics

Parallel Algorithm for DLA

D. Tillberg and JM, PRE **69**, 051403 (2004)

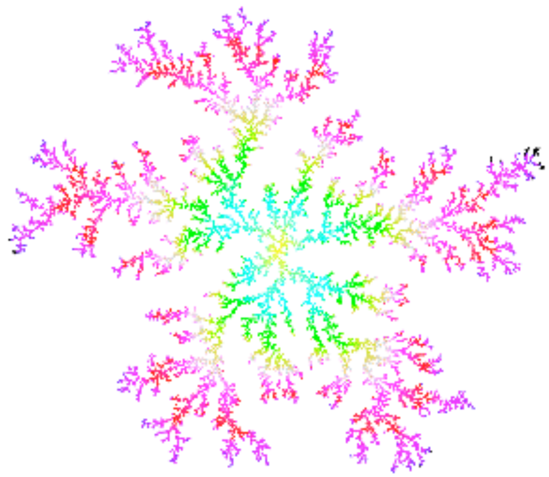
1. Start with seed particle at the origin and *N* walk trajectories
2. In parallel move all particles along their trajectories to tentative sticking points on the existing *semi-secure* cluster, which is initially the seed particle at the origin.
3. New semi-secure cluster obtained by removing all particles that interfere with earlier particles.
4. Continue until all particles are correctly placed.

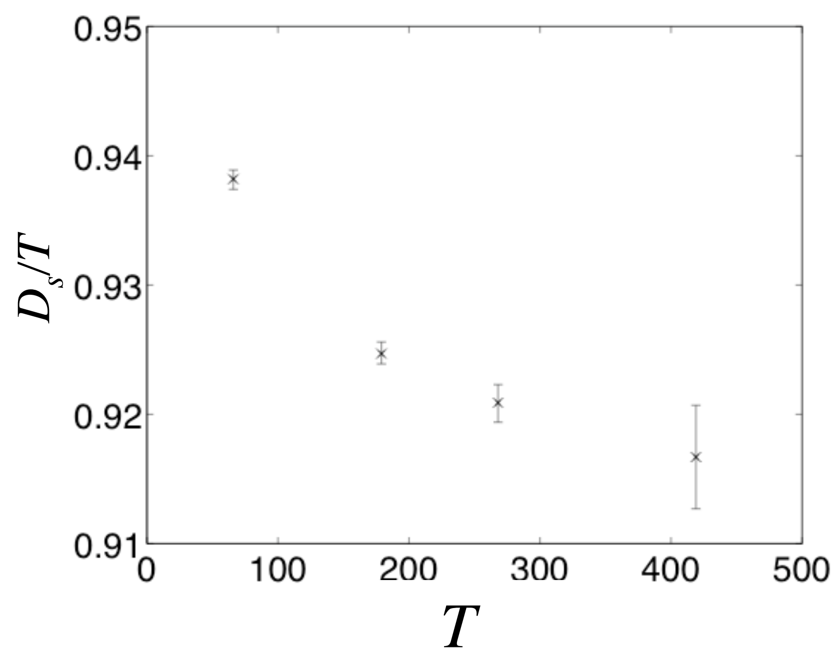


Efficiency of the Algorithm

- DLA is a tree whose structural depth , D_s scales as the radius of the cluster.
- The running time, T of the algorithm is asymptotically proportional to the structural depth.

$$T \sim D_s \sim N^{1/d_f}$$





Depth of DLA

Theorem: Determining the shape of an aggregate from the random walks of the constituent particles is a **P**-complete problem.

Proof sketch: Reduce the Circuit Value Problem to DLA dynamics.

Caveats:

1. **P≠NC** not proven
2. Average case may be easier than worst case
3. Alternative dynamics may be easier than random walk dynamics

Conjecture: $D_c \sim D_s$

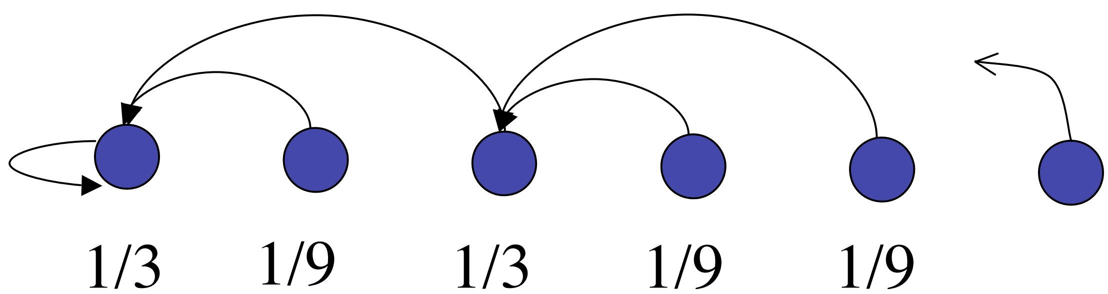
Growing Networks

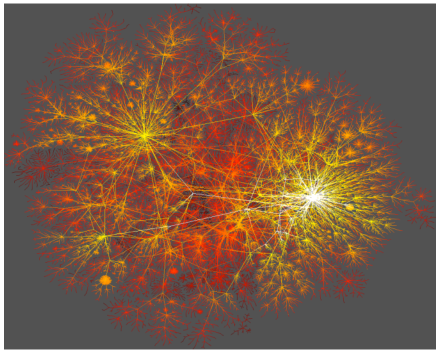
Barabasi and Albert, Science **286**, 509 (1999)
Krapivsky, Redner, Leyvraz, PRL **85**, 4629 (2000)

Add nodes one at a time, connecting new nodes to old nodes according to a "rich get richer" preferential attachment rule:

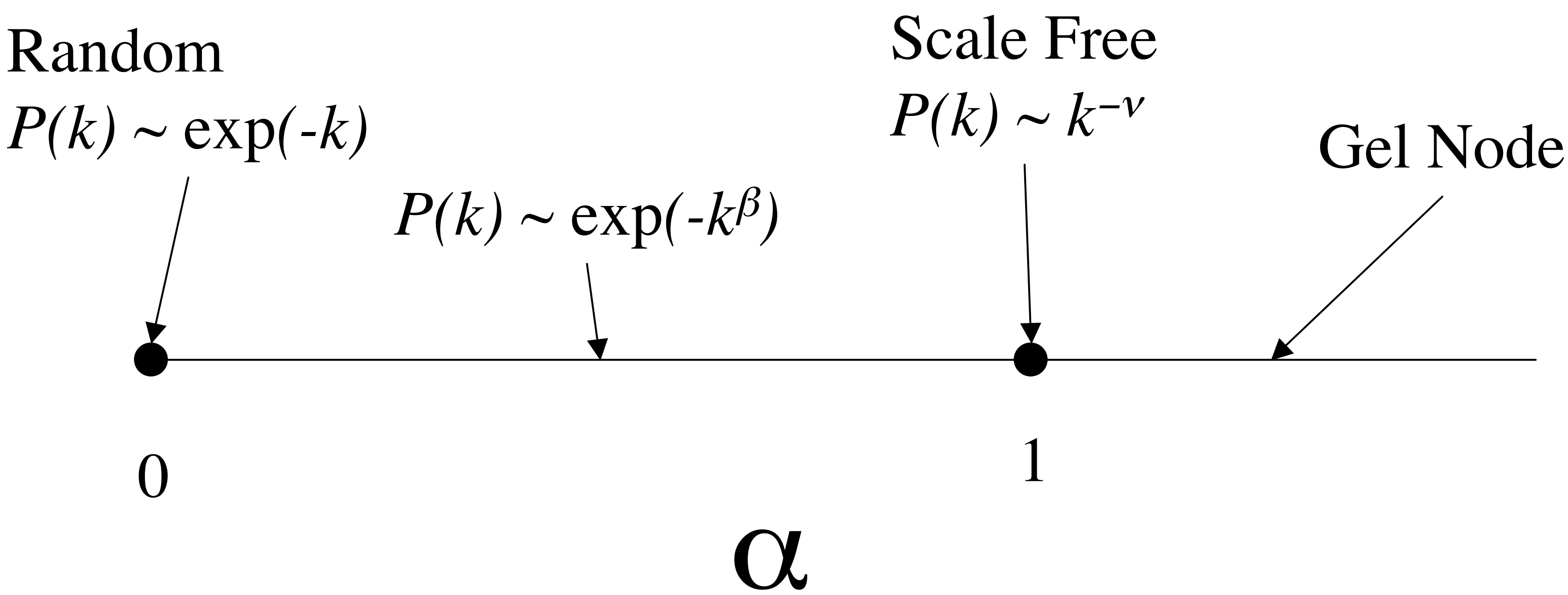
$$\pi_n(t) = \mathrm{Prob}[t \text{ connects to } n] \propto k_n(t)^{\alpha}$$

where $k_n(t)$ is the number of connection to node n at time t.





Behavior of Growing Networks



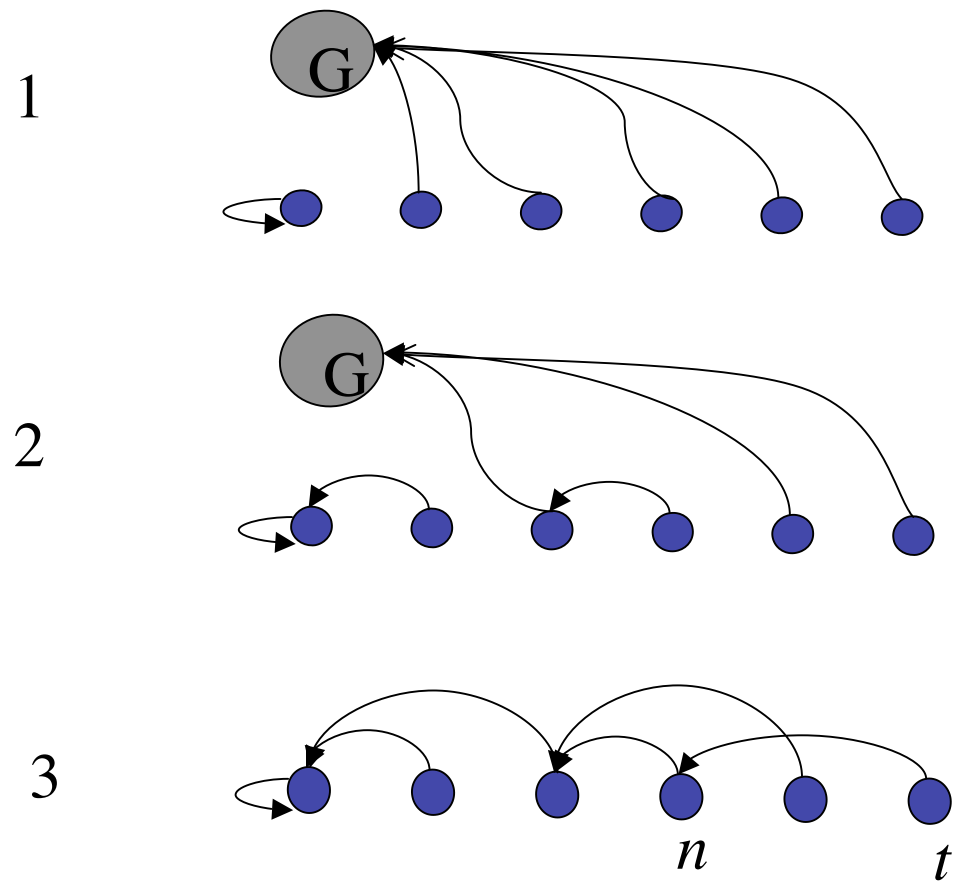
P(k) is the degree distribution

Discontinuous transition at α=1

Parallel Algorithm for Networks

B. Machta and JM, cond-mat/0408372

For the "high temperature" phase $0 \leq \alpha \leq 1$



- Initially all nodes are connected to a "ghost node."
- New connection are made according to bounds on connection probabilities computed from existing node degrees.
- Until all connections are determined.

Bound on step *S* of probability of node *t* connecting to node *n*:

$$p_n^S(t) = k_n^S(t)^\alpha / \tilde{Z}^S(t)$$

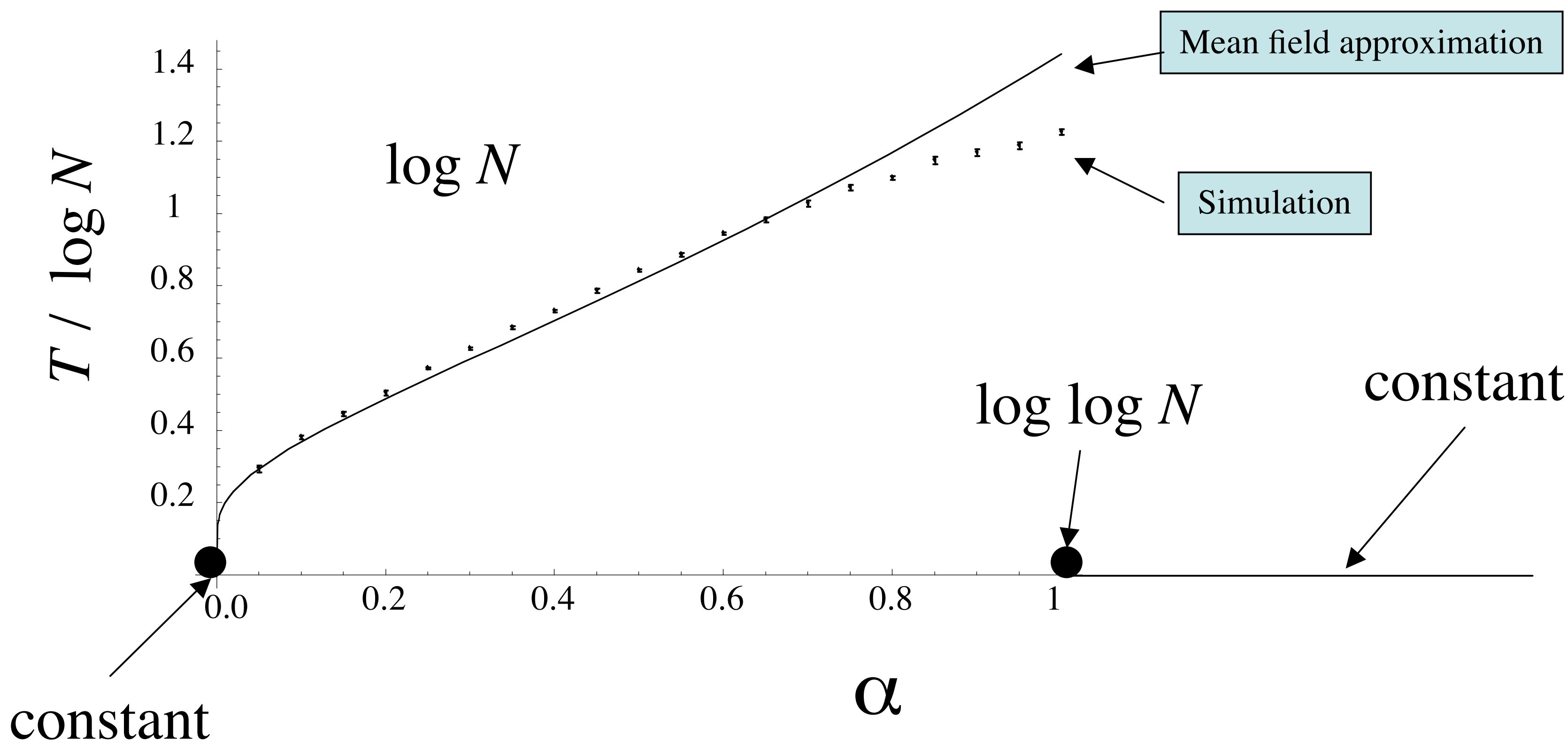
$$\tilde{Z}^S(t) = c k_g^S(t) + \sum_{m=0}^{t-1} k_m^S(t)^\alpha \qquad c = 2^\alpha - 1$$

$$p_n^1(t) \leq p_n^2(t) \leq \cdots \leq p_n^T(t) = \pi_n(t)$$

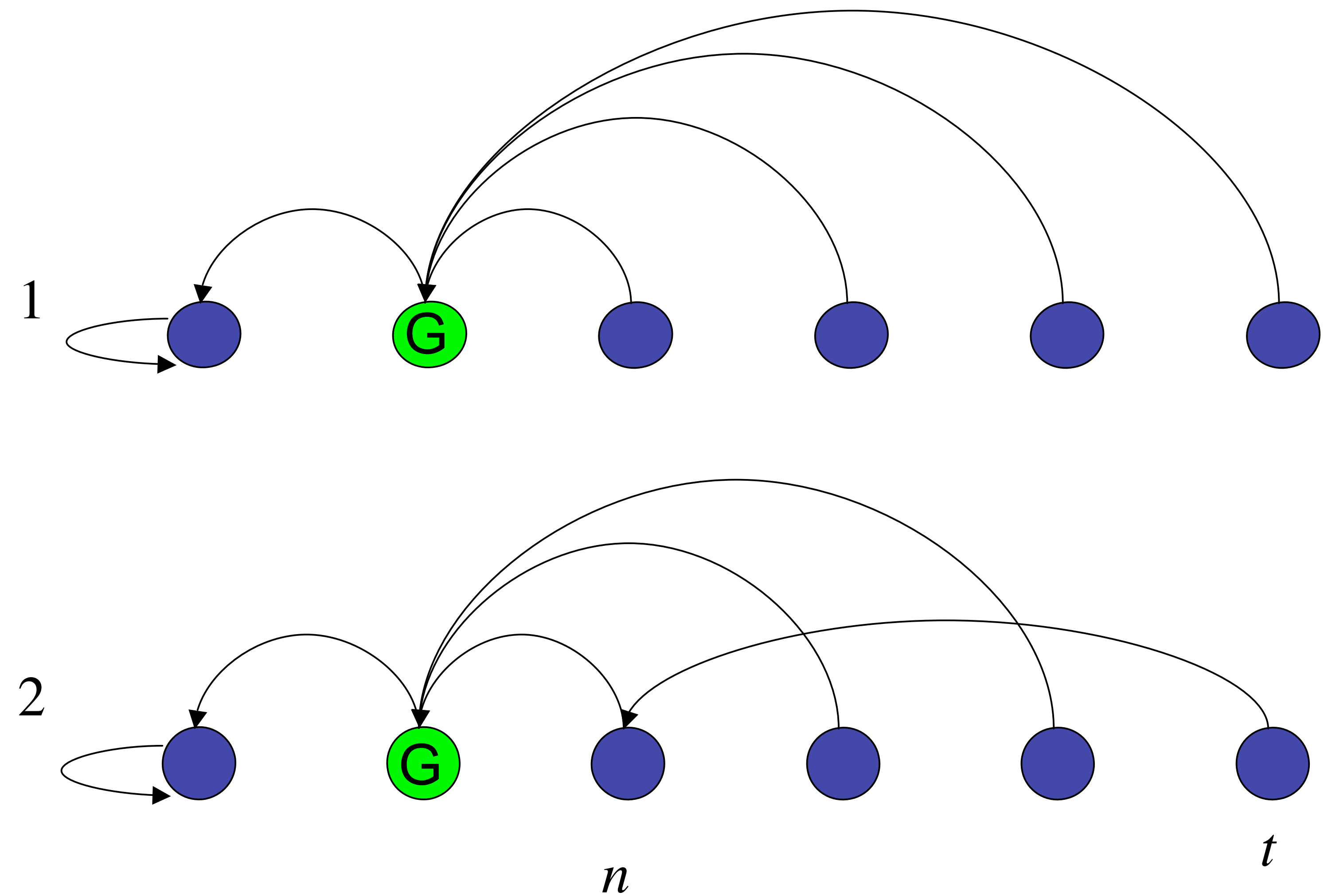
Conditional probability of node *t* connecting to node *n* on step *S* given it was previously connected to *g*:

$$\rho_n^S(t) = \frac{p_n^S(t) - p_n^{S-1}(t)}{1 - \sum_{m=0}^{t-1} p_m^{S-1}(t)}$$

Depth of Growing Networks

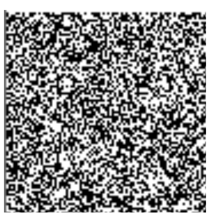


For the “low temperature” phase $\alpha > 1$



What is Physical Complexity?

- “I shall not today attempt further to define the kinds of material I understand to be embraced with that shorthand description. ... But I know it when I see it.”
  - – Justice Potter Stewart on pornography



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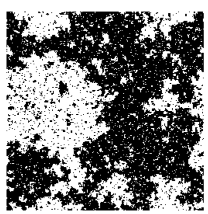

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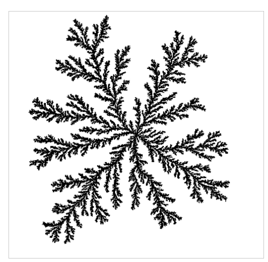

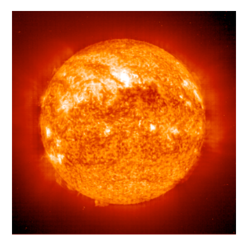

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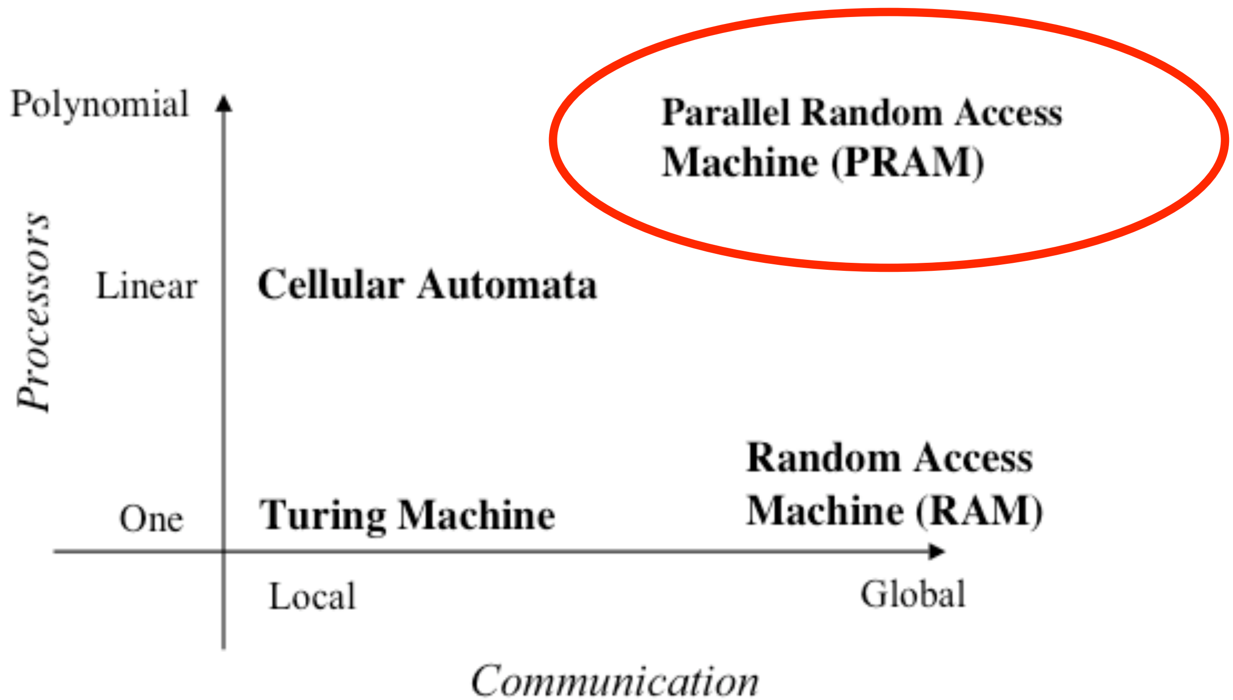
History and Complexity

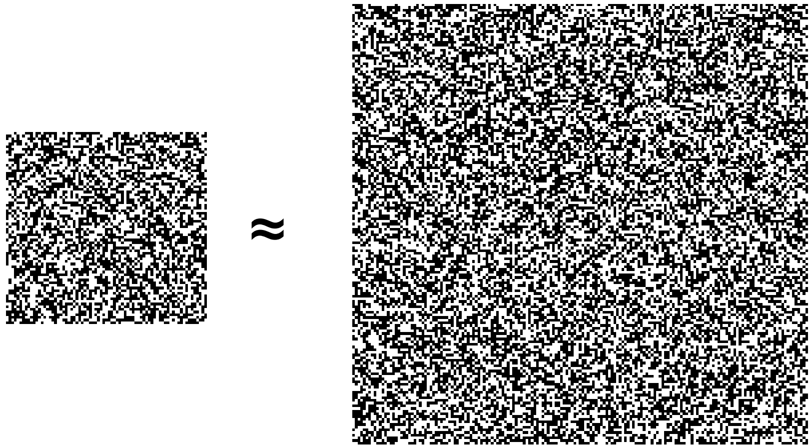
–Charles Bennett

- The emergence of a complex system from simple initial conditions requires a long history.
- History can be quantified in terms of the computational complexity of simulating states of the system.

What Model of Computation?



Discount Hardware



Complexity of a system composed of nearly independent subsystem is given by the most complex subsystem.

Discount Communication

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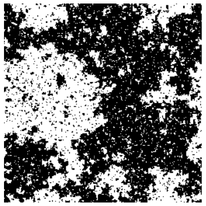
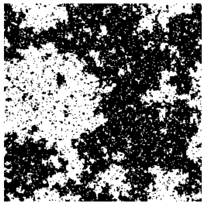
Complexity emerges from interactions, not from signal propagation.

Choose the Fastest Algorithm

 ≈ 

1 sec 1 yr

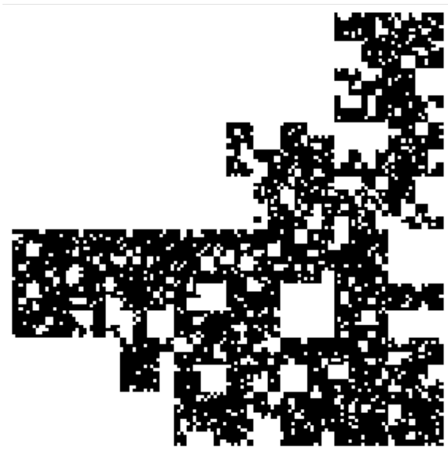
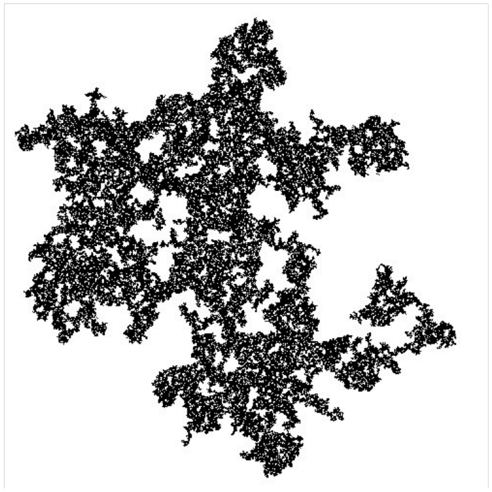
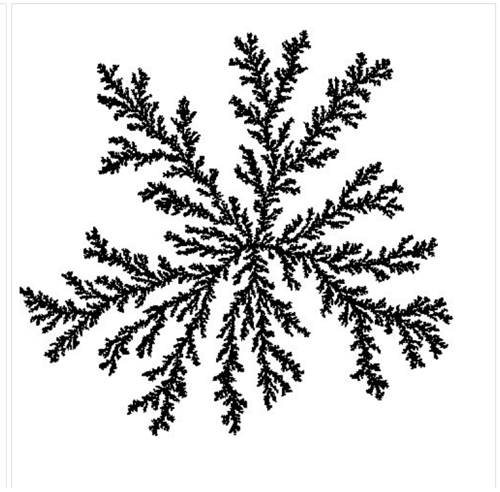
 ≈ 

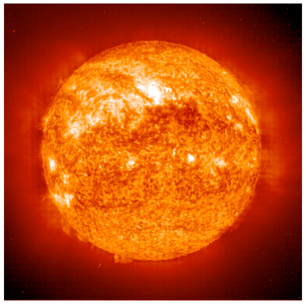
Swendsen-Wang Metropolis

Depth of Physical Systems

The depth of a physical system is the depth of a Boolean circuit (or parallel time on a PRAM) to simulate a typical system state using the most efficient algorithm.

Hierarchy of Depth

constant	polylog	polynomial
		
Mandelbrot percolation	**Invasion percolation**	**DLA**
Growing network $\alpha>1$	Growing network $\alpha\leq1$	$T=T_c$ Ising
$T>T_c$ Ising	Eden growth	
	Ballistic deposition	
	Bak-Sneppen model	
	Internal DLA	



Conclusions

- Computational complexity theory provides interesting perspectives on physical systems.
- DLA has power law depth.
- Growing networks display a complexity transition from logarithmic to constant depth at $\alpha=1$.
- Depth, defined as the minimum number of parallel steps needed to simulate a system, is correlated with intuitive notions of physical complexity.