Energy Cascades and Power Law Tails in Granular Gases

Jon Machta

University of Massachusetts Amherst

- Eli Ben-Naim, *Los Alamos*
- Ben Machta, *Brown & UMass*

- *NSF*, *DOE*
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Outline

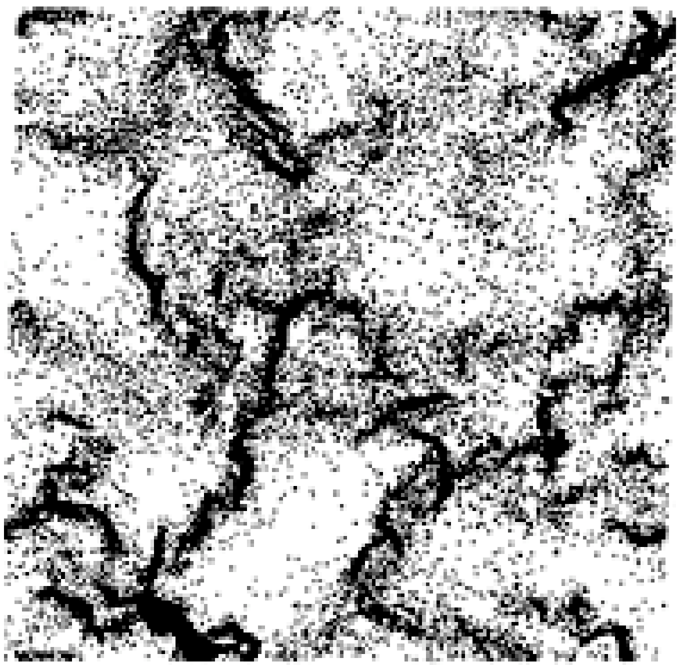
- Background
- Stationary states of the inelastic Boltzmann equation
- Driven steady states
- Decaying states
- Discussion

Granular Materials

- Macroscopic particle systems with dissipative interactions
- Ubiquitous in nature and industry
  - Astrophysical structure
  - Geophysics; sand,gravel
  - Hopper flow, fluidized beds,…
- Different and richer phenomenology than elastic gases
  - Clustering,jamming,non-Maxwellian velocities, violations of equipartition, Maxwell's demons…

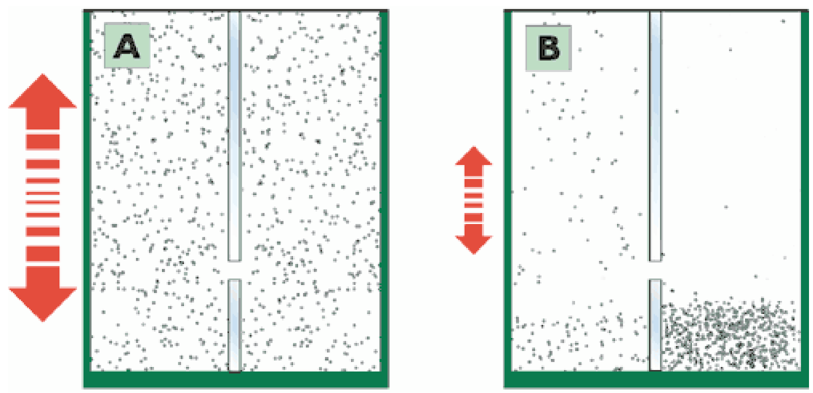
Spontaneous Order

Clustering Instability



Goldhirsch & Zannetti
PRL 70, 1619 (1993)

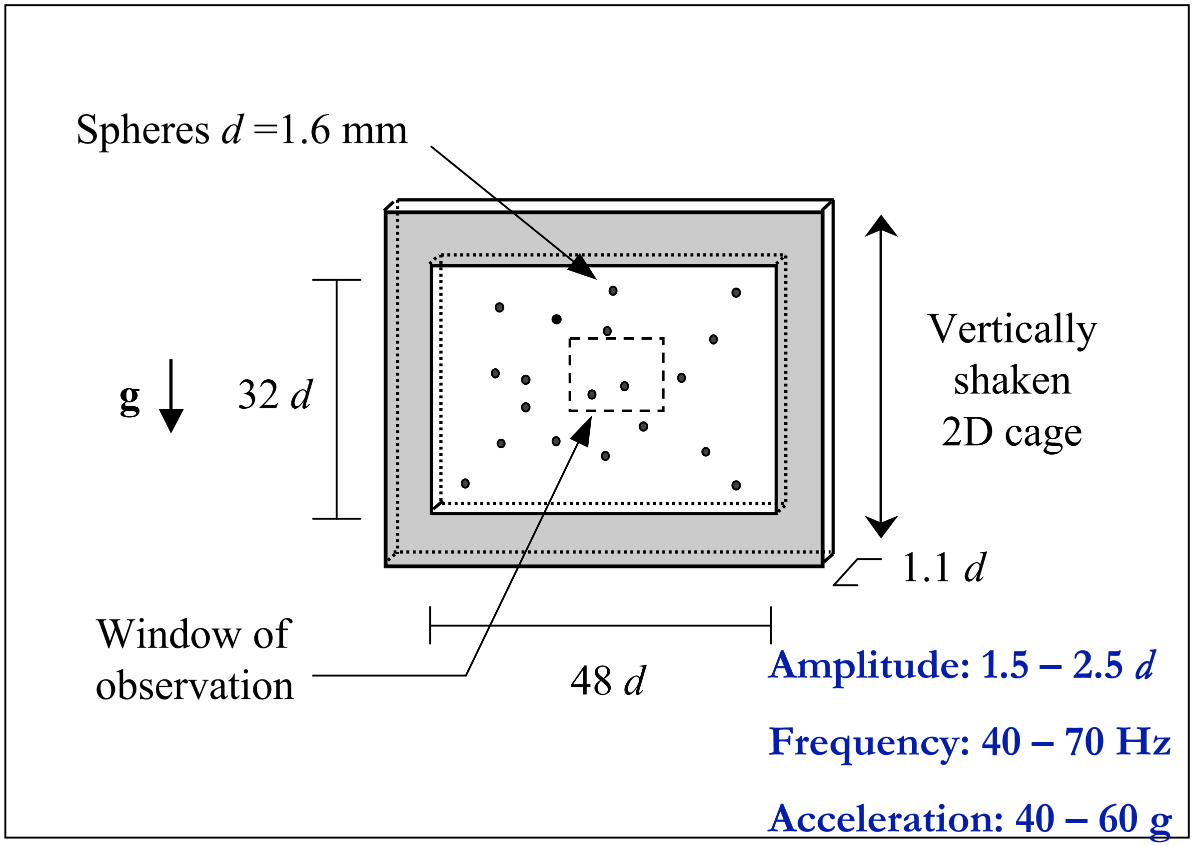
Maxwell's Demon



Eggers PRL 83, 5322 (1999)

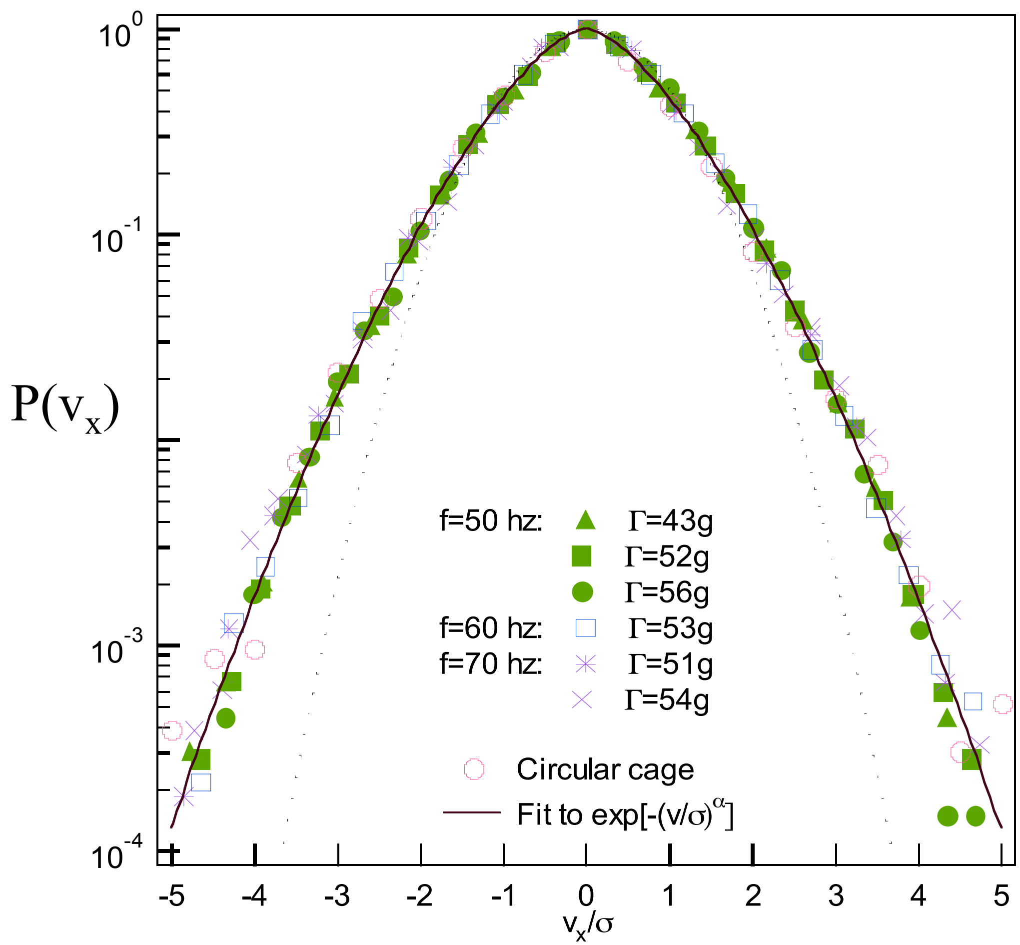
The dense get denser

Driven Granular Gas



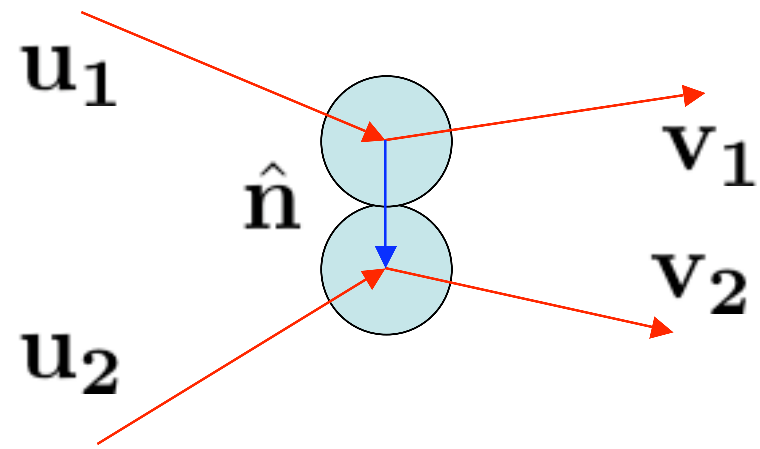
Rouyer & Menon, PRL 85, 3676 (2000)

Non-Maxwellian Distributions



- Distribution independent of material, frequency, amplitude, density, shape of boundary
- Reasonably fit by
  $P(v_x) \sim \exp[-(v_x/\sigma)^\alpha]$
  where α =1.55±0.1
- α=3/2 predicted by kinetic theory (asymptotically).

Inelastic Collisions



$$\mathbf{v}_{1,2} = \mathbf{u}_{1,2} \pm (1-p)(\mathbf{u}_2 - \mathbf{u}_1)\cdot\hat{\mathbf{n}}\,\hat{\mathbf{n}}$$

$$0 \leq p \leq 1/2$$

$$\Delta E = p(1-p)[(\mathbf{u_1} - \mathbf{u_2})\cdot\hat{\mathbf{n}}]^2$$

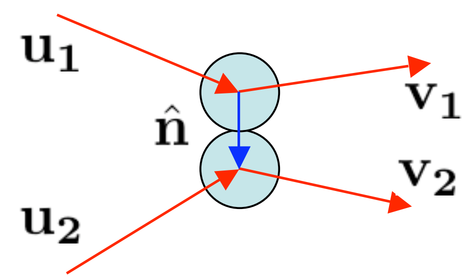
$$r = 1 - 2p \text{ restitution coefficient}$$

Boltzmann Equation

$$\frac{\partial}{\partial t} f(\mathbf{v}) = \int d\hat{\mathbf{n}}\, d\mathbf{u}_1\, d\mathbf{u}_2 |(\mathbf{u}_1 - \mathbf{u}_2)\cdot\hat{\mathbf{n}}|^{\lambda} f(\mathbf{u}_1) f(\mathbf{u}_2)$$
$$\times [\delta(\mathbf{v}-\mathbf{v}_1) - \delta(\mathbf{v}-\mathbf{u}_1)]$$

gain loss

$f(\mathbf{v})$ = probability density for velocity $\mathbf{v}$



λ=1 hard spheres,
λ=0 Maxwell molecules

$$\lambda = 1 - 2(d-1)/s$$
$$V(r) \sim 1/r^s$$

Stationary States of the Boltzmann Equation

$$0 = \int d\hat{\mathbf{n}}\, d\mathbf{u}_1\, d\mathbf{u}_2 |(\mathbf{u}_1 - \mathbf{u}_2) \cdot \hat{\mathbf{n}}|^\lambda f(\mathbf{u}_1) f(\mathbf{u}_2) [\delta(\mathbf{v} - \mathbf{v}_1) - \delta(\mathbf{v} - \mathbf{u}_1)]$$

$$f(\mathbf{v}) = \begin{cases} \exp[-(v/v_0)^2] & \text{elastic} \\ \delta(v) & \text{inelastic} \end{cases}$$

Stationary States: 1D Maxwell Molecules

$$0 = \int du_1\, du_2 f(u_1) f(u_2) \left[\delta(v - v_1) - \delta(v - u_1)\right]$$

$$v_1 = pu_1 + qu_2 \quad p + q = 1$$

Convolution

Fourier transform

$$f(v) = \frac{1}{\pi v_0} \frac{1}{1 + (v/v_0)^2} \sim v^{-2}$$

Cauchy Distribution: infinite energy and dissipation

Linearized Boltzmann Eqn

In the tail, dominant collision between fast (tail) particles and slow particles:

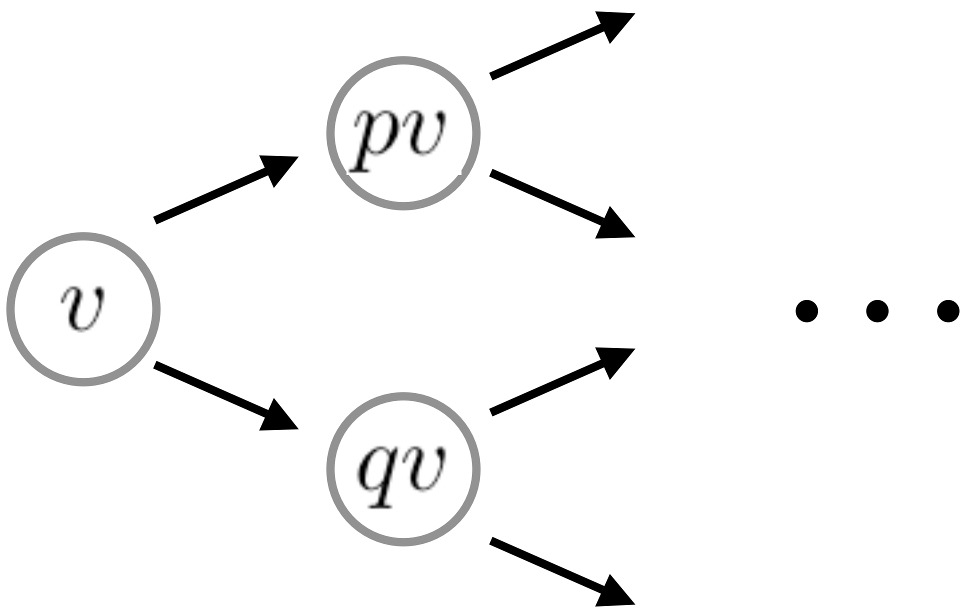
$$0 = \int du\, |u|^\lambda f(u)\, [\delta(v - pu) + \delta(v - qu) - \delta(v - u)]$$

$$\downarrow \quad f(u) \sim u^{-\sigma}$$

$$0 = p^{-1-\lambda+\sigma} + q^{-1-\lambda+\sigma} - 1$$

$$\boxed{\sigma = \lambda + 2}$$

Velocity Cascade



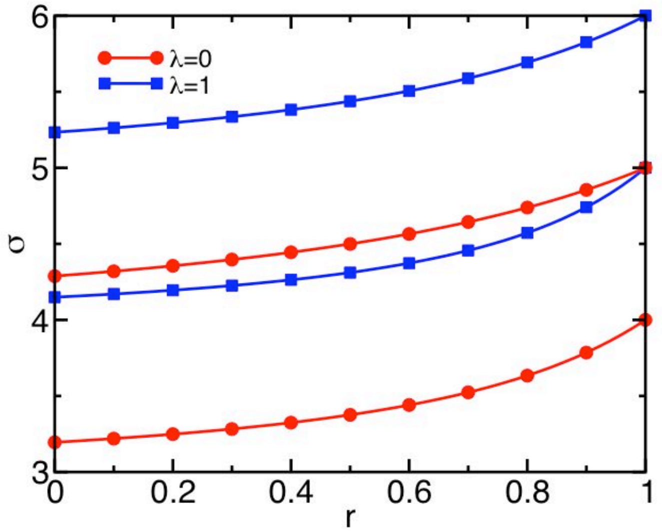
Stationary States d>1

$$0 = \int d\hat{\mathbf{n}}\, d\mathbf{u}_1\, d\mathbf{u}_2 |(\mathbf{u}_1 - \mathbf{u}_2)\cdot\hat{\mathbf{n}}|^\lambda f(\mathbf{u}_1) f(\mathbf{u}_2)[\delta(\mathbf{v}-\mathbf{v}_1) - \delta(\mathbf{v}-\mathbf{u}_1)]$$

$$\frac{1 - {}_2F_1\left(\frac{d+\lambda-\sigma}{2}, \frac{\lambda+1}{2}, \frac{d+\lambda}{2}, 1-p^2\right)}{(1-p)^{\sigma-d-\lambda}} = \frac{\Gamma(\frac{\sigma-d+1}{2})\Gamma(\frac{d+\lambda}{2})}{\Gamma(\frac{\sigma}{2})\Gamma(\frac{\lambda+1}{2})}.$$

Characteristic exponent σ vs restitution coefficient *r* for d=2 and d=3.

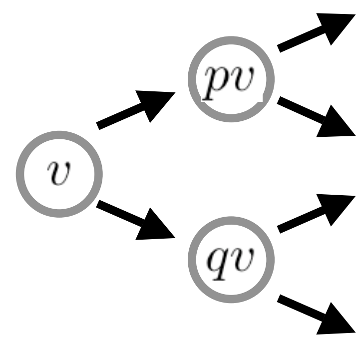
$$d + 1 + \lambda \le \sigma \le d + 2 + \lambda$$



Summary

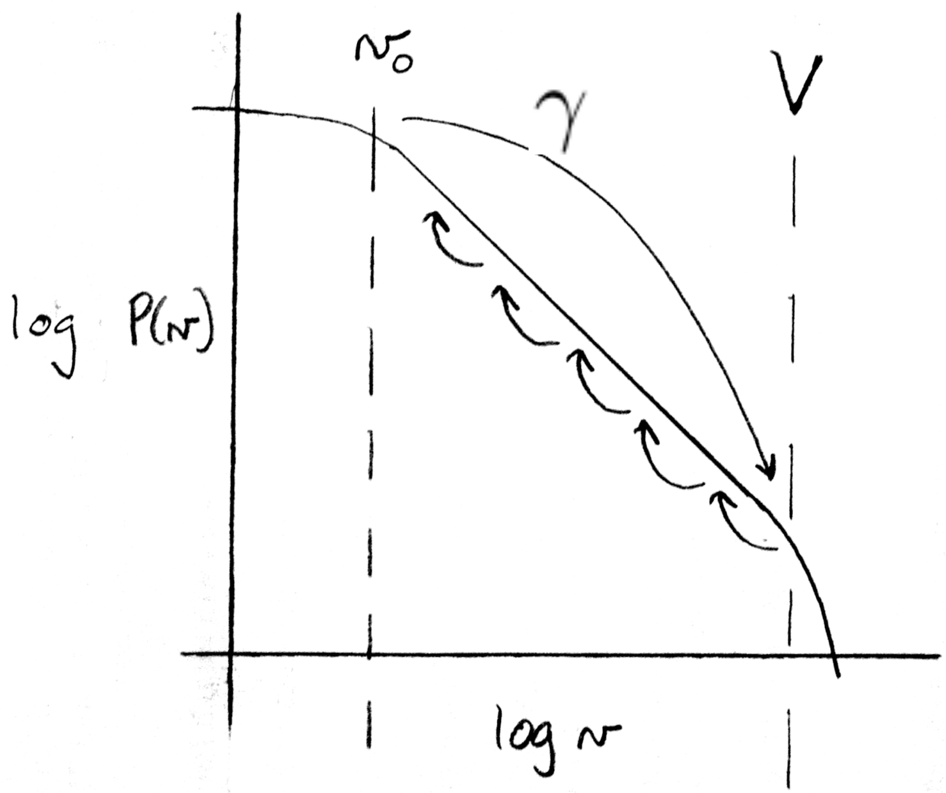
- The Boltzmann equation for inelastic gases has non-trivial stationary states with power law tails and infinite dissipation.
- These states have an infinite velocity cascade from high to low velocity.
- Do these states have any physical significance?
- How can decreasing energy solution be stationary?

$$f(v) \sim v_0^{-d} (v/v_0)^{-\sigma}$$



Driven Steady States

•Boost particles to large velocity *V* at small rate γ to initiate cascade.



γ =injection rate

v_0 =typical velocity

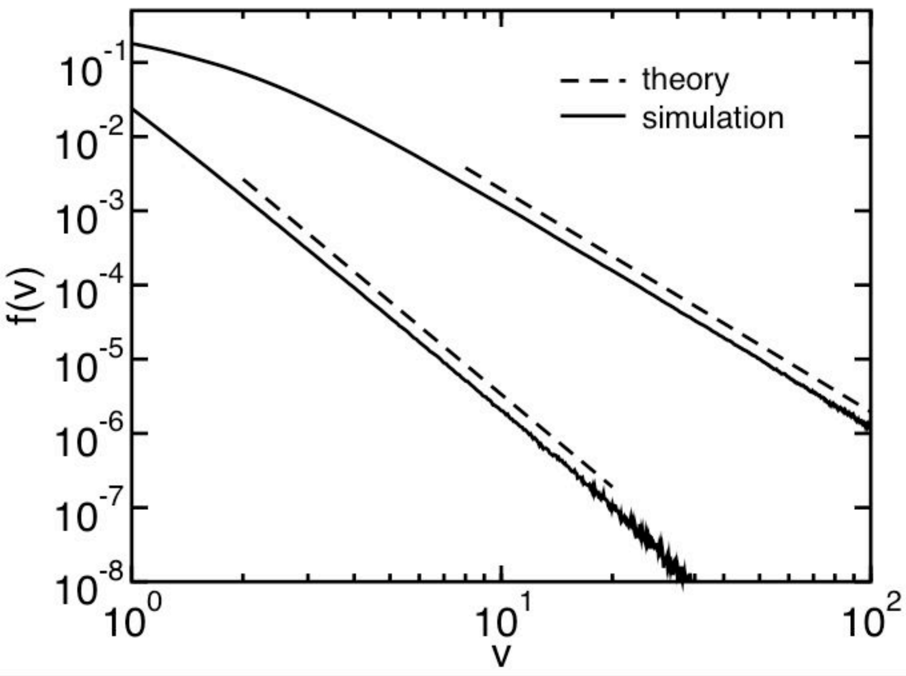
V =cut-off velocity

dissipation =injection⇒

$$\gamma \sim V^{\lambda}(V/v_0)^{d-\sigma}$$

Simulation of Driven Gas

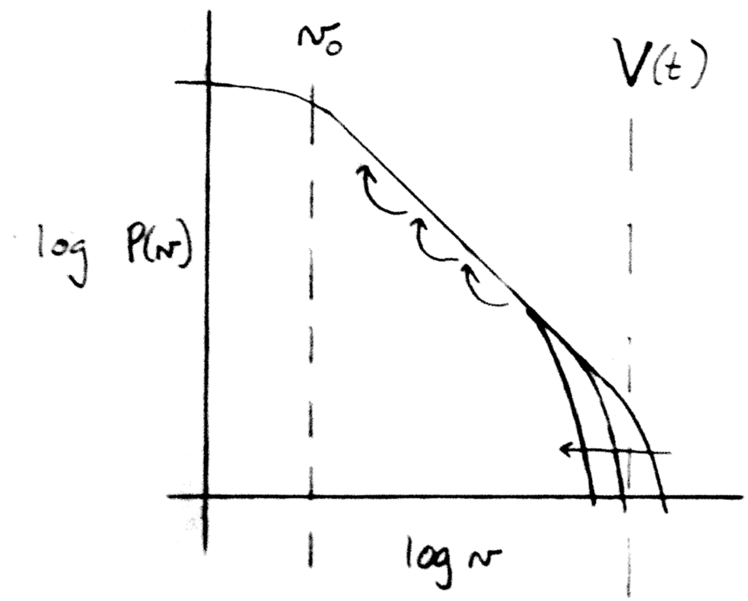
Hard spheres (λ=1) in 1D (top) and 2D (bottom). Dotted line is theory for σ.



Next: Experiments and more realistic simulations

Decaying States

- What happens to steady states when energy injection is turned off?
- Cut-off decreases without modifying the rest of the distribution. (λ>0)



Cut-off vs. Time

energy decrease due to moving cut-off = dissipation

$$\frac{dV}{dt} \sim -V^{1+\lambda}$$

$$V(t) = \left[\frac{V^{\lambda}(0)}{1 + c\lambda V^{\lambda}(0)t}\right]^{1/\lambda}$$

Stationary is not forever even for $V(0) \to \infty$

Cut-off Function

$$d = 1, p = q = 1/2, \lambda = 1$$

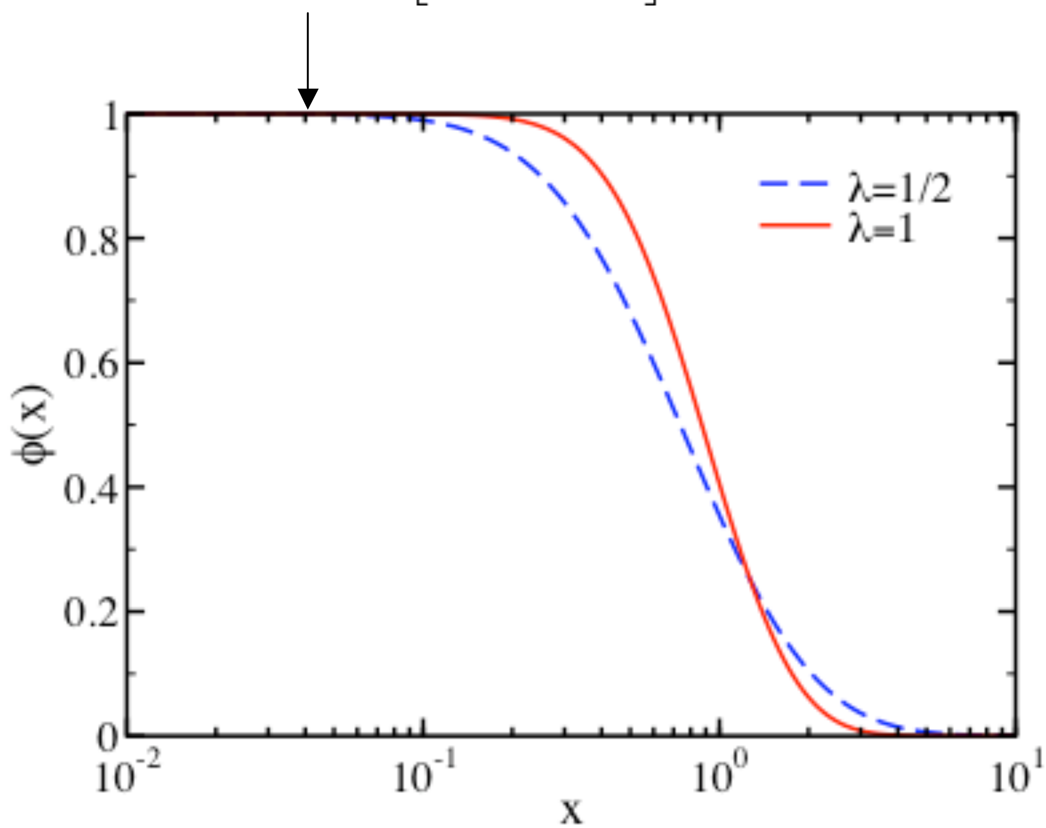
Plug ansatz for cut-off into Boltzmann Equation:

$$f(v,t) \simeq f_s(v)\phi\left(\frac{v}{V}\right)$$

$$\phi'(x) = 2\left[\phi(2x) - \phi(x)\right]$$

Cut-off Function

$$1 - \phi(x) \sim \exp\left[-A\left(\ln x\right)^2\right]$$



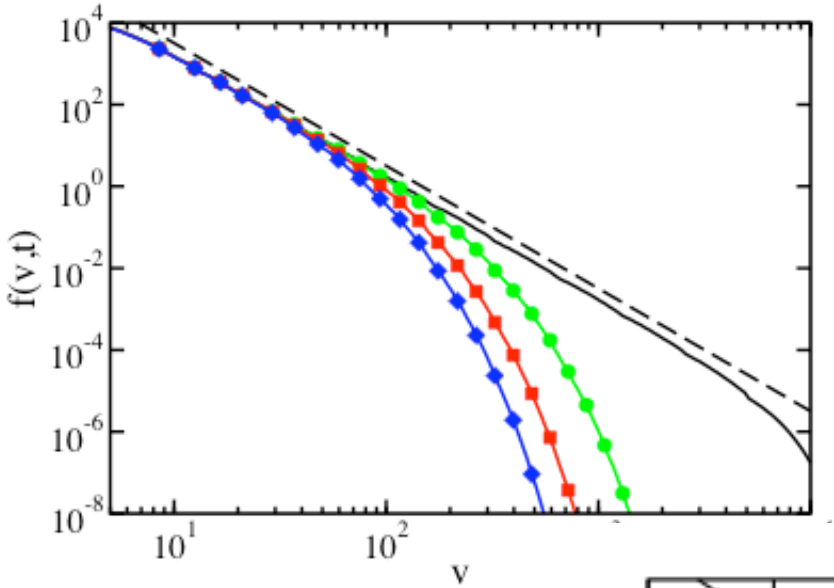
$$f(v,t) \simeq f_s(v)\phi\left(\frac{v}{V}\right)$$

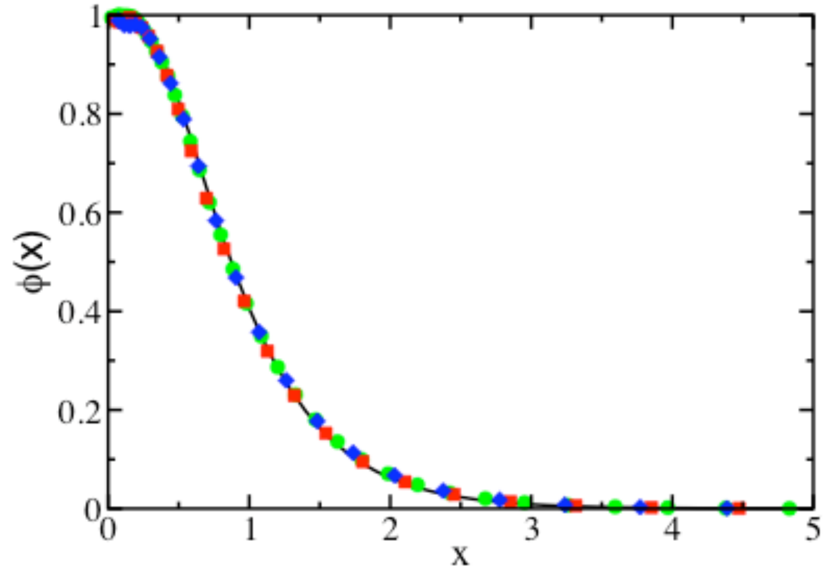
$$\phi'(x) = 2\left[\phi(2x) - \phi(x)\right]$$

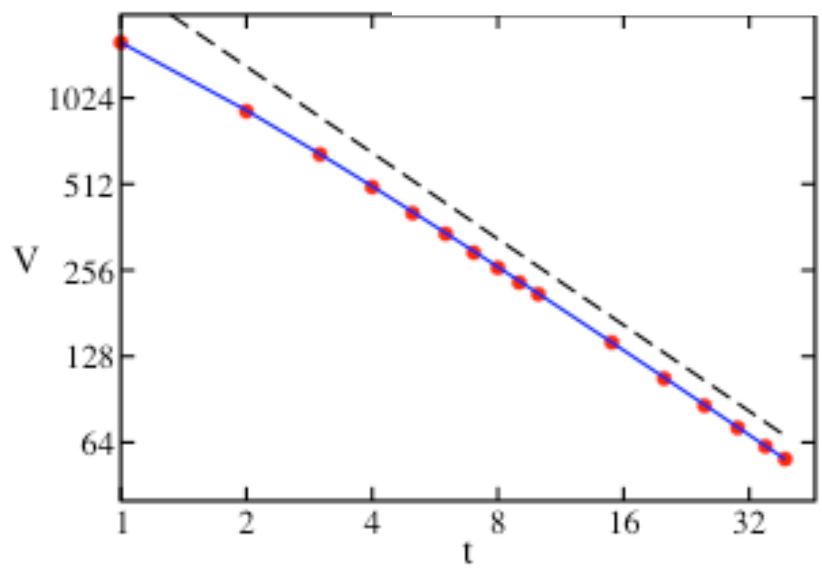
$$\phi(x) = \sum_{n=1}^{\infty} a_n \exp(-2^n x)$$

$$a_n = -\frac{a_{n-1}}{2^{(n-1)} - 1}$$

Simulations







Long Time Behavior

When $V(t) \approx v_0$ there is a crossover to the known, single parameter freely cooling solution with stretched exponential large velocity behavior:

$$f(v) \sim v_0(t)^{-d} e^{-A(v/v_0(t))^{(1+\lambda/2)}}$$

see, e.g Ernst and Brito, cond-mat/0304608

Other Driving Mechanisms

•White Noise:

$$f(v) \sim v_0^{-d} e^{-A(v/v_0)^{(1+\lambda/2)}}$$

see, e.g Ernst and Brito, cond-mat/0304608

•The tail of the velocity distribution appears to depends on the ratio of collision frequency to the injection frequency.

van Zon & MacKintosh, PRL 93,038001 (2004)

Summary

- Stationary states of the inelastic Boltzmann exist with power law tails and energy cascades.
- Driven steady states with cut-off, power law tails can be maintained by rare but energetic injection.
- Decaying states are initially described by a single moving cut-off.
- Next: Experiments and more realistic simulations