Graphical Representations and the Phase Transition of the Ising Spin Glass

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Outline

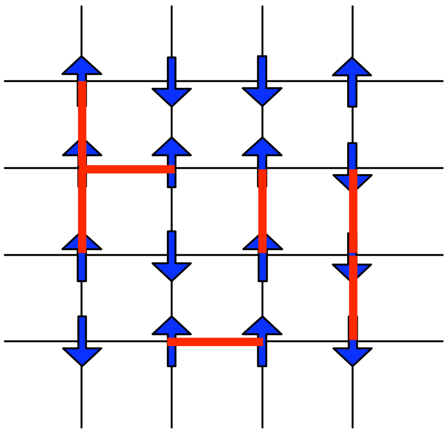
- Graphical representations
- Spin Glasses
- Graphical representations for spin glasses

Graphical Representations

- Tool for rigorous results on spin systems
- Basis for very efficient Monte Carlo algorithms
- Source of geometrical insights into phase transitions

Fortuin & Kastelyn
Coniglio & Klein
Swendsen & Wang

Joint Spin-Bond Distribution



$$\mathcal{W}(\sigma,\omega;p) = p^{|\omega|}(1-p)^{N_b-|\omega|}\Delta(\sigma,\omega)$$

$$\Delta(\sigma,\omega) = \begin{cases} 1 \text{ if for every } (ij)\ \omega_{(ij)} = 1 \rightarrow \sigma_i = \sigma_j \\ 0 \text{ otherwise} \end{cases}$$

Every occupied bond is satisfied

$$\sigma_i = \begin{cases} +1 & \uparrow \\ -1 & \downarrow \end{cases}$$

$$\omega_{ij} = \begin{cases} 1 & \text{(red line)} \\ 0 & \text{(black line)} \end{cases}$$

Marginals

$$\mathcal{W}(\sigma, \omega; p) = p^{|\omega|}(1-p)^{N_b - |\omega|}\Delta(\sigma, \omega)$$

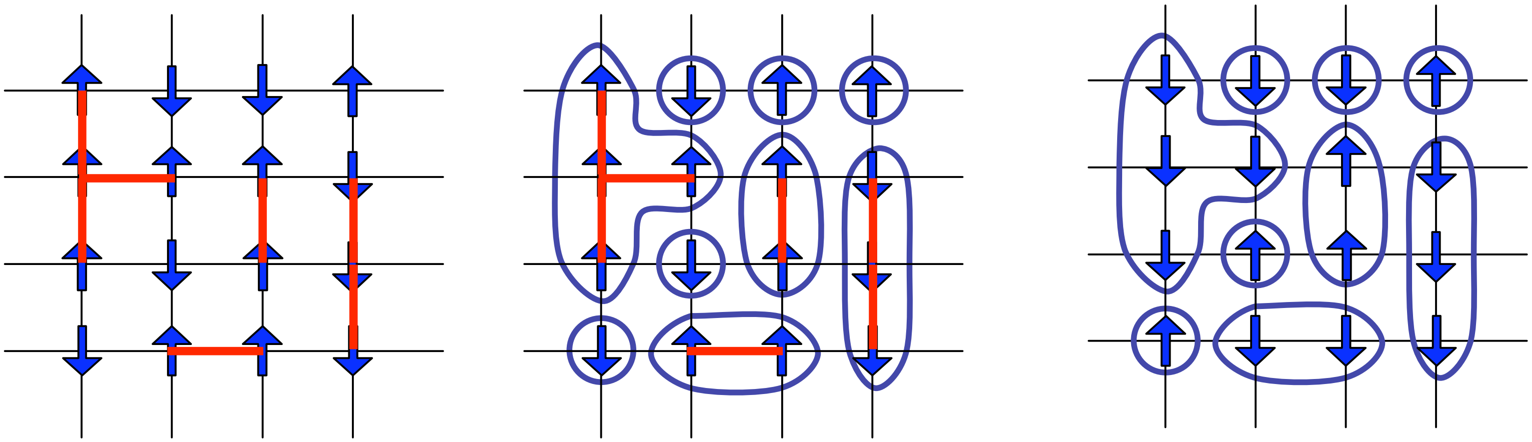
$$\mathcal{W}_{\text{bond}}(\omega; p) = p^{|\omega|}(1-p)^{N_b - |\omega|}2^{\mathcal{C}(\omega)}$$

Fortuin-Kastelyn random cluster model

$$\mathcal{W}_{\text{spin}}(\sigma; p = 1 - e^{-2\beta J}) = e^{\beta J \sum_{(ij)} \sigma_i \sigma_j}$$

Ising model

Swendsen-Wang Algorithm



- Occupy satisfied bonds with probability $p = 1 - e^{-2\beta J}$
- Identify clusters of occupied bonds
- Randomly flip clusters of spins with probability 1/2.

Connectivity and Correlation

$$\langle \sigma_i \sigma_j \rangle = \mathrm{Prob}\{i \text{ and } j \text{ connected}\}$$

- Criticality ⟺ Percolation

Ising Spin Glass

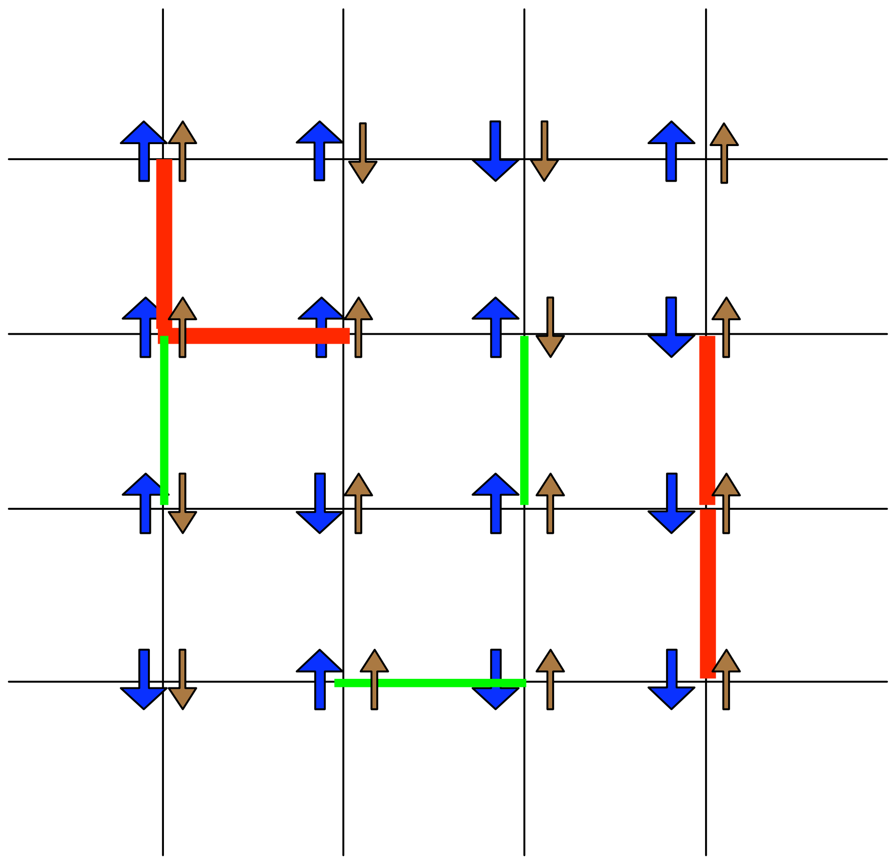
$$\mathcal{H} = -\sum_{(ij)} J_{ij}\sigma_i\sigma_j$$

$J_{ij} = \pm 1$ independent, quenched random couplings

Edwards-Anderson order parameter

$q_i = \sigma_i^{(1)}\sigma_i^{(2)}$ (1) and (2) are independent replicas

Two Replica Graphical Representation



$\sigma_i = 1$

$\tau_i = 1$

$\omega_{ij} = 1$

$\eta_{ij} = 1$

Spin Bond Distribution

$$\mathcal{W}(\sigma, \tau, \omega, \eta; \beta, J)$$
$$= B_{2\beta}(\omega) B_\beta(\eta) \Delta(\sigma, \tau, \omega; J) \Gamma(\sigma, \tau, \eta)$$

$$B_\beta(\eta) = \prod_{(ij)} (1 - e^{-2\beta})^{\eta_{ij}} (e^{-2\beta})^{1-\eta_{ij}}$$

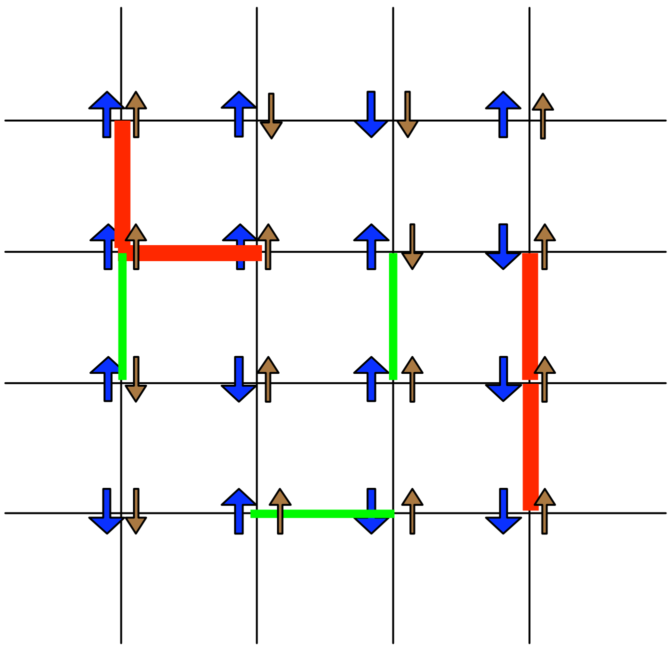
Bernoulli factors for bonds

$$\Delta(\sigma, \tau, \omega; J) = \begin{cases} 1 \text{ if for every } (ij)\ \omega_{ij} = 1 \rightarrow J_{ij}\sigma_i\sigma_j > 0 \text{ and } J_{ij}\tau_i\tau_j > 0 \\ 0 \text{ otherwise} \end{cases}$$

$$\Gamma(\sigma, \tau, \eta) = \begin{cases} 1 \text{ if for every } (ij)\ \eta_{ij} = 1 \rightarrow \sigma_i\sigma_j\tau_i\tau_j < 0 \\ 0 \text{ otherwise} \end{cases}$$

spin bond constraints

Spin Bond Constraints



Red cluster have same *q*

q flips across green bonds

- If bonds satisfied in *both* replicas then

$$\omega_{ij} = 1$$

with probability $p = 1 - e^{-4\beta}$

- If bonds satisfied in only *one* replica then

$$\eta_{ij} = 1$$

with probability $p = 1 - e^{-2\beta}$

Some nice properties

- Correlation function of EA order parameter and connectivity

$\langle q_i q_j \rangle =$

Prob{ *i* and *j* are connected by a path of occupied bonds with an **even** number of green bonds}

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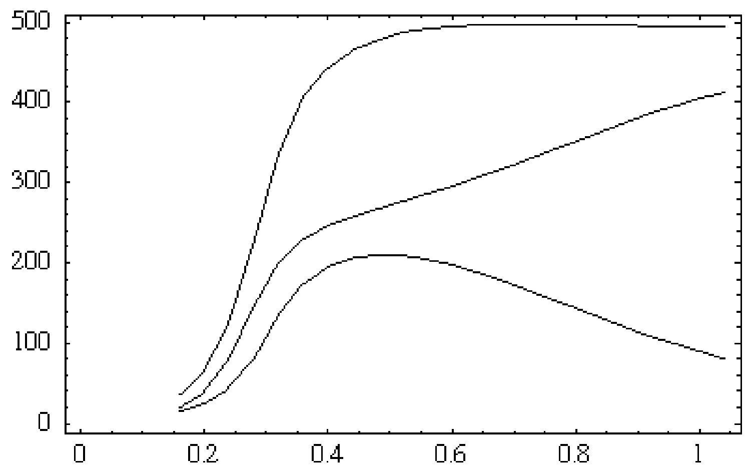
Prob{ *i* and *j* are connected by a path of occupied bonds with an **odd** number of green bonds}

- Spin marginal is two independent Ising spin glasses

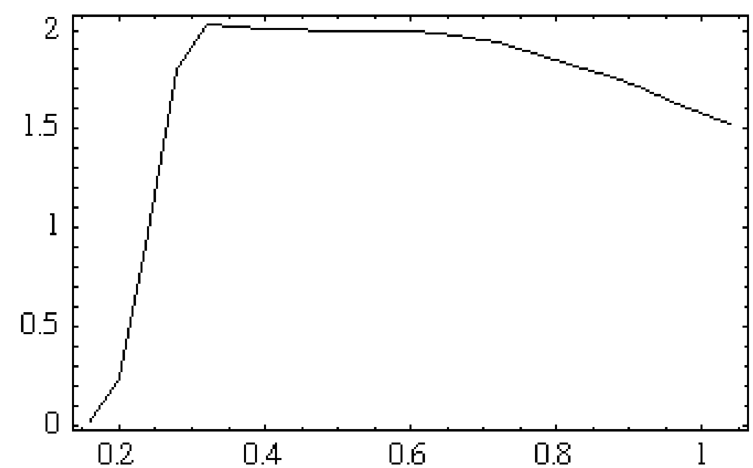
$$\mathcal{W}_{\text{spin}}(\sigma, \tau; \beta, J) = \text{const} \times \exp\left[\beta \sum_{(ij)} J_{ij}(\sigma_i \sigma_j + \tau_i \tau_j)\right]$$

Simulations

8^3 system



Size of first, second largest cluster red cluster and sum of both vs β



Number of wrapping red cluster vs β

Conclusions