Graphical Representations and the Phase Transition of the Ising Spin Glass

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Outline

- Graphical representations
- Spin Glasses
- Graphical representations for spin glasses
Graphical Representations

- Tool for rigorous results on spin systems
- Basis for very efficient Monte Carlo algorithms
- Source of geometrical insights into phase transitions

Fortuin & Kastelyn
Coniglio & Klein
Swendsen & Wang
Joint Spin-Bond Distribution

\[ \mathcal{W}(\sigma, \omega; p) = p^{|\omega|} (1 - p)^{N_b - |\omega|} \Delta(\sigma, \omega) \]

\[ \Delta(\sigma, \omega) = \begin{cases} 
1 & \text{if for every } (ij) \ \omega_{ij} = 1 \rightarrow \sigma_i = \sigma_j \\
0 & \text{otherwise} 
\end{cases} \]

Every occupied bond is satisfied
Marginals

\[ W(\sigma, \omega; p) = p^{\omega}(1 - p)^{N_b - \omega} \Delta(\sigma, \omega) \]

\[ W_{\text{bond}}(\omega; p) = p^{\omega}(1 - p)^{N_b - \omega} 2^{C(\omega)} \]

Fortuin-Kastelyn random cluster model

\[ W_{\text{spin}}(\sigma; p = 1 - e^{-2\beta J}) = e^{\beta J} \sum_{(i,j)} \sigma_i \sigma_j \]

Ising model
Swendsen-Wang Algorithm

- Occupy satisfied bonds with probability $p = 1 - e^{-2\beta J}$
- Identify clusters of occupied bonds
- Randomly flip clusters of spins with probability 1/2.
Connectivity and Correlation

\[ \langle \sigma_i \sigma_j \rangle = \text{Prob}\{i \text{ and } j \text{ connected}\} \]

- Criticality \[\leftrightarrow\] Percolation
Ising Spin Glass

\[ \mathcal{H} = - \sum_{(ij)} J_{ij} \sigma_i \sigma_j \]

\[ J_{ij} = \pm 1 \quad \text{independent, quenched random couplings} \]

Edwards-Anderson order parameter

\[ q_i = \sigma_i^{(1)} \sigma_i^{(2)} \]

(1) and (2) are independent replicas
Two Replica Graphical Representation

\[ \sigma_i = 1 \]  
\[ \tau_i = 1 \]  
\[ \omega_{ij} = 1 \]  
\[ \eta_{ij} = 1 \]
Spin Bond Distribution

\[ \mathcal{W}(\sigma, \tau, \omega, \eta; \beta, J) \]

\[ = B_{2\beta}(\omega)B_{\beta}(\eta)\Delta(\sigma, \tau, \omega; J)\Gamma(\sigma, \tau, \eta) \]

\[ B_{\beta}(\eta) = \prod_{(ij)}(1 - e^{-2\beta})^{\eta_{ij}}(e^{-2\beta})^{1-\eta_{ij}} \]

Bernoulli factors for bonds

\[ \Delta(\sigma, \tau, \omega; J) = \begin{cases} 1 & \text{if for every } (ij) \omega_{ij} = 1 \rightarrow J_{ij}\sigma_i\sigma_j > 0 \text{ and } J_{ij}\tau_i\tau_j > 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ \Gamma(\sigma, \tau, \eta) = \begin{cases} 1 & \text{if for every } (ij) \eta_{ij} = 1 \rightarrow \sigma_i\sigma_j\tau_i\tau_j < 0 \\ 0 & \text{otherwise} \end{cases} \]

spin bond constraints
Spin Bond Constraints

• If bonds satisfied in both replicas then
  \[ \omega_{ij} = 1 \]
  with probability \[ p = 1 - e^{-4\beta} \]

• If bonds satisfied in only one replica then
  \[ \eta_{ij} = 1 \]
  with probability \[ p = 1 - e^{-2\beta} \]

Red cluster have same \( q \) 
\( q \) flips across green bonds
Some nice properties

• Correlation function of EA order parameter and connectivity

\[ \langle q_i q_j \rangle = \]

\[-\text{Prob}\{ i \text{ and } j \text{ are connected by a path of occupied bonds with an \textit{even} number of green bonds}\}

\[-\text{Prob}\{ i \text{ and } j \text{ are connected by a path of occupied bonds with an \textit{odd} number of green bonds}\}

• Spin marginal is two independent Ising spin glasses

\[ \mathcal{W}_{\text{spin}}(\sigma, \tau; \beta, J) = \text{const} \times \exp \left[ \beta \sum_{(ij)} J_{ij} (\sigma_i \sigma_j + \tau_i \tau_j) \right] \]
Simulations

$8^3$ system

Size of first, second largest cluster red cluster and sum of both vs $\beta$

Number of wrapping red cluster vs $\beta$
Conclusions