INTRODUCTION TO G. E. MOORE'S UNPUBLISHED REVIEW OF *THE PRINCIPLES OF MATHEMATICS*

Kevin C. Klement
Philosophy / U. Massachusetts–Amherst
Amherst, MA 01003, USA
klement@philos.umass.edu

Several interesting themes emerge from G. E. Moore's previously unpublished review of *The Principles of Mathematics*. These include a worry concerning whether mathematical notions are identical to purely logical ones, even if coextensive logical ones exist. Another involves a conception of infinity based on endless series neglected in the *Principles* but arguably involved in Zeno's paradox of Achilles and the Tortoise. Moore also questions the scope of Russell's notion of material implication, and other aspects of Russell's claim that mathematics reduces to logic.

We here publish for the first time a lengthy review G. E. Moore wrote of Russell's *The Principles of Mathematics*. The review was intended for the German journal *Archiv für systematische Philosophie*,[1] and was likely composed in the late summer and/or early autumn of 1905.[2] Moore mentions the review in his autobiographical contribution to his Library of Living Philosophers volume, and indeed suggests that he spent a significant amount of

[1] Moore published a review in the *Archiv* the previous year (1904) entitled "Philosophy in the United Kingdom for 1902", and seems to have intended it to be the first in a series of reviews covering British philosophy, including Russell's work. However, no further installments were published.

[2] In a letter to Russell dated 23 October 1905 (RA1 710.052987), Moore mentions having completed it. The review also cites Russell's *OD*, published that month.

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time during his 1904–11 fellowship in Edinburgh studying Russell's book but that he encountered some difficulty doing so.[3] It is not entirely clear why it was never published. Moore's self-perceived difficulty provides one possible explanation: perhaps he was not sufficiently happy with the result. Other explanations may involve his declining friendship and working relationship with Russell,[4] or negative feedback from the journal, from Russell, or elsewhere.[5] This has unfortunately postponed until now the opportunity to examine a direct interaction between these two seminal figures in the early history of analytic philosophy. Russell states his philosophical indebtedness to Moore in the Preface of the *Principles* (p. xviii). Moore's reciprocal admiration for Russell's work is evident even in the brief discussion in his autobiography, and the importance he saw in it is manifest in the opening line of the review. His estimation of the *Principles* as the most important philosophical book published in the UK in 1903 silently places it above his own *Principia Ethica*, published the same year.

Moore's prose in the review has his usual clear and straightforward style, and the review hardly requires additional commentary to be comprehensible. Yet it may be worth highlighting a few places where it might be of interest to contemporary researchers. Moore's questioning of Russell's claim to have established that the propositions of pure mathematics can be derived from logic when Russell excludes certain (apparently) mathematical truths in non-conditional form, can be seen as anticipating a criticism, now called "if-thenism", made by a number of later commentators. According to these critics, Russell did not so much succeed in deriving mathematics from logic, but rather

[3] MOORE, "An Autobiography" (1942). The full passage reads: "At the beginning of the period I spent at Edinburgh what I was chiefly occupied with was trying to understand Russell's *Principles of Mathematics*, a thing which I found very difficult since the book was full of conceptions which were quite new to me. Many parts of it I never did succeed in understanding, but the earlier fundamental parts about logic I think I did in the end succeed in understanding pretty thoroughly. I was helped in understanding by the fact that, as I mentioned before, I did not merely think about and read over and over again what seemed to me to be of cardinal importance, but actually wrote a long review of the book."

[4] Their personal and working relationship seems to have been suffering from tensions from 1899 onward; for discussion, see PRETI, "'He Was in Those Days Beautiful and Slim'" (2008), and LEVY, *G. E. Moore and the Cambridge Apostles* (1979).

[5] Russell was at least aware the unpublished review existed, making note of it in his reply to Moore dated 25 October 1905 (RAI 710.053032).

derived only conditional claims with mathematical axioms as antecedents and mathematical theorems as consequents, thereby greatly reducing the achievement of his form of "logicism". As Moore hints, if the main reason Russell has for not counting, e.g., the claim that "the three angles of every triangle are equal to two right angles" as a proposition of pure mathematics, is that it cannot be established purely logically, the claim that all pure mathematics reduces to logic threatens to become trivial and uninteresting. However, unlike some later thinkers pushing this worry, Moore rightly connects Russell's contention that mathematical truths take the form of universal hypotheticals with his views about how *pure* mathematics gets applied in concrete situations, which helps explain what might otherwise appear to be an oddity in Russell's position.[6] It is perhaps worth mentioning that at one point (p. 144 below), Moore gets Russell's position wrong. Moore asserts that Russell holds all the propositions of the science of logic to be universal hypotheticals, when in fact Russell himself gives examples of non-mathematical truths of the science of logic not taking this form, such as "implication is a relation" (*PoM*, §10).

In the course of this discussion, Moore gives an argument (pp. 142–4) which is uniquely his, and which may be the most interesting part of the review for historians interested in the development of Moore's and Russell's philosophies. Moore makes note of Russell's admission that certain analyses he offers are meant only to meet mathematical standards of definition, not philosophical ones. Russell's definitions of the various cardinal numbers as classes of similar classes are not meant to capture what we ordinarily think when we consider, e.g., that $1 + 1 = 2$. Moore then argues that Russell's definition of 1 from logical primitives, while it may yield something equivalent to the usual notion of 1 (applying to all and only the same collections), still might not yield the very same property. The results Russell proves logically then might not be the very propositions we expected, but instead similar propositions using equivalent, but distinct, notions. To establish the

[6] Later thinkers who push the "if-thenist" worry include Putnam, "The Thesis That Mathematics Is Logic" (1967); Musgrave, "Logicism Revisited" (1977); Coffa, "Russell and Kant" (1981); and Boolos, "The Advantages of Honest Toil over Theft" (1994). There have been many responses, but for those stressing the importance of the pure/applied distinction, see Griffin, "New Work on Russell's Early Philosophy" (1982); Galaugher, *Russell's Philosophy of Logical Analysis 1897–1905* (2013); and Klement, "Russell's Logicism" (2018).

intended mathematical propositions themselves, one would need to be able to establish that the notions are identical and not merely equivalent, or at least show that the equivalences themselves are logical truths, which Moore worries Russell has not done. Moore's suggestion that there might be an entire system of purely logical properties, coextensive but not identical with those of mathematics, is a rather startling one, but from within Moore's own philosophy it is perhaps not an unnatural one. It is reminiscent of his famous "open question argument": if it is an open question whether not a class has the number 1 if and only if that class is a member of the class of all unit classes, then, arguably, having the number 1 and being in that class cannot be identical properties, even if they are coextensive. It is a difficult task to speculate what Russell's response might have been, but this argument has already sparked some debate as to whether or not it shows a deep disagreement between Russell and Moore during this period over the very nature and goals of analysis.[7]

Moore also calls into question (p. 146ff.) what Russell means by claiming that mathematics is deducible from logic, noting that he cannot simply mean that the truths of logic *imply* those of mathematics in Russell's own sense of *material implication*. In that sense, all truths imply all other truths. Moore further claims that material implication is not what we ordinarily mean by implication, and that Russell and others are committed to a different notion of implication. A full century of research into various forms of conditional logics would seem to support Moore's contention, although not as much his suggestion that this further notion is simple and analyzable. It is somewhat disappointing, however, that Moore does not go far in probing to what extent Russell's stronger notion of formal implication (*PoM*, §40) might be serviceable in this regard. Similarly, when it comes specifically to the deducibility of logic from mathematics, Moore does not consider the very straightforward interpretation that this means nothing more nor less than the existence of *deductions* using only logical axioms and inference rules for the various claims of mathematics.

In addition to these topics, the review contains praise (p. 152) for Russell's new theory of denoting in the newly published "On Denoting", and a surprisingly strong statement of disagreement (*ibid.*)

[7] For contrasting standpoints, see LEVINE, "The Place of Vagueness in Russell's Mathematical Development" (2016), and GANDON, "Sidgwick's Legacy?" (2017).

with Russell's earlier theory of denoting concepts, though Moore demurs from elaborating. There is also a very nice statement (p. 146)—perhaps clearer than any similar statement made by Russell himself—of their shared anti-psychologism in logic, according to which the subject matter of logic is not anything to do with human reasoning, thought or psychology.

These topics make up roughly the first half of the review, and most of the remainder (pp. 152–64) is taken up by a lengthy discussion of issues related to infinity and continuity. Although this may not be evident in a contemporary context, Russell's discussion of then-new techniques for solving what had hitherto been regarded as "paradoxes" or "contradictions" of infinity would at the time have been seen as especially important. Moore makes note of two distinct conceptions of infinity discussed by Russell. The first is the notion—now often called "standard infinity" or "Frege infinity"—that applies to a class which does not have any of the inductive natural numbers $0, 1, 2, 3, \ldots$ for its cardinality. The second, which Russell calls "reflectiveness" but is now usually called "Dedekind infinity", is that which applies to a class which can be put in $1-1$ correspondence with a proper part of itself. Moore follows Russell in claiming that these two notions are equivalent, i.e., apply to all and only the same classes. This is now known to be an oversimplification, as in most forms of set theory, the equivalence is only true assuming the axiom of choice (or an equivalent assumption such as Russell's later multiplicative axiom), at least in the weak form of the axiom of countable choice. This is likely something he himself realized before Moore wrote the review, though it is unknown whether or not it was ever communicated to him.[8]

As he does so often, Moore questions whether or not either of these notions of infinity capture our pre-theoretic conception. He goes on to sketch yet another concept of infinity. He defines an *endless* series

[8] Russell had expressed doubts about results now known to be dependent on the axiom of choice as early as 1900 or 1901 (see *Papers* 3: 410, 596), and explicitly formulated his own equivalent multiplicative axiom in his 1904 manuscripts (*Papers* 4: 171–5). Explicit acknowledgement of the importance of this for the equivalence of the two notions of infinity came only after ZERMELO published his 1904 paper "Beweis, dass jede Menge wohlgeordnet werden kann", something that received much discussion in Russell's correspondence with Jourdain over the following year; see GRATTAN-GUINNESS, *Dear Russell—Dear Jourdain* (1977), pp. 46–9. In *Principia Mathematica*, theoretically possible cardinals that are Frege infinite, but Dedekind finite, are discussed and there called "mediate cardinals" (∗124).

as one "which has no beginning, or no end, or which has neither" (p. 154), and defines an infinite series as one that "*either* is itself endless *or* contains an endless series as part of itself" (p. 155). This definition applies to series, but as Moore notes (*ibid.*), one can obtain from it a concept applicable to those classes that are the fields of such series. In contemporary parlance, Moore's definition of infinity is essentially that of a class that has a subclass that can be partially ordered in a way that does not have maximal elements. Moore seems to think that this notion better captures at least one common notion of infinity used in pre-theoretic discourse, and since it is at least intensionally, if not extensionally, distinct from the other notions, one may rationally entertain doubts about whether or not they are equivalent. Moore seems to think it likely that it will turn out to be equivalent as well. In fact, Moore's notion is also not equivalent with the other notions unless the axiom of countable choice is assumed. All classes that are Dedekind infinite are Moore infinite, and all classes that are Moore infinite are Frege infinite, but the axiom is needed to complete the "circle" and obtain that all Frege infinite classes are also Dedekind or Moore infinite.[9] Perhaps Moore's lack of confidence with these issues—not feeling himself to have the technical chops to determine whether these equivalences hold—is one of the reasons he held back the review.[10]

Moore goes on to summarize how various conceptions of infinity can be used to pose Zeno's paradox of Achilles and the Tortoise (pp. 156–62), not only summarizing Russell's discussion but going on to restate what he takes to be a more natural formulation of the paradox involving his own notion of infinity stated in terms of endless series. So stated, the paradox involves the oddity that Achilles must traverse all of an endless series of locations before catching up with the Tortoise at a certain instant. As Moore sorts things out, however, it turns out not to be impossible to traverse every point of an endless series of locations with an endless series of instants, even if all those instants precede a given instant. A series may have an endless part without itself being endless, as, for example, with the series of rational numbers from 0 to 1 inclusive. This series has an end, namely 1, but

[9] For a discussion of these three, along with 23 other senses of "finite" and "infinite", and their mutual interrelations, see De la Cruz, Dzhafarov, and Hall, "Definitions of Finiteness Based on Order Properties" (2006).

[10] Thanks to Jim Levine for suggesting something along these lines to me.

there is an endless series, ½, ⅔, ¾, ⅘, etc., within it. Moore does a nice job summarizing both how the paradox seems puzzling when stated this way, and how it can be solved from within the new mathematics of series.

On the whole, Moore's review sheds new light on his philosophy, and perhaps on Russell's, and their interactions. Those interested in the topics of philosophical analysis, implication, infinity and other topics will no doubt find Moore's perspective valuable. It is not known whether or not Russell himself ever had a chance to read it, but either way it is a shame there is no official reply.

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RUSSELL'S *PRINCIPLES OF MATHEMATICS*[1]

G. E. Moore

[Bertrand Russell. *The Principles of Mathematics*. Vol. 1. Cambridge: Cambridge U. P., 1903. Pp. (2), xxix, 536.]

Of the philosophical books published in the United Kingdom in 1903 the most important is Mr. Russell's *Principles of Mathematics*.[2] In this book, Mr. Russell tells us, he has two main objects. His first object is to establish the two very important propositions (1) "that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts" and (2) "that all its propositions are deducible from a very small number of fundamental logical principles." The examination of the principal branches of pure mathematics, which is necessary to establish these two propositions, occupies the last six Parts of the book, which are entitled respectively "Number", "Quantity", "Order", "Infinity and Continuity", "Space", and "Matter and Motion". In these parts there is much which cannot be easily understood without a special knowledge of Mathematics, and much which has little bearing on philosophy, except so far as it helps to establish Mr. Russell's two main propositions; but there is much also which is of considerable importance for philosophy, quite apart from its bearing on these two propositions: in particular, Mr. Russell examines very carefully the conceptions of Infinity and Continuity, and attempts to shew that they involve no antinomies. Part 1, on the other hand, is devoted to Mr. Russell's second object—"the explanation of the fundamental concepts which mathematics accepts as indefinable", and is almost entirely philosophical in its nature. I shall endeavour to give some account (1) of the meaning and consequences of Mr. Russell's two propositions concerning the relation of Logic and Mathematics (2) of some of the more important points dealt with in Part 1 and (3) of the theory of Infinity and Continuity. Mr. Russell is eminently qualified for his task by a thorough knowledge of Mathematics and by great philosophical acumen; and it is certain that no philosopher ought in future to handle any of the subjects discussed in this book, without taking account of the arguments advanced in it.

1 *The Principles of Mathematics*.

2 [Typeset from a photocopy of the manuscript provided to the Russell Archives by Dorothy Moore in 1971 (RA3 Rec. Acq. 116). The original ms. is in the Moore papers in Cambridge U. Library. The review is © the Estate of G. E. Moore and is published with the Estate's permission. Moore revised the ms. a great deal, the foliation being an indication: *1* 2, 2a, 3–15(14), 16–24(26), 25(27), 26(28), 27–29(30), 30(31), 31(34), 32(30), 32a, 33, 34(?), 35–44. Proofread by A. Duncan and K. Blackwell.]

Mr. Russell maintains, we have seen, the two very important propositions (1) "that all pure mathematics deals exclusively with concepts definable in terms of a very small number of fundamental logical concepts" and (2) "that all its propositions are deducible from a very small number of fundamental logical principles." And in order to bring out the philosophical importance of his book, it will be well to explain as clearly as possible precisely what he means by them. There are several points, which require notice, in order that we may form a just estimate of what he is maintaining.

In the first place, Mr. Russell makes then two assertions with regard to *all* the propositions of pure mathematics. What propositions does he mean to include in this description? It is important to recognize that there are certain kinds of propositions to which his assertions do not, and are not meant to, apply. It might, for instance, be thought that the familiar proposition of Euclid that "The three angles of every triangle are equal to two right angles" was a proposition of *pure* mathematics. It is not, however, one of the propositions of which Mr. Russell is speaking. It cannot be deduced from any logical principles. It follows only if we assume certain of Euclid's axioms, which cannot themselves be deduced from the principles of Logic. All that *can* be deduced from logical principles is that *if* these axioms of Euclid are true, *then* the three angles of every triangle are equal to two right angles; *this* hypothetical proposition *is* one of the propositions to which Mr. Russell's assertions are meant to apply. But then it can likewise be deduced from logical principles that *if* certain axioms, other than Euclid's, are true, *then* it is *not* the case that the three angles of every triangle are equal to two right angles. Is there, we may ask, any means of distinguishing these propositions, which Mr. Russell *does* include among mathematical propositions, from Euclid's proposition, which he does *not* so include, except simply by saying that the former can, and the latter cannot, be deduced from logical principles? To this question Mr. Russell does not enable us to give a definite answer. He does, no doubt, intend to include among mathematical propositions, only propositions which are *true*; and he would, no doubt, himself say that Euclid's proposition, understood as an absolutely universal hypothetical, is certainly false. But his only evidence for its falsity would seem to consist in the fact that it cannot be deduced from logical principles: and this evidence, by itself, is certainly insufficient. There may, perhaps, be other evidence which would entitle us to distinguish this and similar propositions as "non-mathematical" on the ground of their falsity; but Mr. Russell certainly does not give it. Until such evidence is forthcoming, we must therefore be content to say that the mathematical propositions of which Mr. Russell means to assert that they can be deduced from logical principles can only be defined as these—which can be deduced from logical principles. But it must not be thought that this fact destroys the importance of his assertions. It is a very important truth, if *any* propositions at all, such as are commonly recognised as mathematical, can be deduced from logical principles, without the need of further assumptions, and can be defined in terms of logical concepts. The proposition, for instance, that, *if* Euclid's axioms be granted, it will follow that "The three angles of every triangle are equal to two right angles" is a proposition to which, as we have seen, Mr. Russell's assertions *are* meant to apply; and that *this* proposition can be wholly deduced from logical principles and defined in logical terms, is a fact of the greatest importance.

In the second place, it should be noticed that the propositions, which Mr. Russell asserts to be deducible from logical principles, are, without exception, *universal hypotheticals*. They are all of them propositions of the form: If anything whatever were to have a certain property, then that thing would also have a certain other property. And it might again be doubted whether all the propositions of pure mathematics are propositions of this form. It might, for instance, be thought that the proposition $2 \times 2 = 4$ asserts categorically some direct relation between the conception 2×2 and the conception 4, and is not merely identical with the hypothetical: If any terms whatever form two collections, of two terms each, then those terms form a collection of four terms. That this may *possibly* be the case, I do not think Mr. Russell would deny; and if so then we should have to admit that the proposition $2 \times 2 = 4$, so understood, is one to which his assertions do not apply, since it is not a universal hypothetical. But it will, I think, appear doubtful, on further reflection, whether by "$2 \times 2 = 4$" we do mean to assert any other relation between the conception "2×2" and the conception "4" except that which consists in the fact that wherever the one conception applies the other applies too; and, in any case, it is only this fact which is of any practical importance. For if from the fact that $2 \times 2 = 4$ we are to be justified in inferring that a particular pair of pairs forms a collection of four terms, then "$2 \times 2 = 4$" must tell us that *any* pair of pairs whatever forms such a collection. In asserting, therefore, that universal hypotheticals of this kind can be deduced from logical principles, Mr. Russell is making this assertion with regard to the only kind of pure mathematical propositions, which are of any importance. And, understanding that by "propositions of pure mathematics" Mr. Russell means such universal hypotheticals, it is easy to understand the distinction which he draws between "pure" and "applied" mathematics. Mathematics are *applied* whenever it is assumed that some particular kind of entitles actually have some one of those properties, with regard to which pure mathematics proves that anything which has them also has certain other properties. Pure mathematics cannot prove the assumption that the kind of entities in question actually have the properties ascribed to them; but from what pure mathematics does prove, it follows that *if* they have these properties they also have certain others. It is in this sense that Mr. Russell maintains the interesting proposition that the conception of "quantity" does not occur in pure mathematics at all. He shews that when we call things "quantities" we commonly mean to ascribe to them certain properties, which are quite different from any with which pure mathematics deals; and that all mathematical conclusions with regard to quantities in general, or particular kinds of quantities, depend upon the assumption that, *in addition to* the properties distinctive of quantities, the quantities in question possess also certain mathematical properties—certain kinds of order, which belong also to numbers. The assumption that things, which possess the properties distinctive of quantities, possess also these mathematical properties is one which pure mathematics cannot prove; nor do any of its propositions depend in any way upon the fact that there are such things as quantities: but, *if* we assume that there are such things as quantities and that they have certain mathematical properties, then from *this* assumption *together with* what pure mathematics does prove, it follows that they also have certain other properties. Similarly with the space and matter and motion which we actually perceive or believe to exist. That they possess any of the properties, which are dealt with by the pure mathematical sciences of Geometry or Rational Dynamics, it is equally impossible for these sciences to prove and unnecessary for them to assume. Geometry and Rational Dynamics merely prove that *if* anything whatever has certain properties that thing will also have certain others: that actual space, or actual matter and motion have these properties, they do not assume and cannot prove: but, on the other hand, it follows from what they do prove, that *if* actual space, or actual matter and motion, have certain properties they also have certain others. It is not, therefore, the case, that any proposition of Pure Mathematics depends for its proof or its truth upon any theory whatever with regard to the nature of quantity, or of the space or matter or motion which we actually perceive: all the propositions of Pure Mathematics would be equally true and equally demonstrable, even if there were no such things, and are, therefore, true and demonstrable, whatever properties these things may have. But, on the other hand, it is equally important to observe that the fact that the results of mathematics can be applied to actual space and matter and motion, proves conclusively that, in addition to their non-mathematical properties, these entities *do* also possess the mathematical properties with which Pure Geometry and Rational Dynamics deal. Whatever may be their nature in other respects, the space, matter and motion which we actually perceive or believe to exist, certainly have some of these very complicated properties, which can be defined in purely logical terms.

In the third place, it should be observed, that, since the propositions of which Mr. Russell is speaking are universal hypotheticals, it is true, in a sense, that pure mathematics deals with absolutely every conceivable entity—with the whole Universe of being and everything in it. It asserts of absolutely everything that if it has, or were to have, certain properties, it also has, or would have, certain others. When, therefore, Mr. Russell tells us that pure mathematics "deals exclusively" with concepts definable in logical terms, we must understand him to refer only to those properties, of which pure mathematics asserts the universal connection. In another and perfectly good sense it is true that pure mathematics "deals with" all concepts, without exception; since all concepts are included in the Universe of entities, with regard to which pure mathematics asserts that every thing in it, if it have certain properties, must also have certain others.

There still remains, however, a difficulty of some importance with regard to the exact nature of the concepts, which pure mathematics does assert to be universally connected. Mr. Russell asserts, we have seen, *both* that all the propositions of pure mathematics can be deduced from logical principles *and* that all the concepts, which these propositions assert to be universally connected, can be defined in logical terms. But whether the concepts which occur in the propositions which he deduces are the *only* ones, of which pure mathematics asserts the universal connection, it is, as Mr. Russell himself admits, possible to doubt. I may illustrate my meaning by taking as an instance the particular cardinal numbers. Mr. Russell admits that one of the concepts with which pure mathematics does deal is the number "one": "1 + 1 = 2" is, for instance, a proposition of pure mathematics. But his definition in logical terms of the number "one" is by no means simple: it is as follows: The number "one" is the class whose members are all those classes, of which each is such that it has a member *x*, such that the proposition "*y* is a member of the class in question and *y* differs from *x*" is always false, whatever *y* may be. This is the definition in logical terms of the number "one". And whether, whenever we say that we have but *one* penny in our pocket, this definition is a correct *analysis* of the property which we mean to attribute to our penny, it is, Mr. Russell admits, permissible to doubt. It is not plain that what we think to be true of the penny, when we think it is but *one*, is no less than that it is a member of the class of classes of which each is such that it has a member *x*, such that the proposition "*y* is a member of the class in question and *y* differs from *x*" is always false, whatever *y* may be: it is not plain that this is a correct *analysis* of what we think. That it is *equivalent* to what we think, in the sense that anything whatever which has the property which we mean by "one" is also a member of this class of classes, and that anything whatever which is a member of this class of classes also has the property which we mean by "one", there is, indeed, no doubt whatever. But Mr. Russell admits the possibility that it is *only* equivalent—that, possibly, all the members of this class of classes have in common some *other* property, beside the fact that they belong to this class—some other property, which belongs to all of them and only to them, and which may be what we generally mean when we speak of the number "one". Mr. Russell, indeed, boldly asserts his doubt whether there is any such other property; and there is much to be said for his view. But what I wish now to point out is the consequences which follow from the mere possibility that there is such another concept, meant by "one". This other concept would, as I have said, be strictly *equivalent* to the concept which forms Mr. Russell's definition; and, being equivalent, it might, perhaps, in one sense of the word "definition" (that which Mr. Russell seems sometimes to mean by the "mathematical" sense), be itself correctly described as "definable in logical terms". But it is, I think, important to observe that, though propositions, which dealt with it, might thus still be propositions "which dealt exclusively with concepts definable in logical terms", no such proposition would be also "deducible from logical principles". When Mr. Russell asserts that $1 + 1 = 2$ can be deduced from logical principles, his assertion only applies to the proposition in which the concept dealt with is "the class of classes, of which each etc. etc.": it is only *this* proposition which he shews to be deducible from logical principles. If it be true that there is also *another* concept denoted by the word "one", then the proposition that $1 + 1 = 2$, understood as asserting a universal connection between this *other* concept and some others, *cannot* be deduced from logical principles alone. In order to prove it, we should require the additional assumption that this concept is equivalent to the one which occurs in the proposition which *can* be deduced from logical principles. In short, if we are to define "pure mathematics" as consisting only of those propositions which can be deduced from logical principles, this concept would be as completely irrelevant to pure mathematics as is the conception of "quantity": it would be wholly impossible for mathematics to *prove* anything with regard to it, and wholly unnecessary for it to *assume* anything with regard to it. Its relation to pure mathematics would differ from that of "quantity" only in respect of the fact that, whereas in the case of "quantity" the additional assumption required was only that whatever was a quantity also had the properties with which mathematics does deal, in this case the additional assumption required would be not only that everything to which this concept applied also had the properties dealt with by mathematics, but also that everything which had those properties was a thing to which this concept applied. Unless, therefore, it can be shewn that the concepts dealt with in those propositions, which can be deduced from logical principles, are the very ones which occur in the proposition $1 + 1 = 2$, as ordinarily understood, then it must be admitted *either* that the proposition $1 + 1 = 2$, as ordinarily understood, is not a proposition of pure mathematics *or* that Mr. Russell's two assertions do not both apply to all the propositions of pure mathematics. If $1 + 1 = 2$, as ordinarily understood, and similar propositions, *are* propositions of pure mathematics, and if the concepts dealt with in them are (as Mr. Russell admits to be *possible*) *not* those, of which he can deduce from logical principles the universal connection, then for the proof of pure mathematics we require not only a few logical principles, but also an infinite number of wholly independent premisses, asserting equivalence between the concepts dealt with in those propositions which *can* be deduced, and the concepts dealt with in those which *cannot* be deduced, from logical principles.

Something, finally, should be said to explain what those "fundamental logical concepts" are, in terms of which, as Mr. Russell holds, all the concepts dealt with by pure mathematics can be defined. By "Logic", in this connection, Mr. Russell means all those propositions, and only those propositions, which form part of the deductive science commonly called Formal or Symbolic Logic; including, that's to say, not only the theory of the Syllogism, but any additional branches of the subject which have been discovered by modern symbolic logicians. All the propositions of this science he holds to be, like those of mathematics, universal hypotheticals: they all assert that if anything whatever has a certain property it also has a certain other property. These properties, of which Logic asserts the universal connection, and (if the properties themselves are complex) any simple concepts which are involved in their analysis, are what Mr. Russell means by "logical concepts"; and it is among these, therefore, that are to be found those fundamental concepts in terms of which mathematical concepts can be defined. So far all is clear. But which among all these logical concepts are "fundamental"? And, among those that are fundamental, are all, or only some, involved in the definition of mathematical concepts; and, if only some, which? On these points Mr. Russell does not give us very precise information. He does apparently intend to confine the term "fundamental" to *simple* logical concepts; and among these he undoubtedly does regard as fundamental all those which are equivalent to no other logical concepts, except complexes in the analysis of which they are themselves involved. But there appear to be also some simple logical concepts, which are equivalent to complexes, in the analysis of which only *other* simple logical concepts are involved; which have, that is to say, to these complexes the relation which, as we saw, the cardinal numbers would have to the complexes, which Mr. Russell treats as cardinal numbers, in case these complexes are not really identical with what is commonly meant by "one", "two", "three" etc. In the case of the cardinal numbers, it was said, it does appear possible that the ordinary meaning *is* identical with Mr. Russell's complex concepts; but, in the case of Logic, it appears to be certain that there are some simple concepts, which are *merely* equivalent to complexes, in the analysis of which they are not themselves involved. The concept of negation—that which we express by the word "not"—is an example of such simple concepts. That negation can be *mathematically* defined in other terms, Mr. Russell shews; but there seems no doubt that the complex concept, by which he "defines" it, is different from, though equivalent to, what we usually mean by "not". There is, then, at least one simple logical concept, which is equivalent to a complex logical concept, in the analysis of which it is not itself involved. Are there any others; and, if so, what are they? Are all, or none, or only some of them "fundamental"; and, if only some, which? Are all, or none, or only some of them involved in the definition of mathematical concepts; and, if only some, which? To these questions Mr. Russell gives no answer. We can, therefore, only be sure that he has enumerated *some* of the "fundamental logical concepts", which are involved in the definition of mathematical concepts. That he has enumerated all which are "fundamental", and all which occur in the definition of mathematical concepts, we cannot tell. The nature of those which he does enumerate may be briefly indicated as follows:—(1) A relation, which he calls "implication", and which is the relation which holds between two propositions, when, if the one is true, it follows that the other is true too; (2) the relation which we assert to hold between any given entity and a class of entities when we say that the entity in question "is a member of" the class; (3) the concept which we mean by the word "class", when we say that any collection of entities forms "a class"; (4) all the simple concepts (and they are by no means easy to discover) which are involved in the analysis of what we mean when we say of a given class that it has among its members "anything whatever, which possesses a certain property"; the word "property" being used in the widest possible sense, including, for instance, the sense in which we may say that one of the "properties" of every conceivable thing is that "if a foot-rule is longer than the thing in question, the thing in question is shorter than a foot-rule"; (5) the concept which we mean by the word "relation", in the widest possible sense in which we can say that one entity is "related to" another.

To sum up, then: We may say that what Mr. Russell succeeds in proving is that there can be deduced from propositions, which would be admitted to be propositions of merely Formal Logic, in the strictest sense, propositions which are at least equivalent to, and possibly identical with, the most important propositions, which would be admitted to belong to Pure Mathematics; and that the distinctive concepts, which occur in the propositions so deduced, actually involve in their analysis no concepts but what are involved in the analysis of the most purely formal logical propositions. What is new and important, for philosophy, in this result is that propositions, which are at least equivalent to the propositions of such apparently distinct studies as Arithmetic, Geometry and Rational Dynamics, require for their proof no new ideas and no new premisses beyond those which are used by the strictest formal logician; and that the distinctive concepts dealt with by these sciences—such apparently distinct concepts as "number", "space", "matter" and "motion"—are all at least equivalent to concepts in the analysis of which none but the simplest logical concepts are involved. It follows, if we understand by "pure mathematics" those propositions which can be deduced from logical premisses, that pure mathematics differs from Formal Logic in two respects alone: the concepts with which it deals are more complex; and all indemonstrable premisses would be said to belong to Logic. All the propositions of Logic, except its indemonstrable premisses, differ from those of Pure Mathematics in no respect except that they are simpler; and what precise degree of complexity would entitle a proposition to be called "mathematical" rather than "logical", usage would hardly allow us to determine. Moreover, even from the premisses of Logic, mathematical propositions differ in no other respect, except that, whereas the former can be deduced from nothing, the latter can be deduced from the former: the whole of the propositions both of Logic and Mathematics consist of true universal hypotheticals, in which the properties asserted to be universally connected either are, or involve in their analysis, none but a few simple logical concepts, which can be enumerated: every proposition both of Logic and of Mathematics is of this nature, and every true proposition, which is of this nature, belongs either to Logic or to Mathematics. Mr. Russell has accordingly shewn us how to obtain a strict and perfect definition not only of pure mathematics but also of Formal Logic; and it follows, from what he has shown, that Logic ought not to be defined either as a "mental" or as a "normative" science. Even if it be true that the propositions of Logic are in any sense laws in accordance with which we do, or must, or ought to think (and it is highly doubtful whether this is true), it is quite certain that they can be completely defined without mentioning this fact. The propositions of Logic are absolutely universal laws—laws which apply, not only to "mind" or "thought", but also, and equally, to matter and to whatever there may be which is neither mind nor matter; and among the properties, of which they assert the universal connection, the conceptions of "thought" and of "ought" certainly do not occur. Logic itself, therefore, does not tell us either that we do, or must, or ought to think anything whatever; the kind of truths, which it does tell us, can be completely specified without any reference to "thought"; and hence to describe it as enunciating "Laws of Thought" must certainly be misleading, even if, in any sense, the description is true.

But there still remains one point which must be explained, if we are to understand fully Mr. Russell's two main propositions with regard to the relation of Logic and Mathematics. It must be explained what he means by saying that mathematical propositions "*can be deduced from*" the premisses of Logic. And to explain this involves an explanation of the nature of the first of Mr. Russell's "fundamental logical concepts"—the concept which he calls "implication". For, when we say that one proposition "*q*" "can be deduced from" another "*p*", we do at least mean to assert that if *p* is true, it follows that *q* is true too; and we have seen that Mr. Russell gives the name of "implication" to the relation which holds between any two propositions, when, if the first is true, the second is true too: to say "*p implies q*" is strictly equivalent to saying "If *p* is true, *q* is true". The conception of "deduction" does then involve that of "implication"; and Mr. Russell gives us to understand that it involves only one other condition—namely that the proposition *from* which the deduction is made should be true. Any proposition q may, we are told, be correctly said to be "deducible" from a proposition p, provided only that p is *true* and *implies* q. What, then, is meant by "implication"?

Mr. Russell appears to consider that it is a *simple* concept; one, therefore, whose nature cannot be explained by mentioning any concepts which are involved in its analysis. We can only explain *which* simple concept is meant, by pointing out other concepts to which it is *equivalent*. Such, for instance, is the form of explanation used when we say that "p implies q" is equivalent to "If p is true, q is true". This particular explanation, it is true, is one, in the analysis of which there is involved the very concept which it is used to explain: the proposition "If p is true, q is true" is not *merely* equivalent to, but identical with the proposition "'p is true' *implies* 'q is true'." And it might quite well be the case that "implication" was thus only equivalent to complexes, in the analysis of which it is itself involved: that *some* fundamental logical concepts are of this nature is quite certain. But Mr. Russell apparently considers that this is *not* the case with "implication": he regards it as equivalent to a complex, in the analysis of which it is *not* itself involved: it *is*, he holds, "mathematically" definable in terms of *other* logical concepts: its position among logical concepts is, in short, precisely the same as that of the negative "not". The proposition to which Mr. Russell does thus regard "p implies q" as equivalent is: "p is true" and "q is false" are not both true. And it must be admitted that this assertion of equivalence is a great help to understanding what Mr. Russell means by "implication". For it follows that every true proposition is implied by every other proposition, both true and false, and that every false proposition implies every other, both true and false. That these consequences follow is obvious. For, if q is a true proposition, then "p is true" and "q is false", will not *both* be true, whatever p may be and whether it be true or false; and if p is false, then "p is true" and "q is false", will not *both* be true, whatever q may be, and whether it be true or false. And accordingly we see that Mr. Russell means by "implication" a relation which really does hold, as he points out, between the ridiculously false proposition "Socrates is a triangle", and the true proposition "$2 + 2 = 4$": "Socrates is a triangle" really does *imply* that $2 + 2 = 4$. And similarly, since any proposition is "deducible" from every true proposition which implies it, it follows that "$2 + 2 = 4$" is *deducible from* the true proposition "Socrates was a man".

That these consequences, which follow from his assertion that "p implies q" is equivalent to "'p is true' and 'q is false' are not both true", are paradoxical, Mr. Russell himself points out. But what he does not point out is that he himself constantly, in his most important propositions, uses both "implication" and "deduction" in a sense which is *different* from that which he "defines" by this assertion of equivalence—in a sense in which "p implies q" is *not* equivalent to "p is true and q is false are not both true". I will give three important instances of this inconsistency. (1) The first instance is in his use of this very word "equivalence". He tells us that when he says two propositions are "equivalent" he means that they mutually imply one another. But from his definition of "implication" it follows that any two true propositions whatever must mutually imply one another: no pair of true propositions whatever can fail to be "equivalent". Yet, when he tells us that certain true propositions are equivalent, he certainly means, as every one else would mean, that this mutual relation of theirs is one which is *not* shared by all true propositions. It is, in fact, quite certain that we ordinarily use the word "implication" in a sense in which it may be true that a true proposition may imply another, *without* being also implied by it: "one-sided implication" is possible even between true propositions. Mr. Russell is obliged, like every one else, to recognise that it is possible: and yet, in the sense in which he has defined "implication", it is wholly impossible. (2) Mr. Russell maintains that certain true propositions are "indemonstrable"; and he uses the word "demonstration" as a synonym of "deduction". But, in the sense in which he has defined "deduction", it is obvious that no true proposition whatever can be "indemonstrable". It must be "implied" by every true proposition; and hence any true proposition whatever will suffice to demonstrate it. (3) When Mr. Russell tells us that mathematics can be "deduced from" Logic, he certainly means that it has to Logic a relation which by no means all true propositions have to Logic: indeed, he insists, as we have seen, that the propositions of Applied Mathematics can *not* be deduced from Logic; and he certainly does not mean to deny that yet they *may* be true. And conversely he certainly means that Logic has to mathematics a relation, which by no means every science has to it. Yet in the sense in which he has defined "deduction", it is plain that every true proposition of Biology can be "deduced" from Logic as truly as any proposition of pure mathematics; and conversely that pure mathematics can be "deduced" from Anatomy or History as rigorously as they can from Logic. In short, if Mr. Russell really meant by "deduction" no more than what he says he means, his proposition, that pure mathematics can be deduced from Logic, would be profoundly unimportant. It would be asserting between Logic and Mathematics no other relation, than what obviously also holds between Logic and every other science, and between every other science and Mathematics.

It is plain, therefore, that Mr. Russell himself uses the word "implication" in a sense in which "p implies q" is *not* equivalent to "'p is true' and 'q is false' are not both true"; and that this *other* sense is that which is involved in the definition of "deduction". If he is not using the word in this other sense, then it is *false* that one true proposition may imply another, without being implied by it, and it is *false* that pure mathematics can be deduced from Logic in any sense in which it cannot equally well be deduced from any other science. And it seems impossible to doubt that in some sense these propositions are true. It is impossible to doubt that there is a sense of the word "implication" in which it is true that some true propositions imply others, without being implied by them. It is impossible to doubt that there really does hold between the successive steps of a deduction some relation which does *not* hold between every true proposition and every other. There *is*, therefore, some relation commonly meant by the word "implication", which is *not* equivalent to that which Mr. Russell "defines": some relation such that a true proposition is *not* implied by all others, and that a false proposition does *not* imply all others: and of the two relations it is obvious that *this* one has by far the greater philosophical importance. Of what nature this important relation is, it is indeed possible to doubt. It is possible that it is a relation which can be completely defined in terms of *other* concepts, logical or non-logical, among which are included the relation which Mr. Russell "defines". Whether this is so or not, I cannot here discuss; but I myself believe that it is a simple concept, which is not equivalent to any complex, except complexes in the analysis of which it is itself involved. In any case, we must recognise that there is *some* relation, commonly meant by "implication", which is *not* equivalent to that to which Mr. Russell gives the name; and I will in future distinguish them by calling the former "implication, in the ordinary sense", and the latter "implication in Mr. Russell's sense". And it is certain that implication in the ordinary sense, and not merely in Mr. Russell's, is involved in the definition of "deduction": when Mr. Russell asserts that mathematics can be deduced from Logic, he certainly means to assert that the propositions of Logic imply those of mathematics *in the ordinary sense*. There is, however, reason to doubt whether *every* proposition which is implied in this sense by a true proposition, can be correctly said to be "deducible" from it: there is reason to think that something more than this is involved in the definition of "deduction": but this is again a question, which cannot be here discussed. We are left, therefore, in doubt as to precisely what sense is to be attached to Mr. Russell's proposition that pure mathematics can be "deduced" from Logic: we can only be sure that he does mean to assert between them a relation which is *not* equivalent to that which he says he means to assert.

But the distinction just made does enable us to throw some further light on the nature of mathematical propositions. The mathematical propositions, which Mr. Russell claims to deduce, are, we have seen, all of them *universal hypotheticals*; and the distinction which has just been discussed concerns the nature of *particular* hypotheticals. Now Mr. Russell points out that every universal hypothetical is equivalent to a whole group of particular hypotheticals. For instance, the proposition "If anything whatever is a man, then that thing is a mortal" is obviously equivalent to the combined assertion of all the propositions in which it might be asserted of any particular thing, that if that particular thing is a man, it is also a mortal. But interpreting, as he does, all particular hypotheticals as merely asserting implication *in his sense*, it follows that he regards all universal hypotheticals as merely equivalent to a group of assertions of implication *in this sense*. And it is undoubtedly the case that where we use such a form of expression as "If anything is a man, then that thing is a mortal", we do commonly mean no more than this. We mean only that, if we were to assert of any one particular thing both that it was a man and that it was not a mortal, these assertions would not *both* be true: that is to say, we mean only that all such particular propositions as "Kant was a man" imply, each of them, *in Mr. Russell's sense*, the corresponding proposition that the particular subject in question was also a mortal. Accordingly Mr. Russell takes all mathematical and logical propositions merely to assert an universal connection *in this sense*: according to him, they are equivalent to the assertion only that whatever *actually has* a certain property also has a certain other property: the only relation which they assert to hold between the *logical* concepts with which they deal, is that which we assert to hold between the *non-logical* concepts "man" and "mortal" when we say that "If anything is a man, that thing is a mortal"—an assertion which is merely equivalent to the assertion "Every man is mortal". But it is plain that since there is another and quite different relation, which may be expressed in the form of a particular hypothetical—the relation of implication *in its ordinary sense*, we may also have universal hypotheticals which are equivalent to a whole group of assertions of implication *in this sense*. Such universal hypotheticals are, I think, frequently expressed by the form "If anything whatever *were to* have a certain property, it *would* also have another property". We certainly recognise a distinction between such an assertion and the mere assertion that "If anything *has* a certain property, it also *has* another property". We should, for instance, readily allow that, if all men are mortal, then it is true that "whatever *is* a man, *is* also a mortal"; but we should doubt whether we need also admit that whatever *were* to be a man, *would* also be a mortal". However that may be, we certainly recognise that from the assertion "Whatever is a man is also a mortal" it does *not* follow that "Kant was a man" *implies* "Kant was a mortal": the assertion is *not* equivalent to a group of assertions of implication *in the ordinary sense*: it seems obvious that "Kant was a man" does *not imply* "Kant was a mortal", even though it may be true that all men *are* mortal. But I think it is probable that with mathematical propositions the case is different—that they really are equivalent to a group of assertions of implication *in the ordinary sense*. It certainly seems possible that the particular proposition "This is a three-sided figure" really does *imply* that the thing in question is also a triangular figure; and it seems also possible that what mathematics can prove, is that all such propositions, which assert that a particular thing is a three-sided figure, really do imply the corresponding proposition that the thing in question is also a triangle—that it can prove not only "Whatever is trilateral is triangular", but also "Whatever *were* to be trilateral *would be* triangular". That mathematical propositions are more usually expressed in the former form must be admitted. But that what mathematics can prove is something more than this, and more than Mr. Russell claims it can prove, may nevertheless be probable. We certainly seem to see that the conception "2 + 2" and the conception "4" have to one another a relation which the conceptions "man" and "featherless biped" have *not* got, even though it be true that all men are featherless bipeds and all featherless bipeds men: we seem to see that whereas the connection of the two latter conceptions is *merely universal*, the connection of the two former is also *necessary*. Part, at least, of what we mean by this distinction may perhaps be that the two former are connected in a manner which is really equivalent to the assertion of a group of *implications in the ordinary sense*; whereas the latter are *only* connected in a manner which is equivalent to the assertion of a group of implications *in Mr. Russell's sense*. And it seems probable that mathematics can prove that there holds between "2 + 2" and "4" not merely the latter, but also the former, connection.

After discussing "implication", Mr. Russell proceeds to discuss the remainder of the "fundamental logical concepts" enumerated above. The greater part of this discussion is occupied with points bearing on the concepts which I indicated as belonging to group (4); that is to say, it is occupied with the discussion of precisely what is involved in what we mean when we talk of "all the things which are such that, if any of them has or were to have a certain property, it also has or would have a certain other property". And in this connection Mr. Russell discusses many points of great philosophical interest, and brings out clearly several important conclusions; but also much of what he says is very obscure, and it may be doubted whether some of his conclusions are not mistaken. I cannot attempt here now to summarise his discussion; but there is one very striking point, of an exceedingly wide and important bearing, which should, I think, be noticed. Mr. Russell points out that there is a difficulty in discovering the exact meaning of the exceedingly numerous class of propositions which we express by such phrases as "*All* men are mortal", "If *any* man is mortal", "I met *a* man", "*The* man who did this". It is plain that in such cases we do not generally mean merely to assert a simple relation between two predicates, as is assumed by logicians who take an *intensional* view: we do not, for instance, when we say that "All men are mortal", merely mean to assert that the property meant by "humanity" has some *simple* relation to the property meant by "mortality"; we undoubtedly do mean to assert that the two properties are related *in respect of their extension*—that the whole extension of the one is a part of that of the other. But on the other hand it is equally plain that, when we make such a proposition, we by no means always have the whole extension, about which we seem to assert something, before our minds. It would seem, therefore, at first sight, as if we could be cognisant of a proposition *about* a thing, without having that thing before our minds. And in this book Mr. Russell actually adopts this view, and even adds the conclusion, that the thing about which a proposition is, may not be a constituent in that proposition. But this conclusion is undoubtedly wrong; and Mr. Russell has since suggested a much better analysis of the propositions in question (in *Mind*, N.S. No. 56, October, 1905). It would, however, be impossible to explain intelligibly the exact nature and bearings of his new conclusions, without occupying a much greater space than can here be given to them.

It should, finally, be mentioned, before leaving Mr. Russell's discussion of logical concepts, that he discovers several new antinomies—cases, that is to say, in which two propositions appear to be mutually contradictory, and yet, each of them, to be evidently true. The cases in question are, indeed, cases in which each of two mutually contradictory propositions appears actually to *imply* the other. That this should appear to be the case undoubtedly shews that some principle, which appears to be evidently true, is really false; and, until it can be shewn what that principle is, we cannot be sure that it is not an important one. These antinomies are, therefore, worthy of attention, and they are certainly very difficult to explain. Mr. Russell himself, in spite of much discussion, does not arrive at any satisfactory "solution": that is to say, he does not succeed in shewing either that the propositions in question are not mutually contradictory, nor, if they are so, which of them is false, and why, being false, it appears to be true.

It remains, finally, to give some account of Mr. Russell's treatment of the conceptions of "infinity" and "continuity". It has been very commonly supposed that these conceptions involve antinomies; that is to say, that, when any one asserts of space and time, for instance, that they are infinite or continuous, his assertion implies both that they do and that they do not possess some definite property, or implies, at least, that they do not possess some property, which it is very obvious that they do possess. Mr. Russell tries to shew that this is not the case: and the substance of his argument is as follows.

He defines with very great care certain properties which certainly do belong to some of the things which mathematicians would call respectively "infinite" and "continuous"; and which are such, that anything whatever which did possess them would be admitted to deserve to be so called. That is to say, he defines, in the case of each word, one or more properties, which are certainly equivalent to, if not identical with, one at least of the senses in which that word is used in Mathematics. But the properties, which he has thus defined, include, he urges, the only ones from which any plausible argument against infinity or continuity has ever been drawn: all such arguments assume that one or other of the consequences, which follow from the supposition that anything has these properties, is impossible. In order, therefore, to refute these arguments, he need only shew that none of these consequences are really impossible. With this object, he first points out that they are certainly not self-contradictory. The only possible remaining argument against them is, accordingly, that they do contradict some proposition which is evidently true. And he admits that they do contradict propositions which *appear* to be evidently true. But he urges that this appearance is fallacious. It arises, he suggests, from the fact that, owing to our far greater familiarity with *finite* series, we are led to think that every property which belongs to finite series must also belong to *every* series. Certain properties, which must belong to infinite or continuous series, if there be such things, cannot belong to any finite series—cannot, that is, belong to any of the series with which we are most familiar. This fact may very naturally make it appear evident that *no* series can have the properties in question; but it is obviously not a good reason for holding that this is the case; nor does there appear to be any other reason why certain series should not possess properties which no finite series can possess.

In the case of infinity Mr. Russell first defines two properties which can only belong to *whole* numbers, that is to say, to the number of terms in a series or collection. These properties are strictly *equivalent* to one another, that is to say, anything which possesses the one also possesses the other; both are also equivalent to every sense in which a whole number can be said to be "infinite"; and the contradictory of each constitutes a property which is equivalent to every sense in which a whole number can be said to be "finite".

We thus obtain four properties, two of which are equivalent to what is meant when a whole number is said to be "infinite", and two to what is meant when a whole number is said to be "finite"; and the following statements both contain a definition of these properties, and express their equivalence to "finitude" and "infinity", in the sense in which whole numbers can be finite and infinite. (1) A whole number is infinite, if and only if it is the number of terms in a collection, which contains as a mere part of itself another collection which has the same number of terms as the whole collection of which it is a mere part. It is finite if and only if it is the number of terms in a collection, which has *not* the same number of terms as any part of itself. (2) A whole number is infinite, if and only if it does *not* belong to the series of numbers starting from 1, of which each is greater by 1 than the number before it. It is finite, if and only if it *does* belong to this series.

Now Mr. Russell constantly speaks of these four properties as if they constituted the only possible definitions of the words "finite" and "infinite", or at all events the only possible senses in which those words can be truly applied to whole numbers. But, when he uses such language, he says what is not strictly true, and what is, I think, liable to cause serious misunderstanding. In order to avoid such misunderstanding, it will, I think, be well to explain as accurately as possible the relation of these properties to other senses of the words "finite" and "infinite". These other senses may be divided into three groups. (1) In the first place it is plain that not only the number of terms in a series or collection, but also a series or collection itself may be said to be finite or infinite; and, since the properties which Mr. Russell has defined can belong to numbers only, it is plain that the properties which belong to infinite series or collections must be different from them. In the case of collections one such property can be easily discovered: the first property of infinite number is defined by reference to it. Since a whole number is infinite if and only if it is the number of the terms in a collection which has the same number of terms as one of its parts, it is plain we may say that any collection, which has the same number of terms as one of its parts, is itself infinite, in virtue of that fact alone. Since, however, this property of collections is involved in the definition of Mr. Russell's first property, there is obviously no need to consider it separately; and it is also clear that it is strictly equivalent to properties which can be defined by reference to Mr. Russell's two properties: a collection will be finite in this sense if and only if the number of its terms is infinite. But in the case of series, there is an obvious independent definition of infinity, which is more important. It is undoubtedly with reference to series that the word "infinite" is most commonly employed; the property, which it most naturally suggests, is one which can only belong to them; and this property is undoubtedly the most easily intelligible of any which the word can suggest. In short, when we think of "infinity" we most commonly think of "endlessness"—we think of a series which has no beginning, or no end, or which has neither. And it is, in fact, true that every "endless" series must be admitted to be "infinite". But whether "endless" is *equivalent* to any sense of "infinite"—whether there be any sense of "infinite" in which it is also true that every "infinite" series must be "endless", can easily be made to appear very doubtful. It is at all events certain that, if a series can be "endless" at all, it can also be "infinite" in one very plain sense without being "endless". For, if we can conceive a series which has a beginning but no end, we can also conceive that one other term might be added *after* all the terms of this endless series; and this term together with the endless series, would then form a new series, which must be admitted to be infinite, since it contains an infinite series as a part of itself, but which is yet not endless, since it has both a beginning and end. Such an infinite series is, in fact, constituted by any two terms in a compact series *together* with all the points between them: it is, for instance, constituted by all the points on any straight line which are between any two points on the line, *together with* those two points, or by all the rational numbers which are greater than one given rational number and less than another, *together with* the two rational numbers in question. The fact that there are such infinite series, which are *not* endless, shews, therefore, that we can hardly regard mere endlessness as equivalent to infinity; but it also suggests a property of series which *may* be regarded as equivalent to infinity. We may, I think, say that a series is infinite, if and only if it *either* is itself endless *or* contains an endless series as a part of itself. We thus obtain, by reference to the conception of "endlessness", the definition of a property, which does not involve in its definition Mr. Russell's two properties and which does involve properties which are not involved in their definition; and which also is far more in accordance with the ordinary usage of the word infinity. This property is, moreover, strictly equivalent, in the case of series, to that of having an infinite number of terms: the number of terms in a series is infinite if and only if the series is either itself endless or contains an endless series as a part of itself. And we can also obtain a similar property which is equivalent to infinity, in the case of collections: a collection is infinite, if and only if it is either itself an endless series, or contains an endless series as a part of itself. We find, accordingly, that while infinity, in the case of series and collections, is always *equivalent* to properties which can be defined by reference to Mr. Russell's two properties of infinite number, it has one sense, and that the most in accordance with ordinary usage, which involves reference to properties which neither involve nor are involved in the definition of Mr. Russell's two properties. But (2) it is plain, also, that, if what has been said with regard to series and collections is correct, we can obtain a definition of what is meant by "infinite" *whole* numbers, which, while equivalent to Mr. Russell's two properties, is not only different from them, but is effected by reference to properties which neither involve nor are involved in their definition. We may, in fact, say that: A whole number is infinite, if and only if it is the number of terms in a collection, which is either itself an endless series, or contains an endless series as a part of itself; and that: It is finite, if and only if it is the number of terms in a collection which is neither itself an endless series nor contains an endless series as part of itself. Finally (3) there are also, as Mr. Russell subsequently explains, several senses of "finite" and "infinite", which can only be defined by reference to one or other of the mutually equivalent senses, just discussed, in which whole numbers can be finite or infinite. We may, therefore, distinguish such senses from those hitherto discussed, as "derivative" from "primary". But these derivative senses are of some importance. They include among them not only new senses, equivalent to the old, in which *whole* numbers (and hence, by reference to these, series and collections) can be said to be finite or infinite, but also the only senses in which those words could be applied to magnitudes and quantities, and to any numbers *other* than whole numbers; and they include, too, the only senses of finite and infinite, to which "infinitesimal" is correlative.

It appears, therefore, that the four properties defined by Mr. Russell are by no means the only senses in which the words "finite" and "infinite" can be used; they are not even the only senses in which these words can be applied to whole numbers. But it appears also that all the other senses of the words, without exception, are *equivalent* either to these four properties themselves or to properties which involve some one of the four in their definition; and moreover that all other senses of the words, *with a single exception*, involve in their definition either one of these four properties themselves or else some property which is involved in the definition of one of them. The single exception is the sense or senses of finitude and infinity, which are defined by reference to "endlessness": and it may therefore be reasonably expected that, if there is any sense of "infinite" at all, in which impossible consequences follow from the supposition that anything is infinite, these consequences will follow either from the supposition that anything possesses the properties defined by Mr. Russell, or from the supposition that there is such a thing as an endless series or a series which, though not itself endless, contains an endless series. Any consequence, however, which follows from the latter supposition will also follow from the supposition that any series is continuous; since, as will be seen, it follows from the definition of continuity, that if any series are continuous, there must also be series which, though not themselves endless, contain an endless series. It is plain, therefore, that, if Mr. Russell succeeds in shewing that no impossible consequences follow either from the supposition that things possess the properties he has defined, or from the supposition that series may be continuous, he may fairly claim to have proved the possibility of infinity.—We will now, therefore, proceed to examine the consequences of these three suppositions.

And (1) the first of the two properties, which Mr. Russell has defined as equivalent to "infinity" does certainly seem to involve in itself an impossibility. It very naturally seems to be impossible that any collection should contain precisely the same number of terms as a mere part of itself. When it is said that one collection *B* is a mere part of another collection *A*, it is meant that *A* contains all the terms which are contained in *B* and contains also, in addition to these, some terms which *B* does not contain. And that, in spite of this, *A* should contain no more terms—no greater number of terms—than *B* contains, does certainly seem at first sight to be impossible.

It is, Mr. Russell seems to think, only because this appears to be impossible, that it appears impossible, in the well-known puzzle, that Achilles should overtake the tortoise. In this, however, I think he is mistaken: in the puzzle, as usually stated, the conclusion that Achilles cannot overtake the tortoise follows, not from the assumption that a whole cannot have the same number of terms as its parts, but from the assumption that something quite different is impossible; the puzzle, that is to say, calls attention to *another* apparent impossibility, with which I shall presently deal. What Mr. Russell really does is to construct a *new* puzzle, which does, like the old, make it appear impossible that Achilles should overtake the tortoise, but for a different reason: and it is important to emphasize the difference between this new puzzle and the old one, both because it is important to consider separately all the *different* consequences of infinity, which appear to be evidently impossible, and also because, in Mr. Russell's puzzle, the proof that Achilles cannot overtake the tortoise depends upon certain assumptions with regard to the nature of their motion, which are not required by the proof in the old puzzle. The assumptions which Mr. Russell makes are, however, such as commend themselves to common sense: his proof that Achilles cannot overtake the tortoise does appear to be perfectly plausible: and it will, therefore, be worth while to give it; since it shews, that, given certain plausible assumptions with regard to the nature of movement, it really would be impossible for a faster body ever to overtake a slower one, *unless* a whole can have the same number of terms as its parts. The assumptions which Mr. Russell makes are the following: (1) That, during the whole period of their race, both Achilles and the tortoise are constantly moving in the sense that at every moment in this period each of them is in a different position from that in which he is at any other moment in the period; and (2) that Achilles' path in space is continuous in the sense that, in moving from one position to another, he passes through *all* the positions which intervene between the two. From the first of these assumptions it follows that the number of positions occupied by the tortoise during the race must be precisely the same as the number occupied by Achilles, since each occupies just as many different positions as there are moments in the period; and from the second it follows that, in passing from any one point which the tortoise has occupied to any other point which the tortoise has occupied, Achilles must occupy *all* the positions which the tortoise occupied in making the same journey. It follows, therefore, that, if Achilles does overtake the tortoise, he must *both* have occupied all the positions which the tortoise has occupied, as well as others beside, which the tortoise has not occupied, *and also* must have occupied precisely the same number of positions, as the tortoise, and no more. It follows, that is to say, that if any faster and slower bodies can move in the manner which Mr. Russell assumes, then it must be impossible for any of the faster to overtake any of the slower, *unless* it is possible for a whole to have the same number of terms as its part. But that faster bodies do overtake slower ones is certain; and that both may nevertheless have moved in the manner supposed is commonly assumed: and it would appear, therefore, that it must be possible for a whole to have the same number of terms as one of its parts—for it to have, that is to say, precisely that property which, Mr. Russell tells us, every infinite collection must have. And that it is possible for a collection A to contain no *more* terms than B, although it contains all the terms which B contains and also others besides, Mr. Russell urges as follows. The supposition is, in the first place, certainly not self-contradictory. For to say that A contains other terms beside those which B contains is not the same thing as to say that it contains a *greater* *number of* terms than *B*: to say that *B* is a mere part of *A* is quite a different thing from saying that it contains a *smaller number* of terms. We plainly mean something different by saying that *A* contains the *same number of* terms as *B* from what we should mean if we said that *A* contained precisely the *same terms* which *B* contained, and no others: and hence the proposition "*A* contains the same number of terms as *B*" does not directly contradict the proposition "*A* contains all the terms which *B* contains and also some other besides". The appearance of contradiction arises solely from the failure to distinguish the relation of less to greater number with the quite distinct relation of part to whole. Once we have seen that to say "B is a part of *A*" is not the same thing as to say "*B* has a less number of terms than *A*", it seems plain that the proposition "*B* is a part of *A*" does not contradict the proposition "*B* has the same number of terms as *A*"; and once it is seen that these two propositions do not contradict one another, it seems impossible to find any valid reason why they should not both be true. To suppose them both true does not even involve the denial of the axiom that the whole is always *greater* than the part: for to suppose one collection to be *greater* than another is again quite a different thing from supposing it to have a *greater number* of terms. And while it is impossible to find any reason why both should not be true, it is easy to find a good reason why it should *seem* to us that both cannot be true. For in the case of *all* collections which we can examine by enumerating all their terms, the part cannot have the same number of terms as the whole. All such collections are finite; and if we add any new terms to any one of them the new collection thus formed always *has* a different number of terms from that with which we started. It is therefore very natural that we should have come to think that this property must belong to all collections.

(2) The second of the two properties, defined by Mr. Russell as equivalent to infinity, is, it will be remembered, the property of being a whole number, which does not belong to the series of numbers, starting from 1, of which each is greater by 1 than the number before. And Mr. Russell himself appears to think that the supposition that any number possesses this property, either is or directly involves, a *new* supposition, which *appears* to be impossible. But that every number must belong to the series defined, can, I think, hardly appear to any one to be a matter evident in itself and needing no proof; and I am unable to discover what *new* consequence of the contrary supposition Mr. Russell does suppose to appear thus evidently impossible. It is, indeed, plain that this second property does directly involve a consequence, which appears impossible: it involves the consequence that there is a whole number which is not greater by 1 than any other number, and a collection, therefore, which has precisely the same number of terms as the collection which remains when one of its terms is taken away. But this consequence appears impossible for the reason just considered—namely, because it appears impossible that any

collection should have the same number of terms as one of its parts; and since no other apparently impossible consequence appears to be directly involved in this second property of infinite number, I will pass on to the consideration of continuity.

(3) Under the head of "continuity" Mr. Russell gives definitions of two different properties, the definition of the one being comparatively simple, that of the other much more complicated. He would, however, prefer to restrict the name "continuity" to the complicated property, giving to the simpler one the name "compactness". These two properties are so related that every series which is "continuous" (in the restricted sense) is also "compact"; but it is not true that every "compact" series also possesses the more complicated property. The importance of the more complicated property lies in the fact that certain mathematical peculiarities, which have long been recognised as belonging to spatial series, imply that spatial series possess this property; and that, by assuming spatial series to possess it, all such peculiarities can be completely explained. We may, therefore, say that space is not only "compact", but also "continuous" in the restricted sense: but it should be remembered that this kind of "continuity", which does belong to actual space, is defined in purely logical terms and may therefore belong to other series also; and, in fact, it does belong, not only to time, but also to certain numerical series. For our present purpose, however, it is unnecessary to give the very complicated definition of this property; since the assumption that any series possess it appears to involve no paradox beyond what are already involved in the assumption that any series is "compact". I shall, therefore, confine myself to the consequences of the assumption that any series is "compact".

The definition of "compactness" is very easy to understand: A series is compact, if and only if between any two of its terms there is another of its terms. The meaning of this statement seems scarcely to require any explanation; but, in a view of what is to follow, it will be well to define precisely what is meant by "between"—a definition by reference to which we can also define what is meant by a series. Any term *b* is said to be *between* two other terms *a* and *c*, if and only if *a* has to *b*, and also *b* has to *c*, some relation which is both *transitive* (i.e. such that if *a* has it to *b*, and *b* to *c*, *a* also has it to *c*) and *asymmetrical* (i.e. such that if *a* has it to *b*, *b* has *not* got it to *a*). For instance, when it is said that the length of a foot is *between* that of an inch and that of a yard, it is meant only that an inch has to a foot and a foot to a yard one and the same relation, namely, that an inch is *less* than a foot, and a foot than a yard; that an inch is also less than a yard; and that a foot is *not* less than an inch, nor a yard less than either a foot or an inch. A series may then be defined as any set of terms, which are such that, *any* three of the set being taken, one of the three will be *between* the other two in respect of one and the same relation. And we find, accordingly, that a compact series means only a set of terms, which are such that, of *any* two of them, *a* and *b*, *a* has to *b* some transitive asymmetrical relation; and which are such also that, whatever terms of the set *a* and *b* may be, there is some other term of the set *to* which *a* has this same relation which it has to *b*, and which also has to *b* the relation in question.

Now of any such series it may be proved, in the first place, that the number of its terms must be infinite in the sense above discussed, i.e. that they must form a collection which has the same number of terms as one of its parts. The assumption, therefore, that any series is compact does involve the apparently impossible consequence that a whole can have the same number of terms as one of its parts. But it involves also, very directly, two other consequences, both of which appear to be impossible.

(1) The first of these consequences is, I think, that which is assumed to be impossible in the ordinary proof that Achilles cannot overtake the tortoise, and also in Kant's proof that the world must have had a beginning in time: and its apparent impossibility can be best exhibited by considering these proofs. In the former it is pointed out that, by the time Achilles reaches the position from which the tortoise started, the tortoise will be in a new position in front of him, which we may call its *second* position; and that by the time Achilles reaches this *second* position, the tortoise will be in yet another position, which we may call its *third*, and which will still be in front of that which Achilles has now reached, and so on *ad infinitum*. That this will happen, certainly appears to be very obvious; and in assuming that it will, it is not assumed (as it was in Mr. Russell's puzzle) either that in passing from one of the tortoise's positions to another, Achilles must occupy all the positions which the tortoise occupied in the same journey, nor yet that either Achilles or the tortoise is at *every* moment in a different position from that which he occupies in any other moment.

We make no assumption whatever with regard to *all* the positions which the tortoise occupies during the race, but only with regard to *some* of them—all those, namely, which have the same peculiarity as the two which were called above the tortoise's *second* and *third* positions. These two positions are obviously not the second and third among *all* those which the tortoise occupies during the race: *all* its positions probably form a *compact* series: but they are the second and third in a certain *discrete* series, defined as consisting of the tortoise's starting-point together with all those of its subsequent positions which have the same peculiarity as these two have. The peculiarity which these two have is that they are positions of which each was occupied by the tortoise *either* at the moment when Achilles had reached the tortoise's starting-point *or* at a moment when he had reached *one* of the very two positions in question, but one which came before the one occupied by the tortoise at the same moment: and our assumptions are confined to the discrete series of positions, of which each has this same peculiarity—that it is a position occupied by the tortoise *either* at the moment when Achilles occupied the tortoise's starting-point *or* at a moment when he occupied some previous member of the very series in question. And with regard to the discrete series of positions thus defined we make only two assumptions. We assume first that this series does possess *some* members—that *some* of the positions occupied by the tortoise during the race do have the peculiarity above defined; and this assumption is obviously true: the tortoise will at least occupy the positions called above its *second* and *third* positions. And our second assumption is that which was expressed by the words "and so on *ad infinitum*": we assume, namely, that the series of positions in question is *endless*; that is to say, that the time which Achilles takes to get from one of these positions to the next will always be sufficient to enable the tortoise to reach a new position in the same series: and this assumption also seems obviously true: we must indeed admit it to be true, unless we hold that the tortoise remains *absolutely motionless* at one of the positions in question during all the time which Achilles takes in moving from the next preceding of these positions to that one. But now, granting this to be true, it follows strictly that Achilles will *never* get to the end of a series of positions, of which each is *behind* the one occupied by the tortoise at the same moment: he cannot get to the end of them, because the series is endless: there is an endless series of positions, which he must occupy before he can overtake the tortoise, each of which is behind some position simultaneously occupied by the tortoise: and since it *is* thus impossible that he should ever get to the end of positions in which he will be *behind* the tortoise, it certainly *seems* impossible that he should ever get to a position *in front* of the tortoise—overtake *it*, in fact. This, I think, is the reason why it seems impossible that Achilles should overtake the tortoise; and it is, I think, for precisely the same reason that Kant thought it impossible that the world should have had no beginning in time. To suppose that it had none, is, he says, to suppose that before any given point in time, an endless series of states of the world has elapsed: but, he urges, to suppose that a series of things is endless, is to suppose that, if you take them one after another, you will never get to the end of them: and, therefore, he concludes, it is impossible that the whole of an endless series of events should have elapsed before a given point in time. In other words Kant supposes, just what we suppose when we are puzzled by Achilles and the tortoise: namely that, because it is impossible that the *end* of an endless series should ever be reached, it is also impossible that any point which is *beyond* or *after all* the points of an endless series should ever be reached, if before it is reached, *all* the points in the endless series have first to be reached.

What, therefore, is assumed to be evidently impossible in the above arguments is that, if the *end* of a series cannot ever be reached, *all* the points of that series should ever be reached. No reason, however, can be given, why the end of a series should not be reached, if it has an end; and similarly no reason can be given why *all* the terms of a series should not be reached, if there is such a thing as *all* the terms of the series. The above assumption therefore, remains plausible, only if we assume that, if a series *has* no end, there can be no such thing as *all* the terms of that series. Unless it is impossible that there should be such a thing as *all* the terms of a series which has no end, we must admit to be possible, what *appears* to be evidently impossible—namely that *all* the terms of a series which has no end should be reached. But that there is such a thing as *all* the terms of an endless series is one of the consequences which is involved in the supposition that any series is *compact*. For to say that a series is compact is, as we saw, to say that between *any* two of its terms there is a third term. If, therefore, we take any two terms *a* and *b*, there must be some third term, which we may call *c*, between them. But again there must also be another term between *c* and *b*; another again between *this* term and *b*; and so on *ad infinitum*. Accordingly *all* the terms of an endless series must precede any term of a compact series; and not only so, but between any two terms of a compact series, there must be all the terms of a series which has neither beginning nor end. If, therefore, there is such a thing as a compact series, it must be possible that there should be such a thing as *all* the terms of an endless series. And to suppose that there is such a thing is, in the first place, certainly not self-contradictory. For to suppose that a certain set of terms are *all* the terms of some series—are, that is to say, *all* the terms which possess the property of being related by some transitive asymmetrical relation, is obviously not the *same* thing as to suppose that one of them is the *last* in that series—has, that is to say, to no other term of the series the transitive asymmetrical relation in question. And, in the second place, it is easy to see why we should *think* it impossible that any set of terms should be *all* the terms in a series, and yet none of them be the last. For in the case of any series, whose terms we can examine one by one, beginning at the beginning and taking them in order, it is impossible for us thus to examine them *all*, without finding a last term among them: it is impossible for the simple reason that to examine each separately takes a certain space of time, and that an endless series of such spaces of time is not at the command of any of us. But this is obviously no good reason for concluding that the same is true of every series—that all the terms of *every* series must include a last term among them: nor does there seem to be any other reason why we should suppose it impossible that some series have no last term.

But (2) the assumption that any series is "compact" also involves another consequence, closely connected with the last, which might also be thought to be obviously impossible. The last consequence, as we saw, appeared to be most evidently impossible in the case of series, which though not endless themselves, contain an endless series as a part of themselves—in cases, that is to say, where *all* the terms of some endless series are supposed to come *before* some term of another series to which they also belong: and, as we also saw, this property, while it belongs to all compact series, also belongs to some which contain an endless *discrete* series as a part of themselves. It is to precisely the same series that there belongs another property, which it might also seem impossible that any series should possess: this property belongs, that is to say, to all compact series, and to those only among other series which, though not themselves endless, contain an endless discrete series as a part of themselves. This new property is that of including a term which though it has many terms before it, has none which is *the next* before it, or though it has many terms after it has none which is *the next* after it. In the case of compact series *no* term is either next before or next after any other: there are *no* consecutive terms. And in the case of series, which, though not themselves endless, contain an endless discrete series, the first term, which comes after all the terms of the endless series, has no term next before it, although there *is* some term next *after* every one of the terms which come before it. Now there is, I think, a natural tendency to assume that such a state of things is impossible. It does seem to be impossible that in any set of terms, all of which come *before* some other term, and of any two of which must come before the other, absolutely none should be the *next* before—should *immediately* precede—a term which yet all of them do precede. We naturally assume that if a set of terms forms a series at all, there must be some sense in which we can talk of the one which is *next* before any one of them except the first, and of the one which is *next* after any one of them except the last. And if this assumption were correct it *would* be impossible that any series should be compact or should be a discrete series of the kind defined above: for in any such series there must be a least one term of which it is true to say that, though it has terms before it, yet absolutely no term is in any sense the *next* before it. We naturally assume that this is impossible. Yet, obviously, the supposition, which we thus assume to be impossible, is not self-contradictory. To suppose that a set of terms form a series is only to suppose that there is some transitive asymmetrical relation, such that, of *any* two of them, one has this relation to the other. And to suppose that some one of them, *A*, other than the first, has no term *next* before it, is only to suppose that, though some terms have the relation in question to *A*, yet no term has it to *A*, without also having it to some other term, which also has it to *A*: in other words that *between A* and every term which precedes it there is some other term. But these two suppositions obviously do not contradict one another: "*A* has predecessors" is not contradicted by "*A* has no predecessors except such as are followed by others of its predecessors." And, accordingly, the supposition that a set of terms may form a series without being *consecutive* is certainly not self-contradictory; nor does there seem to be any other good reason for asserting that such a series is impossible. And, moreover, here again it is easy to discover a reason why such a series should *seem* to be impossible. For in the case of any set of terms, whose properties we can discover by examining them one by one, it *is* impossible that they should form a series without being consecutive. If, for instance, we have a set of terms in which a given term *c* is preceded by the two terms *a* and *b* and by *no others*, then the supposition that *a*, *b*, *c* form a series *is* incompatible with the supposition that *c* has no immediate predecessor. For, in supposing that they form a series, we suppose that either *a* precedes *b* or *b* precedes *a*. But if *a* precedes *b*, *a* cannot follow *b*; and since no terms except *a* and *b* precede *c*, there can be no term which both follows *b* and precedes *c*; which is the same thing as to say that *c* is *consecutive* to *b*. And similarly if *b* precedes *a*, *b* cannot follow *a*, and since no terms except *a* and *b* precede *c*, there can be no term which both follows *a* and precedes *c*; which is the same thing as to say that *c* is *consecutive* to *a*. And the same result will hold good if we suppose the term *c* to be preceded only by three terms or by four terms, or by any merely finite number of terms: in any such case the supposition that *c* together with its predecessors forms a series *necessitates* the conclusion that one of these predecessors *immediately precedes c*. We are therefore familiar with an immense number of cases in which it is impossible that a set of terms should form a series, unless each term either immediately precedes or immediately follows some other; this must be so with any set of terms, whose number is finite: and it seems natural enough that this fact should have led us to suppose that the same must be true of any set of terms whatever.

We have thus examined all the senses of "infinity" which were distinguished above as "primary". And in none of these series, it appears, does the supposition that things are "infinite", involve any self-contradictory consequences. It does, indeed, involve consequences which *appear* to be impossible: but there seems very good reason to suspect that this appearance is fallacious. Yet it would seem, as was said, to be only from the apparent impossibility of these consequences that any plausible arguments against infinity have ever been drawn. We may conclude, then, that such arguments are one and all fallacious: and, considering how many philosophical theories have been supported by the supposition that infinity is impossible, such a conclusion is not without importance. If it were only for the extremely careful analysis, which renders such a conclusion possible, Mr. Russell's book certainly deserves the attention of philosophers.

[WORKS CITED]

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