10.3 Expanding Substitutional to Intensional Models

We have one part left of our rectange:

$$\begin{array}{ccc} A_1, \ldots, A_n \vdash_{\text{fK}} B & \Longrightarrow & A_1, \ldots, A_n \vDash_{\text{iK}} B \\ \Uparrow & & \Downarrow \\ \text{The tree for } A_1, \ldots, A_n, {\sim}B \text{ closes} & \Longleftarrow & A_1, \ldots, A_n \vDash_{\text{tK}} B \end{array}$$

In particular, the right side remains: that if an argument is iK valid then it is tK valid.

As with certain other results, it suffices to show the contrapositive, that *if an argument is tK invalid, then it is iK invalid.* Showing this is in effect to show that every argument that has a tK counterexample model has an iK counterexample model. We shall in fact prove something stronger:

(tK-iK Conversion) For every tK model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a^t} \rangle$, there is an iK model $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{I}, \mathbf{a^i} \rangle$ such that for every world $\mathbf{w}$ in $\mathbf{W}$ and sentence A, $\mathbf{a^t_w}(A) = \mathbf{a^i_w}(A)$ (and hence, $\mathbf{a^t_w}(A) = \text{T}$ iff $\mathbf{a^i_w}(A) = \text{T}$).

In other words, for every tK model, there is a corresponding iK model that makes the same sentences true at the same worlds.

Proof: Let $\langle \mathbf{W}, \mathbf{R}, \mathbf{a^t} \rangle$ be an arbitrary tK model. Let us define an iK model $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{I}, \mathbf{a^i} \rangle$ as follows:

- The set of worlds $\mathbf{W}$ and accessibility relation $\mathbf{R}$ for the two models are identical. (It follows that if the tK model was, e.g., a tT model or tS5 model, then the corresponding iK model is an iT or iS5 model, etc.)

- We replace the substitutional assignment function $\mathbf{a^t}$ with an intensional one $\mathbf{a^i}$, defined such that, for each $\mathbf{w}$ in $\mathbf{W}$:
  - For each term t, $\mathbf{a^i_w}(t) =$ the set of terms s for which $\mathbf{a^t_w}(t = s) = \text{T}$.
  - For each predicate letter P, $\mathbf{a^i_w}(P) =$ the set of n-tuples $\langle \mathbf{o}_1, \ldots, \mathbf{o}_n \rangle$ where each $\mathbf{o}_1, \ldots, \mathbf{o}_n$ is itself a set of terms, and there are terms $t_1, \ldots, t_n$ such that $\mathbf{a^i_w}(t_1) = \mathbf{o}_1$ and …and $\mathbf{a^i_w}(t_n) = \mathbf{o}_n$ and $\mathbf{a^t_w}(P(t_1, \ldots, t_n)) = \text{T}$.
  - The other values of $\mathbf{a^t}$ are determined by the rules governing all intensional models.

- The domain structure $\mathbb{D}$ of the iK model is such that $\mathbf{D}$ is the set of all sets of terms $\mathbf{o}$ such that there is a term t where $\mathbf{a^i_w}(t) = \mathbf{o}$ for some $\mathbf{w}$ in $\mathbf{W}$, and for each $\mathbf{w}$ in $\mathbf{W}$, $\mathbf{Dw}$ is the set of all sets of terms $\mathbf{o}$ such that there is a term t where $\mathbf{a^i_w}(t) = \mathbf{o}$ and $\mathbf{a^t_w}(\text{E}t) = \text{T}$.

- The set of privileged intensions $\mathbf{I}$ is the set of all functions $\mathbf{i}$ from worlds to set of terms such that there is a constant c for which for each $\mathbf{w}$ in $\mathbf{W}$, $\mathbf{a^i_w}(c) = \mathbf{i}(\mathbf{w})$ (or, in other words, $\mathbf{I}$ is the set of all intensions that are intensions of constants).

In order for the model just defined to count as an iK model at all, the following must hold of it:

(1) It must be that $\mathbf{a^i_w}(t) =$ some member of $\mathbf{D}$ for each term t. This is obvious from how we defined $\mathbf{D}$.

(2) It must be that $\mathbf{a^i_w}(P) =$ a set of n-tuples taken from $\mathbf{D}$ for each predicate letter P. This is obvious from how we specified $\mathbf{a^i}$ and $\mathbf{D}$ above.

(3) For each constant c, $\mathbf{a}(c)$ must be a member of $\mathbf{I}$. This is obvious from how we specified $\mathbf{I}$ above.

(4) Most interestingly, it must be that $\mathbf{a^i_w}(\text{I}) =$ the set of all pairs $\langle \mathbf{o}, \mathbf{o} \rangle$ where $\mathbf{o}$ is a member of $\mathbf{D}$. This can be seen to hold as follows. First, for each $\mathbf{o}$ in $\mathbf{D}$, $\langle \mathbf{o}, \mathbf{o} \rangle$ must be in $\mathbf{a^i_w}(\text{I})$. Note $\mathbf{o}$ is $\mathbf{a^i_w}(t)$ for some term t, and $\mathbf{a^t_w}(t = t) = \text{T}$, since our tK model must obey (t=) on page 48. Thus, $\langle \mathbf{o}, \mathbf{o} \rangle$ is in $\mathbf{a^i_w}(\text{I})$ by our general specification of $\mathbf{a^i_w}(P)$ for all predicate letters P above. Note also that it is impossible for $\mathbf{a^i_w}(\text{I})$ to contain other ordered pairs $\langle \mathbf{o}_1, \mathbf{o}_2 \rangle$ where $\mathbf{o}_1 \neq \mathbf{o}_2$. This would only be possible if there were terms t_1 and t_2 where $\mathbf{a^i_w}(t_1) = \mathbf{o}_1$ and $\mathbf{a^i_w}(t_2) = \mathbf{o}_2$ and $\mathbf{a^t_w}(t_1 = t_2) = \text{T}$ but $\mathbf{a^i_w}(t_1) \neq \mathbf{a^i_w}(t_2)$. This is impossible, since $\mathbf{a^i_w}(t_1)$ is the set of terms s where $\mathbf{a^t_w}(t_1 = s) = \text{T}$, and $\mathbf{a^i_w}(t_2)$ is the set of terms s where $\mathbf{a^t_w}(t_2 = s) = \text{T}$, and by the other clause of (t=), which supports symmetry and transitivity of identity, these must be the same set.

Now, in order to show our main result, we need to show that for every world $\mathbf{w}$ in $\mathbf{W}$ and sentence A, $\mathbf{a^t_w}(A) = \mathbf{a^i_w}(A)$. We prove this by wff-induction on the length of A.

Base step: A simple statement is either "$\bot$" or an atomic statement $P(t_1, \ldots, t_n)$. In the first case, it holds for all intensional and substitutional models that $\mathbf{a^t_w}(\bot) = \mathbf{a^i_w}(\bot) = \text{F}$. For atomics, it suffices to show that $\mathbf{a^t_w}(A) = \text{T}$ iff $\mathbf{a^i_w}(A) = \text{T}$.

For the left to right conditional, if $\mathbf{a^t_w}(P(t_1, \ldots, t_n)) = \text{T}$, then $\langle \mathbf{a^i_w}(t_1), \ldots, \mathbf{a^i_w}(t_n) \rangle$ is in $\mathbf{a^i_w}(P)$, and hence $\mathbf{a^i_w}(P(t_1, \ldots, t_n)) = \text{T}$ by the semantics for atomic statements under intensional models.

In the other direction, if $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(P(t_1,\ldots,t_n)) = \mathrm{T}$, then $\langle \mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(t_1),\ldots,\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(t_n)\rangle$ is in $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(P)$, and so there must be *some* terms $s_1,\ldots,s_n$ for which $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(s_1) = \mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(t_1)$ and $\ldots$ and $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(s_n) = \mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(t_n)$ for which $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(P(s_1,\ldots,s_n)) = \mathrm{T}$. But notice also that $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(s_1) = \mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(t_1)$ iff $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(t_1 = s_1)$ and so on for the others, and hence, because our tK models obeys (t=), we have that $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(P(t_1,\ldots,t_n)) = \mathrm{T}$.

Induction step: Assume as inductive hypothesis that for every sentence B shorter than A, $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(B) = \mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(B)$ for all $\mathbf{w}$ in $\mathbf{W}$. We need to show that it holds for A as well. Let $\mathbf{w}$ be an arbitrary world in $\mathbf{W}$.

If A takes the form $B \to C$ or $\Box B$, then its truth value is determined in terms of the truth values of B and C or the truth value of B across worlds in precisely the same way in intensional and substitutional models. Hence, $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(A) = \mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(A)$ follows straightaway by the inductive hypothesis.

All that remains is to show that if A takes the form $\forall x\, Bx$, then $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\forall x\, Bx) = \mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(\forall x\, Bx)$. Again it suffices to show that $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\forall x\, Bx) = \mathrm{T}$ iff $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(\forall x\, Bx) = \mathrm{T}$.

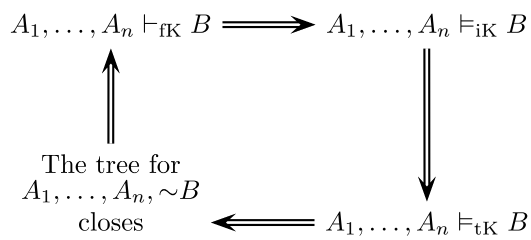
For the left to right, assume that $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\forall x\, Bx) = \mathrm{T}$. This means that for every constant c, if $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\mathrm{E}c) = \mathrm{T}$ then $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(Bc) = \mathrm{T}$. Take an arbitrary intension $\mathbf{i}$ in $\mathbf{I}$. By our characterization of $\mathbf{I}$, $\mathbf{i}$ is the intension of some constant c. By our characterization of $\mathbf{Dw}$, if $\mathbf{i}(\mathbf{w})$ is in $\mathbf{Dw}$, then $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\mathrm{E}c) = \mathrm{T}$, and so $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(Bc) = \mathrm{T}$. By the inductive hypothesis, $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(Bc) = \mathrm{T}$, and since $\mathbf{i}$ is the intension of c, for the hybrid $B\mathbf{i}$, we have $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(B\mathbf{i}) = \mathrm{T}$. In sum, for every $\mathbf{i}$ in $\mathbf{I}$, if $\mathbf{i}(\mathbf{w})$ is in $\mathbf{D}(\mathbf{w})$ then $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(B\mathbf{i}) = \mathrm{T}$. By the semantics of quantified statements in intensional models, it follows that $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(\forall x\, Bx) = \mathrm{T}$.

For right to left, assume that $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(\forall x\, Bx) = \mathrm{T}$. This means that for every $\mathbf{i}$ in $\mathbf{I}$, if $\mathbf{i}(\mathbf{w})$ is in $\mathbf{Dw}$ then $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(B\mathbf{i}) = \mathrm{T}$. Let c be any constant. By our characterization of $\mathbf{I}$, the intension $\mathbf{i}$ of c is a member of $\mathbf{I}$. By our characterization of $\mathbf{Dw}$, if $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\mathrm{E}c) = \mathrm{T}$ then $\mathbf{i}(\mathbf{w})$ is in $\mathbf{Dw}$. Hence, if $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\mathrm{E}c) = \mathrm{T}$ then $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(B\mathbf{i}) = \mathrm{T}$. Since $\mathbf{i}$ is in the intension of c, if $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(B\mathbf{i}) = \mathrm{T}$ then $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(Bc) = \mathrm{T}$. By the inductive hypothesis, if $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(Bc) = \mathrm{T}$ then $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(Bc) = \mathrm{T}$. In sum, if $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\mathrm{E}c) = \mathrm{T}$ then $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(Bc) = \mathrm{T}$. Since c was arbitrary, this holds for all constants c. Thus, by the semantics of quantified statements in substitutional models, it follows that $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(\forall x\, Bx) = \mathrm{T}$.

This completes the induction, and also the proof of (tK-iK Conversion), as listed earlier.

Some corollaries:

- Suppose that a given tK model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a}^{\mathbf{t}}\rangle$ is a counterexample to the argument $A_1,\ldots,A_n/\therefore B$. This means that there is a world $\mathbf{w}$ in $\mathbf{W}$ for which $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(A_1) = \mathrm{T}$ and $\ldots$ and $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(A_n) = \mathrm{T}$ but $\mathbf{a}^{\mathbf{t}}_{\mathbf{w}}(B) = \mathrm{F}$. It follows by (tK-iK Converstion) above that there is an iK model $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{I}, \mathbf{a}^{\mathbf{i}}\rangle$ such that $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(A_1) = \mathrm{T}$ and $\ldots$ and $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(A_n) = \mathrm{T}$ but $\mathbf{a}^{\mathbf{i}}_{\mathbf{w}}(B) = \mathrm{F}$. This model is an iK counterexample to $A_1,\ldots,A_n/\therefore B$.
- In other words, if $A_1,\ldots,A_n \nvDash_{\mathrm{tK}} B$, then $A_1,\ldots,A_n \nvDash_{\mathrm{iK}} B$.
- By contraposition, if $A_1,\ldots,A_n \vDash_{\mathrm{iK}} B$ then $A_1,\ldots,A_n \vDash_{\mathrm{tK}} B$. This completes:



- Following the rectangle around the other way, we also have that if $A_1,\ldots,A_n \vDash_{\mathrm{tK}} B$, then $A_1,\ldots,A_n \vDash_{\mathrm{iK}} B$, and so $A_1,\ldots,A_n \vDash_{\mathrm{tK}} B$ iff $A_1,\ldots,A_n \vDash_{\mathrm{iK}} B$.
- We also get the tK soundness of fK, i.e., that if $A_1,\ldots,A_n \vdash_{\mathrm{fK}} B$, then $A_1,\ldots,A_n \vDash_{\mathrm{tK}} B$.
- We get the completeness of iK, i.e,, that if $A_1,\ldots,A_n \vDash_{\mathrm{iK}} B$, then $A_1,\ldots,A_n \vdash_{\mathrm{fK}} B$, etc.
- Similar results follow for fT/tT/iT and for fB/tB/iB and for fS5/tS5/iS5, etc.; the modifications necessary the proofs given are trivial.
- If we wished to include objectual semantics, we need only treat constants as rigid on our trees, adopt system rK (or system oK if there are additional terms), consider only those iK models, riK models, in which the class $\mathbf{I}$ contains rigid intensions, consider only those tK models, rtK models, in which $\mathbf{a}_{\mathbf{w}}(b = c) = \mathbf{a}_{\mathbf{v}}(b = c)$ for any worlds $\mathbf{w}$ and $\mathbf{v}$ in $\mathbf{W}$ for all constants b and c, and so on, we we'd get the same results. (I didn't think it worth our time to look at all the details.)

Homework: Explain, briefly and informally, why the tK-iK conversion process just sketched, when applied to a rtK model, would generate an riK model.