8 Semantics for Quantified Modal Logic

The proper way to do formal semantics for Quantified Modal Logic is very controversial.

This is in part because semantics for quantified logic is controversial in general, but the issues become even more difficult when modal or other intensional contexts are involved. Garson considers five major approaches; we shall only have time to look at three in any detail.

8.1 Common Core

What the various approaches have in common is the notion of a model, and the resulting notions as *counterexample*, *validity*, *logical equivalence*, *satisfiability*, etc.

These are defined in terms of models precisely as we did for propositional modal logic and this needs not be repeated here.

Moreover, all the approaches we shall consider have in common that every model provides at least three things: **W**, a collection of worlds, **R**, an accessibility relation between the worlds in **W**, and **a**, an assignment function determining a semantic value for the meaningful parts of the language. *Some* of the kinds of models we shall consider provide more than just these three things, but all provide *at least* these.

Moreover, all take truth values T, F as the semantic values of sentences. They differ in how to characterize the semantic rules governing *atomic sentences* (including identity statements) and *quantified sentences*, but for other kinds of sentences, the following provisos are held in common (carried over from propositional modal semantics):

$\mathbf{a_w}(\bot) = \mathrm{F}$

$$\mathbf{a_w}(A \to B) = \begin{cases} \mathrm{T} & \text{if } \mathbf{a_w}(A) = \mathrm{F} \text{ or} \\ & \quad \mathbf{a_w}(B) = \mathrm{T} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

$$\mathbf{a_w}(\Box A) = \begin{cases} \mathrm{T} & \text{if } \mathbf{a_v}(A) = \mathrm{T} \text{ for every } \mathbf{v} \text{ in} \\ & \quad \mathbf{W} \text{ such that } \mathbf{wRv} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

We can obtain the expected results for $\mathbf{a_w}(\sim A)$, $\mathbf{a_w}(A \vee B)$, $\mathbf{a_w}(A \,\&\, B)$, $\mathbf{a_w}(A \leftrightarrow B)$ and $\mathbf{a_w}(\Diamond A)$, as before. (See page 18 of the handouts.)

What differs between the various approaches is how to give similar specifications for $\mathbf{a_w}(P(t_1, \ldots, t_n))$ and $\mathbf{a_w}(\forall x\, Ax)$.

8.2 Substitutional Semantics

One of the two most influential approaches is substitutional semantics. The basic idea here is that a quantified statement is true when all of its *instances* are true.

In other words, what makes "∀x Fx" true (assuming it is) is that "Fa" is true, "Fb" is true, "Fc" is true, "Fa′" is true, and so on. (If we wish to cohere with free logic, we need only consider those constants that denote something.)

This makes the truth of a quantified sentence depend upon *the truth of other sentences*, not directly on entities themselves.

It is called "substitutional semantics," because the truth of a quantified statement depends on the truth of all the *substitution instances* for the variables.

On this approach, the function $\mathbf{a_w}$ assigns truth values directly to atomic statements. It may assign either T or F (always one or the other but never both) to any atomic statement, with the only restrictions noted below. A **tK model** is any model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$ obeying the restrictions on the left as well as the following:

(t=) For every world **w** in **W**, for all terms of the language, $\mathbf{a_w}(\mathrm{I}(t, t)) = \mathrm{T}$ (i.e., $\mathbf{a_w}(t = t) = \mathrm{T}$) and, if $\mathbf{a_w}(\mathrm{I}(t, s)) = \mathrm{T}$ then $\mathbf{a_w}(P(u_1, u_2, \ldots, u_n, s, v_1, v_2, \ldots, v_m)) = \mathbf{a_w}(P(u_1, u_2, \ldots, u_n, t, v_1, v_2, \ldots, v_m))$.

$$\mathbf{a_w}(\forall x\, Ax) = \begin{cases} \mathrm{T} & \text{if for every constant } c, \text{ if} \\ & \quad \mathbf{a_w}(\mathrm{E}c) = \mathrm{T} \text{ then } \mathbf{a_w}(Ac) = \mathrm{T} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

(The proviso (t=) ensures that the principles of identity logic are tK-valid.) If we want the model to support the classical quantifier rules, we need only assume that $\mathbf{a_w}(\mathrm{E}c) = \mathrm{T}$ for all constants c and worlds **w** in **W**.

It follows by the definition of "∃" and the other semantic rules that:

$$\mathbf{a_w}(\exists x\, Ax) = \begin{cases} \mathrm{T} & \text{if for some constant } c, \\ & \quad \mathbf{a_w}(\mathrm{E}c) = \mathrm{T} \text{ and } \mathbf{a_w}(Ac) = \mathrm{T} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

With tK models defined, we can define tT, tB, tS4, or tS5 models, etc., as kinds of tK models whose accessibility relations have certain properties (reflexivity, symmetry, transitivity, etc.), as well as the correlated specific notions of validity, equivalence, etc.

Substitutional semantics for quantifiers is very simple, easy to understand, and straightforward to apply. Unfortunately, however, it is not without disadvantages:

- It does not explain or utilize the notion of the *reference* of terms. A tK model is free to stipulate the truth value of $t = s$ however it wishes provided it obeys (t=).

- It makes the truth conditions for certain statements viciously circular if (Def E) is adopted. Notice that whether or not $\forall x\, Ax$ is true depends on whether Ec is true for all constants c, and if this were interpreted as $\exists x\, x = c$, i.e., $\sim\forall x \sim x = c$, its truth value would depend on another universal statement (and that one on itself!) For this reason, "E" for existence must be taken as an undefined predicate letter instead, and a model is free to stipulate what constants make Ec true as it wishes.

- It completely ignores things that are not named by any constant or term. $\forall x\, Fx$ can be true even if not all things are F provided that none of the things that aren't F have names.

- A related problem: it cannot quantify over more entities than there are constants. While we have allowed infinitely many constants in our language by using $'$ and $''$ etc., we still only have $\aleph_0$, and hence could not quantify over $2^{\aleph_0}$ or more things.

Despite these worries, substitutional semantics is in many ways the simplest and is also the easiest to correlate with trees. We shall also see that while it does have differences, the notion of validity it defines ends up being coextensional with other notions of validity.

Garson also sketches a more sophisticated kind of substitutional semantics involving "sK models". I didn't think it worth our time to look at the details.

8.3 Objectual Semantics

Objectual semantics is probably the most common way of doing formal semantics for quantified logic, not just for quantified modal logic, but for all kinds of logical systems involving quantifiers.

The core difference is that, rather than defining the truth of a quantified sentence of the form $\forall x\, Ax$ in terms of the truth of other sentences, it defines in it terms in virtue of a *domain of quantification*, i.e., a collection of objects or entities regarded as the possible "values" of the variable x.

Each predicate letter is assigned its own semantic value, its "extension", for each possible world.

Then, e.g., "$\forall$x Fx" is regarded as true in world **w** if and only if every object in the domain of quantification for variable "x" that exists in world **w** is in the extension of "F" at **w**.

Garson's way of filling out the details (based on work by Smullyan) is idiosyncratic as far as objectual semantics goes. Objectual semantics usually defines truth in terms of satisfaction, i.e., whereupon certain assignments of values to variables "satisfy" a given open sentences, and certain other assignments not "satisfying" them. Instead of dealing with open sentences and satisfaction, Garson deals with "hybrids" understood as the result of substituting an actual object in for one or more of the argument places within an atomic sentence, resulting in something that is partly linguistic, partly extra-linguistic. The results are the same as for the traditional approach, however.

These "hybrids" are indeed odd, but could perhaps be made more palatable by understanding them as set-theoretic constructs. Garson does not portray them this way. Most likely this is order to limit the amount of set theory the reader needs to know (which is a lot for other ways of doing objectual semantics). Presumably he also wishes to preserve greater similarity with the substitutional approach.

We make this more precise as follows.

An **oK model** takes the form, $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{a} \rangle$, where **W** is a set of worlds, **R** is an accessibility relation beween those worlds, **a** is an assignment function (now expanded so that predicate letters, constants and even nonlinguistic items are also given semantic values by it), and $\mathbb{D}$ is what Garson calls a "domain structure," itself consisting of two things, (i) a set **D** of objects (the set of all entities existing at any possible world in **W**) and (ii) an function that assigns to each world **w** in **W** a subset of **D** of objects existing at that world, written **Dw**.

In addition to those in the "common core," the following constraints are placed on the assignment function **a** in an oK model:

$\mathbf{a_w}(t)$ is some member of **D** for each term t (it need not be a member of **Dw**)

$\mathbf{a_w}(\mathbf{d}) = \mathbf{d}$ itself, for each member **d** of **D**

$\mathbf{a_w}(P)$ = a set of n-tuples taken from **D**, for each predicate letter **D** (e.g., a set of ordered pairs for dyadic relational predicates, a set of ordered triples for triadic relational predicates, etc.)

$\mathbf{a_w}(\mathrm{I})$ = the set of all ordered pairs of the form $\langle \mathbf{d}, \mathbf{d} \rangle$ where $\mathbf{d}$ is a member of $\mathbf{D}$

For any $P(\mathbf{o}_1, \ldots, \mathbf{o}_n)$, where each of $\mathbf{o}_1, \ldots, \mathbf{o}_n$ is *either* a term of the language, *or*, a member of $\mathbf{D}$, so that $P(\mathbf{o}_1, \ldots, \mathbf{o}_n)$ is either an atomic sentence or a hybrid:

$$\mathbf{a_w}(P(\mathbf{o}_1, \ldots, \mathbf{o}_n)) = \begin{cases} \mathrm{T} & \text{if } \langle \mathbf{a_w}(\mathbf{o}_1), \ldots, \mathbf{a_w}(\mathbf{o}_n) \rangle \\ & \quad \text{is a member of } \mathbf{a_w}(P) \\ \mathrm{F} & \text{otherwise} \end{cases}$$

We can use the notation $[Ax]^{\mathbf{d}}/x$, or simply $A\mathbf{d}$, for the hybrid that results when the *object* $\mathbf{d}$ replaces the variable x in the open sentence Ax. We then have:

$$\mathbf{a_w}(\forall x\, Ax) = \begin{cases} \mathrm{T} & \text{if } \mathbf{a_w}(A\mathbf{d}) = \mathrm{T} \\ & \quad \text{for every } \mathbf{d} \text{ in } \mathbf{Dw} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

In order to validate the quantifier rules, even the weakened free logic rules, objectual semantics requires that one interpret constants c rigidly, that is:

For every worlds $\mathbf{w}$ and $\mathbf{v}$ in $\mathbf{W}$, $\mathbf{a_w}(c) = \mathbf{a_v}(c)$.

We need not assume the same for other terms, if there are any.

The above provisos, along with (Def E) and (Def ∃), yield the following results:

$$\mathbf{a_w}(\exists x\, Ax) = \begin{cases} \mathrm{T} & \text{if there is some } \mathbf{d} \text{ in } \mathbf{Dw} \\ & \quad \text{such that } \mathbf{a_w}(A\mathbf{d}) = \mathrm{T} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

For any $\mathbf{o}$, where o is either a term or member of $\mathbf{D}$:

$$\mathbf{a_w}(\mathrm{E}\mathbf{o}) = \begin{cases} \mathrm{T} & \text{if } \mathbf{a_w}(\mathbf{o}) \text{ is in } \mathbf{Dw} \\ \mathrm{F} & \text{otherwise} \end{cases}$$

We can define oT, oB, oS4, oS5 models, etc., as subspecies of oK models $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{a} \rangle$ whose accessibility relations have the requisite features.

If we wish to validate the classical quantifier rules, we can also define a **qoK model** (or qoT model, qoB model, etc.) as an oK model (oT model, oB model, etc.) with a constant domain, i.e., one in which $\mathbf{Dw} = \mathbf{D}$ for all worlds $\mathbf{w}$ in $\mathbf{W}$. It follows from the result above then that $\mathbf{a_w}(\mathrm{E}t) = \mathrm{T}$ for every term t for every world $\mathbf{w}$ in any qoK model.

We can see why objectual semantics requires taking constants as rigid by considering the following instance of (∃In), derived from (∀Out):

$$\mathrm{Ec} \;\&\; \Box\, \mathrm{c} = \mathrm{c} \vdash_{\mathrm{fK}} \exists \mathrm{x} \Box\, \mathrm{x} = \mathrm{c}$$

Suppose that our model contains two worlds, $\mathbf{w}$ and $\mathbf{v}$, but suppose that constant "c" is non-rigid, and denotes one existent object at $\mathbf{w}$ but another existent object at $\mathbf{v}$. Here the premise is true; c exists and in every world, "c = c" is true. However, objectual semantics defines the truth of a quantified statement in terms of objects, and no *object* is necessarily c. The problem is avoided if a constant must name the same object in every possible world.

While in many ways more attractive than substitutional semantics, objectual semantics might also be seen as having drawbacks:

- It necessitates taking constants as rigid designators, not allowing us the flexibility to explore other options.
- It seems to presuppose the intelligibility of transworld identity, insofar as the semantic value of a constant is the same in each world, and hence, at least if there are to be any interesting necessary statements formed with constants, the very same object must exist within the domains of multiple worlds. Some find this unintelligible.
- It licenses the substitution of identicals (at least for constants) in all modal contexts.

Kripke, who first did formal semantics for modal logic in a rigorous way, was happy to bite these bullets (and indeed, argued that the thesis that names are rigid designators is independently plausible). Others, including Garson, however, have sought alternative approaches.

8.4 Intensional Semantics

The words "intension" and "extension" have been used in different ways by different philosophers and logicians since the 17th century. Originally, they were applied only to predicates, and the extension of a predicate was defined as the class of things of which it holds, and the intension was understood as the property or trait the predicate represents.

A more recent explanation of the distinction comes from Carnap. Working within the context of modal logic, Carnap defined the extension of a term as its semantic value at a given world (e.g., the actual world), and its intension as its pattern of extensions at different worlds, understood technically as a function from possible worlds to extensions. For instance, the intension of the predicate "is a horse" would be understood as a function that maps each world to the set of things that are horses at that world.

This distinction could be applied to other linguistic categories as well. The intension of a sentence, often called a **proposition**, could be understood as a function from worlds to truth values.

The intension of a term, which Carnap called **an individual concept**, could be understood as a function from worlds to individual objects.

(This way of characterizing the distinction is perhaps the most popular, but is not uncontroversial or universally accepted. However, Garson seems to adopt this terminology uncritically.)

We might have two (non-rigid) terms that have the same semantic value in a given world but different semantic values at other worlds. Suppose w is the actual word, "i" means "the inventor of bifocals" and "p" means "the first Postmaster General". In that case $\mathbf{a_w}(\mathrm{i}) = \mathbf{a_w}(\mathrm{p})$ but it might be that $\mathbf{a_v}(\mathrm{i}) \neq \mathbf{a_v}(\mathrm{p})$.

The intension of "i", which Garson writes simply "$\mathbf{a}(\mathrm{i})$" can be thought of as a function that maps each world to the person (or thing) that invented bifocals at that world.

Carnap's intensions roughly played the role in his philosophy that senses played in Frege's.

The core idea of intensional semantics is to think of the truth conditions of a quantified statement not in terms of a range of instances of the quantified statement, or in terms of a range of objects quantified over, but rather in terms of a range of individual concepts.

To avoid collapsing back into the defects of the substitutional approach, we need to countenance intensions that are not the intensions "of" any term of the object language. These cannot be represented using the notation "$\mathbf{a}(t)$". Instead, in the metalanguage, we simply write $\mathbf{i}$ for some function from worlds to members of $\mathbf{D}$, and $\mathbf{i}(\mathbf{w})$ for the member of $\mathbf{D}$ that the individual concept $\mathbf{i}$ picks out for world $\mathbf{w}$.

The impetus for the approach can be appreciated by returning to why it is that objectual semantics needed to assume that constants are rigid.

Again, assume that it were possible for "c" to denote one entity in $\mathbf{w}$ and another in $\mathbf{v}$. Suppose our model has only two possible worlds. In both, the denotation of "c" is human. In that case, we have both "Ec" and "□Hc". But it would not follow from this that any one *thing* is human in all possible worlds. However, "$\exists x \Box Hx$" is provable from these in fK, so we are faced with a paradox. To preserve the validity of the inference, we ruled out the possibility that "c" could denote different entities in different worlds to prevent this sort of situation from arising.

This would be objectionable to someone who denied the intelligibility of trans-world identity. They might assume instead that so long as the object denoted by "c" at $\mathbf{v}$ is the modal *counterpart* of the object denoted by "c" at $\mathbf{w}$, "□Hc" and "$\exists x \Box Hx$" may be true without their being any one *thing* that is human in multiple worlds.

(Consider also a temporal reading of "□" for someone who believes in temporal parts as ontologically fundamental. The claim that "$\exists x \Box Hx$", meaning "there is something that is always human," should not require that the same thing (temporal part) exist at multiple times, only that there be different things at different times (making up the same organism) each of which is human.)

Returning to our original example, it might be pointed out that while there is no *thing* that is human at every world, there is an individual concept, viz., $\mathbf{a}(\mathrm{c})$, whose value is human at every world. In intensional semantics, this may be enough to make "$\exists x \Box Hx$" true.

As attractive as this proposal sounds, one might worry that it makes it too easy for a *de re* modal claim to come as true. Suppose, e.g., that "S" is the predicate meaning "is a spy". Suppose "s" stands for the non-rigid term "the shortest spy". Suppose there are two worlds in our model; who the shortest spy is is different in the two worlds, but in each world, of course, the shortest spy is a spy. In this model, there is an individual concept, viz., $\mathbf{a}(\mathrm{s})$ whose value is a spy at every world. If this concept is allowed to count as one the quantifiers quantify over, then we shall have to accept the truth of:

$$\exists x \Box Sx$$

(Or, *something is necessarily a spy.*) And not just the relatively innocuous *de dicto* claim:

$$\Box \exists x Sx$$

(*Necessarily, something is a spy.*)

To avoid this problem, Garson defines an intensional model as specifying a *selected* subset of individual concepts, or functions from $\mathbf{W}$ into $\mathbf{D}$, that count as those the quantifiers quantify over. The presumption is that such a model would select individual concepts whose values at different worlds can in a sense be understood as parts of the same substance or system (e.g., counterparts of the same object in the modal case, temporal parts of the same continuous being in the temporal case, etc.)

However, the *definition* of an intensional model doesn't enforce any particular requirements on which individual concepts count and which do not. Any way of carving out a select group will generate an intensional model. Since not all ways of carving out a select group will warrant the inference from Es & □ Ss to ∃x □ Sx, this argument will still have counterexamples, and hence will not be logically valid.

We can fill in the details as follows. Let an iK model $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{I}, \mathbf{a} \rangle$ consist a set of worlds $\mathbf{W}$, an accessibility relation $\mathbf{R}$, a domain structure $\mathbb{D}$, an assignment function $\mathbf{a}$ and finally, $\mathbf{I}$, which is understood as a subset of the functions from $\mathbf{W}$ into $\mathbf{D}$, i.e., a subset of possible individual concepts for the models: these are the "priviledged" intensions in terms of which we understand quantification.

The semantic rule for atomic statements can be given as in objectual semantics, but we need to expand the notion of a hybrid to allow hybrids containing individual concepts as components (rather than either objects or terms), those of the form $A\mathbf{i}$. We then assume that:

$$\mathbf{a_w}(\mathbf{i}) = \mathbf{i}(\mathbf{w})$$

For quantified statements, we assume:

$$\mathbf{a_w}(\forall x\, Ax) = \begin{cases} \text{T} & \text{if for every } \mathbf{i} \text{ in } \mathbf{I}, \\ & \quad \text{if } \mathbf{i}(\mathbf{w}) \text{ is in } \mathbf{Dw} \text{ then} \\ & \quad \mathbf{a_w}(A\mathbf{i}) = \text{T} \\ \text{F} & \text{otherwise} \end{cases}$$

To validate the quantifier rules (∀In) and (∀Out), as stated using constants, we need to assume that the intension of every constant is a member of the privileged group, $\mathbf{I}$, i.e.:

For every constant c, there is a function $\mathbf{i}$ that is in $\mathbf{I}$, such that $\mathbf{a_w}(c) = \mathbf{i}(\mathbf{w})$ for all worlds $\mathbf{w}$ in $\mathbf{W}$.
(In short, $\mathbf{a}(c)$ is a member of I.)

(We need not make the same assumption about other terms t that are not constants c.)

From (Def ∃) and the above, it follows that:

$$\mathbf{a_w}(\exists x\, Ax) = \begin{cases} \text{T} & \text{if there is some } \mathbf{i} \text{ in } \mathbf{I}, \\ & \quad \text{where } \mathbf{i}(\mathbf{w}) \text{ is in } \mathbf{Dw} \text{ and} \\ & \quad \mathbf{a_w}(A\mathbf{i}) = \text{T} \\ \text{F} & \text{otherwise} \end{cases}$$

As you would expect, we can similarly characterize iT, iB, iS4 and iS5 models (and the corresponding conceptions of validity) as subspecies of iK models in which the accessibility relation has the appropriate characteristic or characteristics.

Other sorts of subspecies of iK models can be defined based on particular theories about which intensions or individual concepts should count as members of the priviledged group.

One obtains what is, for all intents and purposes, the objectual approach by considering only those iK models whose privileged class is the class of all constant intensions: those that pick out the same object in every possible world so that $\mathbf{i}(\mathbf{w}) = \mathbf{i}(\mathbf{v})$ for every $\mathbf{w}$ and $\mathbf{v}$. (Of course, here we are back to the presupposition that the same object may exist at multiple worlds, etc.)

A much more liberal approach would be to consider those models (which we might call cK models) in which all individual concepts are considered privileged (or are included in $\mathbf{I}$): the result is what Garson calls "conceptual semantics." Here, note that the argument from "Es & □ Ss" to "∃x □ Sx" is cK-valid.

Intensional semantics seem clearly to be Garson's favored approach. He stresses the following:

- It is flexible. Other approaches such as the objectual approach and the conceptual approach can be accommodated within its general rubric.
- It does not force us, just to do formal semantics for modal logic, to decide upon controversal philosophical issues such as the intelligibility of trans-world identity or the the rigidity of constants/proper names.
- While not sharing the conceptual and expressive limitations of the substitutional approach, it generates precisely the same list of valid arguments and statements. In order words:

  $$A_1, A_2, \ldots, A_n \vDash_{\text{iK}} B \;\text{ iff }\; A_1, A_2, \ldots, A_n \vDash_{\text{tK}} B$$

  (This is proven later in the book.) This is especially welcome since substitutional semantics is the easiest to correlate with the results of Trees.