1.	$\Diamond(A \,\&\, B)$	
2.	$\quad \Box, A \,\&\, B$	
3.	$\quad A$	3 (&Out)
4.	$\Diamond A$	1,2–3 ($\Diamond$Out)
5.	$\quad \Box, A \,\&\, B$	
6.	$\quad B$	5 (&Out)
7.	$\Diamond B$	1,5–6 ($\Diamond$Out)
8.	$\Diamond A \,\&\, \Diamond B$	4,7 (&In)
9.	$\Diamond(A \,\&\, B) \rightarrow (\Diamond A \,\&\, \Diamond B)$	1–8 (CP)

The converse of the above, however, does not hold, but we do have:

$$\vdash_K (\Box A \,\&\, \Diamond B) \rightarrow \Diamond(A \,\&\, B)$$

1.	$\Box A \,\&\, \Diamond B$	
2.	$\Box A$	1 (&Out)
3.	$\Diamond B$	1 (&Out)
4.	$\quad \Box, B$	
5.	$\quad A$	2 ($\Box$Out), (Reit)
6.	$\quad A \,\&\, B$	4,5 (&In)
7.	$\Diamond(A \,\&\, B)$	3,4–6 ($\Diamond$Out)
8.	$(\Box A \,\&\, \Diamond B) \rightarrow \Diamond(A \,\&\, B)$	1–7 (CP)

4 Extensions of K (T, B, S4, S5)

Logical systems capturing stronger or at least more specific conceptions of necessity can be obtained by adding additional inference rules or axioms to K.

Consider the following schemata:

(M): $\Box A \rightarrow A$

(B): $A \rightarrow \Box\Diamond A$

(4): $\Box A \rightarrow \Box\Box A$

(5): $\Diamond A \rightarrow \Box\Diamond A$

According to (M), *whatever is necessary is true*. This axiom should hold for any system codifying a conception of *alethic* modality, where necessity is understood in terms of necessary truth. An obvious derived rule, which we'll also abbreviate as (M), allows us to conclude A from $\Box A$ within the same subproof. (Notice that rule ($\Box$Out) of K does not allow this.) 'M' stands for *modality*, where this word is understood in the narrow sense.

According to (B), *what is actually true is necessarily possible*. This is named after L. E. J. Brouwer.

According to (4), *if something is necessary, it is necessarily necessary*. This is one of the original axioms used by C. I. Lewis in codifying modal logic, and retains the number used in his numbering of axioms.

According to (5), *if something is possible, it is necessarily possible*. This, the strongest of these principles, coheres with the understanding of necessity as truth in *all* possible worlds absolutely. For something to be possible is for it to be true at some world. If it is true at some world, it is true at all worlds that it is true at some world, which is what it is for it to be necessarily possible. Again, the numbering comes from Lewis.

We can define a number of systems by adding the instances of one or more of the above to K:

System T: the system obtained from K by allowing every instance of (M) as an an axiom.

System M: another name for System T.

System B: the system obtained from T by allowing every instance of (B) as an axiom.

System S4: the system obtained from T by allowing every instance of (4) as an axiom.

System S5: the system obtained from T by allowing every instance of (5) as an axiom.

Note that these systems add every instance of certain axiom schemata, not individual axioms. Axiom (M) covers not only

$\Box p \rightarrow p$

but everything of that form, including

$\Box{\sim}p \rightarrow {\sim}p$
$\Box\Box p \rightarrow \Box p$
$\Box({\sim}p \rightarrow \Diamond q) \rightarrow ({\sim}p \rightarrow \Diamond q)$, etc.

In the first proof scheme below, we shall write "$\Box{\sim}A \rightarrow {\sim}A$", and cite (M). This is not confusing a statement with its negation but rather making use of a more specific schema all of whose instances are also instances of a more general schema.

Note that axioms do not count as "hypotheses" or "premises". Whatever can be proven using only the inference "rules" of a system along with its axioms counts as a theorem of that system.

Obviously, all theorems and derived rules of PL and K hold also for these systems. We say that one system is *an extension* of another when its theorems include all those of the other. Hence, T, B, S4 and S5 are extensions of K; similarly, B, S4 and S5 are extensions of T, and we can even show that S5 is an extension of both B and S4, since their defining axiom schemata are theorem schemata of S5.

In T and its extensions, we can prove that whatever is actual is possible:

($\Diamond$In) $\vdash_T A \rightarrow \Diamond A$

1.	$\mid A$	
2.	$\mid \Box{\sim}A \rightarrow {\sim}A$	(M)
3.	$\mid {\sim}{\sim}A$	1 (DN)
4.	$\mid {\sim}\Box{\sim}A$	2,3 (MT)
5.	$\mid \Diamond A$	4 (Def $\Diamond$)
6.	$A \rightarrow \Diamond A$	1–5 (CP)

The above is also a theorem schema of B, S4 and S5 as well. As the abbreviation implies, we shall also cite this for a derived rule ($\Diamond$In), allowing us to infer $\Diamond A$ from A.

Here are some useful derived rules of K.

(${\sim}\Diamond$In) From $\Box{\sim}A$ infer ${\sim}\Diamond A$. (Really just DN.)

(${\sim}\Box$In) From $\Diamond{\sim}A$ infer ${\sim}\Box A$. *Proof scheme:*

$\Diamond{\sim}A$	
$\mid \Box A$	
$\mid \mid \Box$	
$\mid \mid A$	($\Box$Out)
$\mid \mid {\sim}{\sim}A$	(DN)
$\mid \Box{\sim}{\sim}A$	($\Box$In)
$\mid \Diamond{\sim}A$	(Reit)
$\mid {\sim}\Box{\sim}{\sim}A$	(Def $\Diamond$)
$\mid \bot$	($\bot$In)
${\sim}\Box A$	(IP)

In system B we can prove that whatever is possibly necessary is actual.

(B′) $\vdash_B \Diamond\Box A \rightarrow A$

1.	$\mid \Diamond\Box A, {\sim}A$	
2.	$\mid {\sim}A \rightarrow \Box\Diamond{\sim}A$	(B)
3.	$\mid \Box\Diamond{\sim}A$	1b,2 (MP)
4.	$\mid \mid \Box$	
5.	$\mid \mid \Diamond{\sim}A$	3 ($\Box$Out)
6.	$\mid \mid {\sim}\Box A$	5 (${\sim}\Box$In)
7.	$\mid \Box{\sim}\Box A$	4–6 ($\Box$In)
8.	$\mid {\sim}\Diamond\Box A$	7 (${\sim}\Diamond$In)
9.	$\mid \bot$	1a,8 ($\bot$In)
10.	$\Diamond\Box A \rightarrow A$	1–9 (CP)/(IP)

In S4 we can prove that whatever is possibly possible is possible.

(4′) $\vdash_{S4} \Diamond\Diamond A \rightarrow \Diamond A$

1.	$\mid \Diamond\Diamond A, {\sim}\Diamond A$	
2.	$\mid \Box{\sim}A$	1b (${\sim}\Diamond$)
3.	$\mid \Box{\sim}A \rightarrow \Box\Box{\sim}A$	(4)
4.	$\mid \Box\Box{\sim}A$	2,3 (MP)
5.	$\mid \mid \Box$	
6.	$\mid \mid \Box{\sim}A$	4 ($\Box$Out)
7.	$\mid \mid {\sim}{\sim}\Box{\sim}A$	6 (DN)
8.	$\mid \Box{\sim}{\sim}\Box{\sim}A$	5–7 ($\Box$In)
9.	$\mid \Box{\sim}\Diamond A$	8 (Def $\Diamond$)
10.	$\mid {\sim}\Box{\sim}\Diamond A$	1a (Def $\Diamond$)
11.	$\mid \bot$	9,10 ($\bot$In)
12.	$\Diamond\Diamond A \rightarrow \Diamond A$	1–11 (CP)/(IP)

Examining the moves from step 2 to step 4 to step 6 shows that in S4 one can bring in lines that begin with '$\Box$' into boxed subproofs without losing the '$\Box$'. This would also have the effect that one could apply ($\Box$Out) to a line to bring it into a boxed subproof within a boxed subproof of the subproof where a given statement with '$\Box$' as its main operator is found.

By comparing ($\Diamond$In), (4′) and (B′) to (M), (4) and (B), respectively, you can see a pattern. These statements are called the *duals* of each other. They are equivalent in systems built on K. S4 could be formulated instead by adding (4′) as an axiom schema to T.

Here's a proof of the dual of (5) in S5:

(5′) $\vdash_{S5} \Diamond\Box A \rightarrow \Box A$

1.	$\mid\ \Diamond\Box A, {\sim}\Box A$	
2.	$\mid\ \Diamond{\sim}A$	1b (${\sim}\Box$)
3.	$\mid\ \Diamond{\sim}A \rightarrow \Box\Diamond{\sim}A$	(5)
4.	$\mid\ \Box\Diamond{\sim}A$	2,3 (MP)
5.	$\mid\ \mid\ \Box$	
6.	$\mid\ \mid\ \Diamond{\sim}A$	4 ($\Box$Out)
7.	$\mid\ \mid\ {\sim}\Box A$	6 (${\sim}\Box$In)
8.	$\mid\ \Box{\sim}\Box A$	5–7 ($\Box$In)
9.	$\mid\ {\sim}\Box{\sim}\Box A$	1a (Def $\Diamond$)
10.	$\mid\ \bot$	8,9 ($\bot$In)
11.	$\Diamond\Box A \rightarrow \Box A$	1–10 (CP)/(IP)

Here, examining the steps from 2 to 4 to 6 shows that in (S5) we can bring statements whose main operators are ‘◇’ into boxed subproofs unmodified.

Earlier it was mentioned that S5 is an extension of both S4 and B. We can show this by showing that the axiom schemata of these systems are theorem schemata of S5.

$\vdash_{S5} A \rightarrow \Box\Diamond A$

1.	$A \rightarrow \Diamond A$	($\Diamond$In)
2.	$\Diamond A \rightarrow \Box\Diamond A$	(5)
3.	$A \rightarrow \Box\Diamond A$	1,2 (HS)

$\vdash_{S5} \Box A \rightarrow \Box\Box A$

1.	$\Box A \rightarrow \Box\Diamond\Box A$	(B)
2.	$\mid\ \Box$	
3.	$\mid\ \Diamond\Box A \rightarrow \Box A$	(5′)
4.	$\Box(\Diamond\Box A \rightarrow \Box A)$	2–3 ($\Box$In)
5.	$\Box(\Diamond\Box A \rightarrow \Box A) \rightarrow (\Box\Diamond\Box A \rightarrow \Box\Box A)$	(Dist)
6.	$\Box\Diamond\Box A \rightarrow \Box\Box A$	4,5 (MP)
7.	$\Box A \rightarrow \Box\Box A$	1,6 (HS)

It follows that all theorems of B and S5, including (4′) and (B′), etc., are also theorems of S5.

S5 is the logical system for modality assumed in most philosophical discussions of necessity and possibility.

Iterated modalities in S4 and S5 can be eliminated or introduced harmlessly. In S4 (and hence S5 as well), any two repeated modal operators of the same type are equivalent. We have not only:

$\vdash_{S4} \Box A \leftrightarrow \Box\Box A$

$\vdash_{S4} \Diamond A \leftrightarrow \Diamond\Diamond A$

But also:

$\vdash_{S4} \Box A \leftrightarrow \Box\Box\Box A$

1.	$\Box A \rightarrow \Box\Box A$	(4)
2.	$\Box\Box A \rightarrow \Box\Box\Box A$	(4)
3.	$\Box A \rightarrow \Box\Box\Box A$	1,2 (HS)
4.	$\Box\Box\Box A \rightarrow \Box\Box A$	(M)
5.	$\Box\Box A \rightarrow \Box A$	(M)
6.	$\Box\Box\Box A \rightarrow \Box A$	4,5 (HS)
7.	$\Box A \leftrightarrow \Box\Box\Box A$	3,6 ($\leftrightarrow$In)

Similarly, we have:

$\vdash_{S4} \Diamond A \leftrightarrow \Diamond\Diamond\Diamond A$

Iterated modalities can even be eliminated or introduced inside the scope of other modal operators, e.g.:

$\vdash_{S4} \Box(A \rightarrow \Box\Box B) \leftrightarrow \Box(A \rightarrow \Box B)$

1.	$\mid\ \Box(A \rightarrow \Box\Box B)$	
2.	$\mid\ \mid\ \Box$	
3.	$\mid\ \mid\ A \rightarrow \Box\Box B$	1 ($\Box$Out)
4.	$\mid\ \mid\ \Box\Box B \rightarrow \Box B$	(M)
5.	$\mid\ \mid\ A \rightarrow \Box B$	3,4 (HS)
6.	$\mid\ \Box(A \rightarrow \Box B)$	2–5 ($\Box$In)
7.	$\Box(A \rightarrow \Box\Box B) \rightarrow \Box(A \rightarrow \Box B)$	1–6 (CP)
8.	$\mid\ \Box(A \rightarrow \Box B)$	
9.	$\mid\ \mid\ \Box$	
10.	$\mid\ \mid\ A \rightarrow \Box B$	8 ($\Box$Out)
11.	$\mid\ \mid\ \Box B \rightarrow \Box\Box B$	(4)
12.	$\mid\ \mid\ A \rightarrow \Box\Box B$	10,11 (HS)
13.	$\mid\ \Box(A \rightarrow \Box\Box B)$	9–12 ($\Box$In)
14.	$\Box(A \rightarrow \Box B) \rightarrow \Box(A \rightarrow \Box\Box B)$	8–13 (CP)
15.	$\Box(A \rightarrow \Box\Box B) \leftrightarrow \Box(A \rightarrow \Box B)$	7–14 ($\leftrightarrow$In)

In S5, we get an even stronger result: any series of iterated modal operators, whether the same or different types, reduces to the final one in the series. Here are some examples:

(Note that in these examples we start using "(M)", "(4)", "(5)", etc., not just as names of axioms or theorems, but as derived rules for the results of *modus ponens* steps where an instance of these axioms/theorems plays the role of the conditional involved. We shall continue to do so.)

$\vdash_{S5} \Box\Diamond\Diamond\Box A \leftrightarrow \Box A$

1.	$\Box\Diamond\Diamond\Box A$	
2.	$\Diamond\Diamond\Box A$	1 (M)
3.	$\Diamond\Box A$	2 (4′)
4.	$\Box A$	3 (5′)
5.	$\Box\Diamond\Diamond\Box A \rightarrow \Box A$	1–4 (CP)
6.	$\Box A$	
7.	$\Diamond\Box A$	6 ($\Diamond$In)
8.	$\Diamond\Diamond\Box A$	7 ($\Diamond$In)
9.	$\Box\Diamond\Diamond\Box A$	8 (5)
10.	$\Box A \rightarrow \Box\Diamond\Diamond\Box A$	6–9 (CP)
11.	$\Box\Diamond\Diamond\Box A \leftrightarrow \Box A$	5,10 ($\leftrightarrow$In)

$\vdash_{S5} \Diamond(\Box\Diamond A \;\&\; \Diamond\Box B) \leftrightarrow \Diamond(\Diamond A \;\&\; \Box B)$

1.	$\Diamond(\Box\Diamond A \;\&\; \Diamond\Box B)$	
2.	$\Box, \Box\Diamond A \;\&\; \Diamond\Box B$	
3.	$\Box\Diamond A$	2 (&Out)
4.	$\Diamond\Box B$	2 (&Out)
5.	$\Diamond A$	3 (M)
6.	$\Box B$	4 (5′)
7.	$\Diamond A \;\&\; \Box B$	5,6 (&Out)
8.	$\Diamond(\Diamond A \;\&\; \Box B)$	1,2–7 ($\Diamond$Out)
9.	$\Diamond(\Box\Diamond A \;\&\; \Diamond\Box B) \rightarrow \Diamond(\Diamond A \;\&\; \Box B)$	1–8 (CP)
10.	$\Diamond(\Diamond A \;\&\; \Box B)$	
11.	$\Box, \Diamond A \;\&\; \Box B$	
12.	$\Diamond A$	11 (&Out)
13.	$\Box B$	11 (&Out)
14.	$\Box\Diamond A$	12 (5)
15.	$\Diamond\Box B$	13 ($\Diamond$In)
16.	$\Box\Diamond A \;\&\; \Diamond\Box B$	14,15 (&In)
17.	$\Diamond(\Box\Diamond A \;\&\; \Diamond\Box B)$	10,11–16 ($\Diamond$Out)
18.	$\Diamond(\Diamond A \;\&\; \Box B) \rightarrow \Diamond(\Box\Diamond A \;\&\; \Diamond\Box B)$	10–17 (CP)
19.	$\Diamond(\Box\Diamond A \;\&\; \Diamond\Box B) \leftrightarrow \Diamond(\Diamond A \;\&\; \Box B)$	9,18 ($\leftrightarrow$In)

It may help for your homework to know that Garson uses notation like "M + (4) + (B)" to speak of the system obtained from System M (also known as System T) by adding both (4) and (B) as axiom schemata. Similarly "K + (5)" would be the system obtained from K by adding only (5) as an axiom schema. These might also be abbreviated as "M4B" or "K5", respectively, for short. In your homework, you'll prove that M + (4) + (B) is equivalent to system S5: i.e., that they have the same theorems. (We have already seen that (4) and (B) are theorems of S5. All that remains is show that (5) follows from (M), (4) and (B), along with the rules of K.)

In this way of writing, S5 could also be called M5 or K5M, etc.. The label S5, however, has become the standard name for historical reasons (again, going back to the work of C. I. Lewis).