9.3 Tree Conversion

Converting trees to proofs in quantified modal logic is both possible and straightforward. Indeed, it is not worth our time to go over all the details.

The basic procedure is the same as for propositional logic.

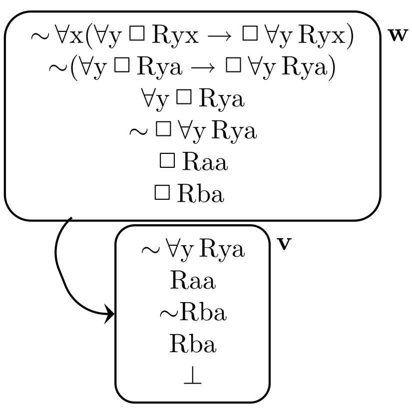
The tree rules for identity logic correspond directly to inference rules or axioms.

The rules for true ∀-statements and false ∃-statements correspond to (∀Out) or (Q∀Out); in the latter case, you can use (∼∃Out) to change $\sim\exists x\, Ax$ to $\forall x \sim Ax$ beforehand.

The rules for true ∃-statements and false ∀-statements correspond to the (∃Out) or (Q∃Out) derived rules. (In the latter case, you can use (∼∀Out) to change $\sim\forall x\, Ax$ to $\exists x \sim Ax$ beforehand.)

The only tricky thing has to do with occasions in which we have to re-apply a rule of the former kind in an earlier world box after applying a rule of the latter kind in a later world box. If converted directly, the constant introduced for (∃Out) or (Q∃Out) may not be new, thus making the derived rule inapplicable.

Earlier we saw this example (for classical quantification):



The way around this is to note that since "b" was new when we introduced it, it must not occur in any of the *hypotheses* above when converted. (Hypotheses occur only with premises, the negation of the conclusion, branches and the first lines in a world box; but this structure must already be in place when the rule was first used.)

Rather than using it, we may take the result of the (Q∃Out) as a new assumption. It leads to "⊥", so we may take this as a *reductio* of the result. Since the constant does not occur in any hypotheses, we may use (∀In). The result will be the opposite of the existential statement or false universal statement with which we began.

For the above, we get:

1.	w	$\sim\forall x(\forall y\,\Box Ryx \to \Box\forall y\,Ryx)$	
2.		$\exists x \sim(\forall y\,\Box Ryx \to \Box\forall y\,Ryx)$	
3.		$\sim(\forall y\,\Box Rya \to \Box\forall y\,Rya)$	2 (Q∃Out)
4.		$\forall y\,\Box Rya$	3 (→F)
5.		$\sim\Box\forall y\,Rya$	3 (→F)
6.		$\Box Raa$	4 (Q∀Out)
7.		$\Box Rba$	4 (Q∀Out)
8.		$\Diamond\sim\forall y\,Rya$	5 (∼□)
9.	v	$\Box, \sim\forall y\,Rya$	
10.		Raa	6 (□Out)
11.		$\sim Rba$	
12.		Rba	7 (Reit), (□Out)
13.		$\bot$	11,12 (⊥In)
14.		Rba	11–13 (IP)
15.		$\forall y\,Rya$	14 (Q∀In)
16.		$\bot$	9,15 (⊥In)
17.		$\bot$	8,9–16 (◇ ⊥)
18.		$\forall x(\forall y\,\Box Ryx \to \Box\forall y\,Ryx)$	1–17 (IP)

This or a similar procedure should handle all tree conversions.

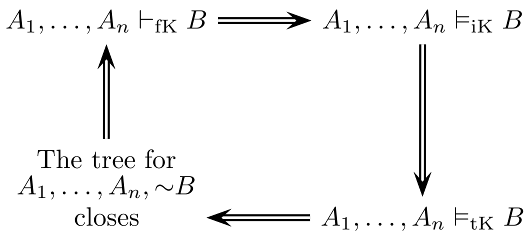
10 Metatheory for Quantified Modal Logic

For the most part, the results we are after are similar to those we saw in our earlier unit. Indeed, I don't think it is worth our time to examine the metalinguistic proofs in as much detail as we did then, since much of it would be redundant.

(Assigning you to fill in the details might make an excellent final exam question, however.)

Things are a bit more complicated as well by the fact that we've looked at three ways of doing formal semantics for quantified modal logic. Proving soundness and completeness results separately for each would be time consuming.

In class, I shall focus on establishing the following square of results.



(As well as similar results for fT, fB, fS4, fS5, qK, qS5, etc.)

Put together, these results establish both soundness and completeness for fK vis-à-vis both substitutional and intensional semantics, as well as the extensional equivalence of those two approaches to semantics.

(The results will also point the way towards finding similar results for objectual semantics, since oK models can be defined as a subspecies of iK models.)

For all intents and purposes, we have already sketched the proof for the left side of this square, since closed trees can be converted into reductio proofs. Provided we have drawn the right kind of trees, the results hold not just for fK, but also fT, FB, fK4, fS5, etc., as well as qK, qB, qS5 and the others.

What remains are the three other sides. We shall content ourselves with mere sketches.

10.1 Tree Modeling

Taking things somewhat out of order from Garson's presentation of this material (jumping ahead to chap. 16), we now move on to the bottom part of the square, i.e, that if a given argument is valid on substitutional/truth-value semantics, then the tree for it closes.

As before, we establish this by proving the *contrapositive*, i.e., that if tree for an argument $A_1, \ldots, A_n / \therefore B$ has an open branch, then it is invalid, i.e., there is a tK (or tT or tS5, etc.) counterexample to it.

The argument for this is almost exactly the same as the argument we saw for it in propositional logic. We define the **Tree Model** as the model $\langle \mathbf{W}, \mathbf{R}, \mathbf{a} \rangle$ where $\mathbf{W}$ is the set of worlds that have world boxes on the tree, $\mathbf{R}$ is the accessibility relation between them as shown on the Tree, and $\mathbf{a}$ makes an atomic statement true for a given world if and only if it appears in the world box for a given world. To count as a tK model, the model must obey the rule (t=) stated on page 48 of the handouts. Notice that this will be guaranteed by the application of the identity tree rules.

If rules for rigid constants systems have been applied, then the Tree Model will count as a rtK model, i.e., a tK model in which $\mathbf{a_w}(b = c) = \mathbf{a_v}(b = c)$ for all worlds $\mathbf{w}$ and $\mathbf{v}$ in $\mathbf{W}$, since this atomic statement will appear in all world boxes if it appears in any.

Again, we use wff-induction to show that for every sentence C, if C appears on the open branch in the world box for a given world $\mathbf{w}$ in $\mathbf{W}$, then $\mathbf{a_w}(C) = \text{T}$. The base case is true by the definition of the tree model, and the fact that the branch is open.

For the induction step, we assume as inductive hypothesis that this holds for all sentences shorter than C, and need to prove that it holds for C as well. For the cases in which C takes one of the forms $\Box D$, $D_1 \to D_2$, $\sim P(t_1, \ldots, t_n)$, $\sim \Box D$, $\sim(D_1 \to D_2)$, $\sim\bot$, we get the result the same as with our propositional soundness proof. We need only show the same for statements of the forms $\forall x\, Dx$ and $\sim \forall x\, Dx$. (We are assuming that all defined notation has been eliminated.).

Suppose C takes the form $\forall x\, Dx$. If all tree rules have been applied, then for every constant c, if Ec appears on the tree branch in the world box for $\mathbf{w}$, then so does Dc. By the inductive hypothesis, this means if $\mathbf{a_w}(\text{E}c) = \text{T}$ then $\mathbf{a_w}(Dc) = \text{T}$. By the semantics for the quantifier in substitutional models, this means that $\mathbf{a_w}(\forall x\, Dx) = \text{T}$, i.e., $\mathbf{a_w}(C) = \text{T}$.

Suppose C takes the form $\sim \forall x\, Dx$. If all tree rules have been applied, there is a constant c such that both Ec and $\sim Dc$ appears on the tree in the world box. It follows by inductive hypothesis that $\mathbf{a_w}(\text{E}c) = \text{T}$ and $\mathbf{a_w}(\sim Dc) = \text{T}$. It follows that $\mathbf{a_w}(Dc) = \text{F}$. Hence, it is not the case that for all c, if $\mathbf{a_w}(\text{E}c) = \text{T}$ then $\mathbf{a_w}(Dc) = \text{T}$. By the semantics for the quantifier, $\mathbf{a_w}(\forall x\, Dx) = \text{F}$, and therefore $\mathbf{a_w}(\sim \forall x\, Dx) = \text{T}$, i.e., $\mathbf{a_w}(C) = \text{T}$.

This completes the induction. Hence the premises at the top of the tree are true in the starting world, and the negation of the conclusion is also true, and thefore, the conclusion is false. It follows that the tree model is a countermodel.

10.2 Soundness

We now turn to the *top* of our rectange, and establish that if a given argument $A_1, \ldots, A_n / \therefore B$ has an fK (or fT, fB, fS4, fS5) proof, then it is is iK (or iT, iB, iS4, iS5, respectively) valid, i.e., for all worlds $\mathbf{w}$

in $\mathbf{W}$ in an iK model $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{I}, \mathbf{a} \rangle$, if $\mathbf{a_w}(A_1) = \mathrm{T}$ and ... and $\mathbf{a_w}(A_n) = \mathrm{T}$ then $\mathbf{a_w}(B) = \mathrm{T}$ as well.

Again, we assume we have a proof in the appropriate system from $A_1, \ldots, A_n$ to B. We require the proof be written out in full, with no derived rules or notation or theorems cited. Assume also that for an arbitrary intensional model $\langle \mathbf{W}, \mathbf{R}, \mathbb{D}, \mathbf{I}, \mathbf{a} \rangle$ and world $\mathbf{w}$ in $\mathbf{W}$, $\mathbf{a_w}(A_1) = \mathrm{T}$ and ... and $\mathbf{a_w}(A_n) = \mathrm{T}$. We need to show that $\mathbf{a_w}(B) = \mathrm{T}$ as well.

We prove by proof induction that every line in the proof has the property ϕ, i.e., that it is true in all worlds $\mathbf{v}$ in $\mathbf{W}$ that makes all the relevant assumptions for that line true. The relevant assumptions are defined just as we did for propositional logic; see page 34 of the handouts.

We assume as inductive hypothesis that ϕ holds of all steps prior to the current step C in the proof. We need to show that it holds of C as well. Suppose that $\mathbf{v}$ makes all relevant assumptions true. Let us show that $\mathbf{a_v}(C) = \mathrm{T}$ We tackle this by cases. If line C is a hypothesis, an (MP) step, a (DN) step, a (CP) step, a ($\Box$Out) step, a ($\Box$In) step, a (Reit) step, or a modal axiom, we obtain the result in precisely the same way as in propositional logic.

There are four more cases to consider:

- *New case 1*: C is an (=In) step, and takes the form $t = t$. Since $\langle \mathbf{a_v}(t), \mathbf{a_v}(t) \rangle$ is in $\mathbf{a_v}(I)$, it follows that $\mathbf{a_v}(C) = \mathrm{T}$.

- *New case 2*: C is an (=Out) step. Hence, C takes the form $P(u_1, \ldots, u_n, t, v_1, \ldots, v_m)$, and we have, in the same subproof, $s = t$ and $P(u_1, \ldots, u_n, s, v_1, \ldots, v_m)$. By the inductive hypothesis, $\mathbf{a_v}(s = t) = \mathrm{T}$, which means that $\mathbf{a_v}(s) = \mathbf{a_v}(t)$. Also by the inductive hypothesis, $\mathbf{a_v}(P(u_1, \ldots, u_n, s, v_1, \ldots, v_m)) = \mathrm{T}$, so $\langle \mathbf{a_v}(u_1), \ldots, \mathbf{a_v}(u_n), \mathbf{a_v}(s), \mathbf{a_v}(v_1), \ldots, \mathbf{a_v}(v_m) \rangle$ is in $\mathbf{a_v}(P)$. But this is the same as $\langle \mathbf{a_v}(u_1), \ldots, \mathbf{a_v}(u_n), \mathbf{a_v}(t), \mathbf{a_v}(v_1), \ldots, \mathbf{a_v}(v_m) \rangle$, and so $\mathbf{a_v}(P(u_1, \ldots, u_n, t, v_1, \ldots, v_m)) = \mathrm{T}$, i.e., $\mathbf{a_v}(C) = \mathrm{T}$.

- *New case 3*: C is an ($\forall$Out) step. Hence C takes the form $\mathrm{E}c \to Dc$, and we have a previous step of the form $\forall x\, Dx$, and, by the inductive hypothesis $\mathbf{a_v}(\forall x\, Dx) = \mathrm{T}$. There are two subcases to consider. Suppose $\mathbf{a_v}(c)$ is not in $\mathbf{Dv}$. Then $\mathbf{a_v}(\mathrm{E}c) = \mathrm{F}$ and hence, $\mathbf{a_v}(\mathrm{E}c \to Dc) = \mathrm{T}$. Suppose instead that $\mathbf{a_v}(c)$ is in $\mathbf{Dv}$. For each constant c, the intension $\mathbf{a}(c)$ of c in the model is a member $\mathbf{i}$ of $\mathbf{I}$, and because $\mathbf{a_v}(c)$ is in $\mathbf{Dv}$, $\mathbf{i}(\mathbf{v})$ is in $\mathbf{Dv}$. Hence, by the semantics for quantified statements, for the hybrid $D\mathbf{i}$, we have $\mathbf{a_v}(D\mathbf{i}) = \mathrm{T}$. Since $\mathbf{i}$ is the intension of c, it follows that $\mathbf{a_v}(Dc) = \mathrm{T}$, and therefore, $\mathbf{a_v}(\mathrm{E}c \to Dc) = \mathrm{T}$. Either way, $\mathbf{a_v}(C) = \mathrm{T}$.

- *New case 4.* C is an ($\forall$In) step. Hence, C takes the form $\forall x\, Dx$ and we have a previous line of the form $\mathrm{E}c \to Dc$, where c does not appear in any hypothesis. Since c is a constant, there is an intension $\mathbf{i}$ in $\mathbf{I}$ such that $\mathbf{a_v}(\mathrm{E}\mathbf{i} \to D\mathbf{i})$; in fact, since no special assumptions were made about c, the same must be true for any intension $\mathbf{i}$ in $\mathbf{I}$.[1] So for every such intension, either $\mathbf{a_v}(\mathrm{E}\mathbf{i}) = \mathrm{F}$ or $\mathbf{a_v}(D\mathbf{i}) = \mathrm{T}$. For those such that $\mathbf{i}(\mathbf{v})$ is in $\mathbf{Dv}$, it cannot be that $\mathbf{a_v}(\mathrm{E}\mathbf{i}) = \mathrm{F}$, and so, for those, $\mathbf{a_v}(D\mathbf{i}) = \mathrm{T}$. By the semantics for the quantifier, we get that $\mathbf{a_v}(\forall x\, Dx) = \mathrm{T}$, i.e., $\mathbf{a_v}(C) = \mathrm{T}$.

That exhausts the rules present in fK at least. Hence, ϕ holds for every line C in our proof. This is true of the last line, i.e., B. Hence B is true in every world $\mathbf{v}$ in our model that makes the premises true. Hence the argument from $A_1, \ldots, A_n$ to B is valid in the appropriate sense.

If we want to cover those proofs that utilize additional like axiom schemata like (RC), we must compare their validity not to fK or qK models directly, but only a limited subset, such as, with (RC), those in which $\mathbf{a_w}(c) = \mathbf{a_v}(c)$ for every constant c and worlds $\mathbf{w}$ and $\mathbf{v}$ in the model. It then follows that any instance of (RC) has ϕ, because any instance of (RC) is true in every world in every model with rigid constants. We can show this as follows.

Suppose for *reductio* that $\mathbf{a_v}((b = c \to \Box\, b = c)\ \&\ (b \neq c \to \Box\, b \neq c)) = \mathrm{F}$ in some such world. Then it must be that either $\mathbf{a_v}(b = c \to \Box\, b = c) = \mathrm{F}$ or $\mathbf{a_v}(b \neq c \to \Box\, b \neq c) = \mathrm{F}$. Neither option is possible in the kind of model we have to consider. If it were the case that $\mathbf{a_v}(b = c \to \Box\, b = c) = \mathrm{F}$, then $\mathbf{a_v}(b = c) = \mathrm{T}$ but $\mathbf{a_v}(\Box\, b = c) = \mathrm{F}$. This means that $\mathbf{a_v}(b) = \mathbf{a_v}(c)$. It also means that there is some world $\mathbf{u}$ such that $\mathbf{vRu}$ and $\mathbf{a_u}(b = c) = \mathrm{F}$. This means that $\mathbf{a_u}(b) \neq \mathbf{a_u}(c)$. But this is impossible, since $\mathbf{a_u}(b) = \mathbf{a_v}(b)$ and $\mathbf{a_u}(c) = \mathbf{a_v}(c)$.

For homework you need to show that it is also impossible that $\mathbf{a_v}(b \neq c \to \Box\, b \neq c) = \mathrm{F}$, which is the same as exercise 15.3 in the book.

This completes our discussion of soundness.

[1] A more rigorous proof of this is given in the book.